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ON THE LIE STRUCTURE OF LOCALLY MATRIX ALGEBRAS

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Let A be a unital locally matrix algebra over a field $\mathbb{F}$ of characteristic different from 2. We find a necessary and sufficient condition for the Lie algebra $A/\mathbb{F}\cdot 1$ to be simple and for the Lie algebra of derivations $\mathrm{Der}(A)$ to be topologically simple. The condition depends on the Steinitz number of A only.

Key words and phrases: locally matrix algebra, derivation.

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INTRODUCTION

Let F be a ground field of characteristic different from 2 and let $\mathbb{N}$ be the set of all positive integers. Recall that an associative F-algebra A is called a *locally matrix algebra* (see [9,10]) if for an arbitrary finite subset of A there exists a subalgebra $B \subset A$ containing this subset and such that $B \cong M_n(F)$ for some $n \in \mathbb{N}$. We call a locally matrix algebra *unital* if it contains unit 1.

Let $\mathbb{P}$ be the set of all primes. An infinite formal product of the form

$$s = \prod_{p\in\mathbb{P}} p^{r_p}, \quad \text{where} \quad r_p \in \mathbb{N}\cup\{0,\infty\} \quad \text{for all} \quad p\in\mathbb{P}, \tag{1}$$

is called *Steinitz number*; see [12]. Denote by symbol $\mathbb{SN}$ the set of all Steinitz numbers. Let

$$s_1 = \prod_{p\in\mathbb{P}} p^{r_p}, \; s_2 = \prod_{p\in\mathbb{P}} p^{k_p} \in \mathbb{SN}.$$

Then

$$s_1\cdot s_2 = \prod_{p\in\mathbb{P}} p^{r_p+k_p}, \quad \text{where} \quad k_p\in\mathbb{N}\cup\{0,\infty\},$$

and $t+\infty = \infty + t = \infty+\infty = \infty$ for all $t\in\mathbb{N}$.

Let A be a countable-dimensional unital locally matrix algebra. In [5], J.G. Glimm defined the Steinitz number $\mathbf{st}(A)$ of the algebra A and proved that the algebra A is uniquely determined by $\mathbf{st}(A)$.

In [2], we extended Glimm's definition to unital locally matrix algebras of arbitrary dimensions. For a unital locally matrix algebra A denote by $D(A)$ the set of all numbers $n\in\mathbb{N}$ such that there exists a subalgebra $A'\subseteq A$, $1\in A'$, and $A'\cong M_n(F)$. The *Steinitz number* $\mathbf{st}(A)$ *of the algebra* A is the least common multiple of the set $D(A)$. It turned out that a unital locally matrix algebra A of dimension $>\aleph_0$ is no longer determined by its Steinitz number $\mathbf{st}(A)$; see [2,3].

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An associative algebra A gives rise to the Lie algebra

$$A^{(-)} = (A, [a,b] = ab - ba).$$

Along with the Lie algebra $A^{(-)}$ we will consider its square $[A, A]$. Let $Z(A)$ denote the center of the associative algebra A. In [6], I.N. Herstein showed that if A is a simple associative algebra then the Lie algebra

$$[A, A] \diagup Z(A) \cap [A, A] \tag{2}$$

is simple. Since a locally matrix algebra A is simple, it follows that the Lie algebra (2) is simple.

Let $M_n(\mathbb{F})$ be a matrix algebra, $\mathfrak{gl}(n) = \big(M_n(\mathbb{F})\big)^{(-)}$, $Z(M_n(\mathbb{F})) = \mathbb{F} \cdot 1$. The Lie algebra

$$\mathfrak{pgl}(n) = \mathfrak{gl}(n) \diagup \mathbb{F} \cdot 1$$

is simple unless $p = \operatorname{char} \mathbb{F} > 0$ and p divides n; see [11]. We will show that for an infinite-dimensional unital locally matrix algebra A simplicity of the Lie algebra

$$A^{(-)} \diagup \mathbb{F} \cdot 1 \tag{3}$$

depends only on the Steinitz number $\mathbf{st}(A)$.

For a Steinitz number (1) denote $\nu_p(s) = r_p$.

Theorem 1. *The Lie algebra* (3) *is simple if and only if*

$$\operatorname{char} \mathbb{F} = 0$$

or

$$\operatorname{char} \mathbb{F} = p > 0 \quad \textit{and} \quad \nu_p(\mathbf{st}(A)) = 0 \quad \textit{or} \quad \infty.$$

Recall that a linear transformation $d : A \to A$ of an algebra A is called a *derivation* if

$$d(ab) = d(a) \cdot b + a \cdot d(b)$$

for arbitrary elements $a,\ b \in A$. The vector space $\operatorname{Der}(A)$ of all derivations of an algebra A is a Lie algebra with respect to commutation; see [7]. If A is an associative algebra then for an arbitrary element $a \in A$ the operator

$$\operatorname{ad}(a) : A \to A, \quad x \mapsto [a, x],$$

is an *inner derivation*. The subspace $\operatorname{Inder}(A) = \{\operatorname{ad}(a) \mid a \in A\}$ of all inner derivations is an ideal of the Lie algebra $\operatorname{Der}(A)$.

Let X be an arbitrary set. The set $\operatorname{Map}(X, X)$ of all mappings $X \to X$ is equipped with Tykhonoff topology; see [13]. The subspace of all derivations $\operatorname{Der}(A)$ of an algebra A is closed in $\operatorname{Map}(A, A)$ in Tykhonoff topology. It makes the Lie algebra $\operatorname{Der}(A)$ a topological algebra.

Theorem 2. *Let A be a unital locally matrix algebra. Then*

(1) *the Lie algebra* $[\operatorname{Der}(A), \operatorname{Der}(A)]$ *is topologically simple;*

(2) *the Lie algebra* $\operatorname{Der}(A)$ *is topologically simple if and only if* $\operatorname{char} \mathbb{F} = 0$ *or* $\operatorname{char} \mathbb{F} = p > 0$ *and* $\nu_p(\mathbf{st}(A)) = 0$ *or* ∞.

1 PROOF OF THE THEOREM 1

Proof. Let A be a unital locally matrix $\mathbb{F}$-algebra. We will show that

$$A = [A, A] + \mathbb{F} \cdot 1 \quad \text{if and only if} \quad \operatorname{char} \mathbb{F} = 0$$

$$\text{or} \quad \operatorname{char} \mathbb{F} = p > 0 \quad \text{and} \quad \nu_p(\mathbf{st}(A)) = 0 \text{ or } \infty.$$

Consider a matrix algebra $M_n(\mathbb{F})$. Then $\big[M_n(\mathbb{F}), M_n(\mathbb{F})\big] = \big\{a \in M_n(\mathbb{F}) \,\big|\, \operatorname{tr}(a) = 0\big\} = \mathfrak{sl}(n)$. If $\operatorname{char} \mathbb{F} = 0$ or $\operatorname{char} \mathbb{F} = p > 0$ and p does not divide n then $\operatorname{tr}(1) = n \neq 0$, and therefore $a = \Big(a - \frac{1}{n} \operatorname{tr}(a) \cdot 1\Big) + \frac{1}{n} \operatorname{tr}(a) \cdot 1 \in \mathfrak{sl}(n) + \mathbb{F} \cdot 1$ for an arbitrary element $a \in A$. It implies that

$$M_n(\mathbb{F}) = \big[M_n(\mathbb{F}), M_n(\mathbb{F})\big] + \mathbb{F} \cdot 1.$$

If p divides n then $\operatorname{tr}(1) = 0$, hence

$$\big[M_n(\mathbb{F}), M_n(\mathbb{F})\big] + \mathbb{F} \cdot 1 = \mathfrak{sl}(n) \neq M_n(\mathbb{F}).$$

If $\operatorname{char} \mathbb{F} = 0$ or $\operatorname{char} \mathbb{F} = p > 0$ and p does not divide $\mathbf{st}(A)$ then for an arbitrary matrix subalgebra $1 \in A_1 \subset A$, $A_1 \cong M_n(\mathbb{F})$, the characteristic p does not divide n. Hence

$$A_1 = [A_1, A_1] + \mathbb{F} \cdot 1.$$

So, $A = [A, A] + \mathbb{F} \cdot 1$.

Suppose now that p^∞ divides $\mathbf{st}(A)$. Consider a matrix subalgebra

$$1 \in A_1 \subset A, \quad A_1 \cong M_n(\mathbb{F}).$$

The number n divides $\mathbf{st}(A)$. Since p^∞ divides $\mathbf{st}(A)$ it follows that pn also divides $\mathbf{st}(A)$. Hence, there exists a subalgebra $A_2 \subset A$ such that

$$A_1 \subset A_2, \quad A_2 \cong M_m(\mathbb{F}) \quad \text{and} \quad p \quad \text{divides} \quad m/n.$$

Let C be the centralizer of the subalgebra A_1 in A_2. We have

$$A_2 = A_1 \otimes_{\mathbb{F}} C, \quad C \cong M_{m/n}(\mathbb{F})$$

(see [4,8]). For arbitrary elements $a \in A_1$, $b \in C$ we have

$$\operatorname{tr}_{A_2}(a\, b) = \operatorname{tr}_{A_1}(a) \cdot \operatorname{tr}_C(b),$$

where tr_{A_2}, tr_{A_1}, tr_C are traces in the subalgebras A_2, A_1, C, respectively. We have

$$\operatorname{tr}_C(1) = \frac{m}{n} = 0.$$

Hence,

$$\operatorname{tr}_{A_2}(A_1 \otimes 1) = \operatorname{tr}_{A_1}(A_1) \cdot \operatorname{tr}_C(1) = \{0\},$$

and therefore $A_1 \subseteq [A_2, A_2]$. We showed that if p^∞ divides $\mathbf{st}(A)$ then $A = [A, A]$.

Suppose now that $\nu_p(\mathbf{st}(A)) = k$, $1 \leq k < \infty$. Consider a subalgebra

$$1 \in A_1 \subset A, \quad A_1 \cong M_{p^k}(\mathbb{F}).$$

Choose an element $a \in A_1$ such that $\mathrm{tr}_{A_1}(a) \neq 0$. We claim that

$$a \notin [A, A] + \mathbb{F} \cdot 1. \tag{4}$$

Indeed, if the element a lies in the right hand side then there exists a subalgebra $A_2 \subset A$ such that

$$A_1 \subset A_2, \quad A_2 \cong M_n(\mathbb{F}) \quad \text{and} \quad a \in [A_2, A_2] + \mathbb{F} \cdot 1.$$

Since p divides n it follows that $\mathrm{tr}_{A_2}(1) = 0$, hence $\mathrm{tr}_{A_2}(a) = 0$.

As above, let C be the centralizer of the subalgebra A_1 in A_2, so that $A_2 = A_1 \otimes_{\mathbb{F}} C$. The algebra C is isomorphic to the matrix algebra $M_m(\mathbb{F})$, where $m = n/p^k$. The number m is coprime with p, hence $\mathrm{tr}_C(1) = m \neq 0$. Now,

$$\mathrm{tr}_{A_2}(a) \;=\; \mathrm{tr}_{A_2}(a \otimes 1) \;=\; \mathrm{tr}_{A_1}(a) \;\cdot\; \mathrm{tr}_C(1) \;\neq\; 0.$$

This contradiction completes the proof of the claim (4).

If $A = [A, A] + \mathbb{F} \cdot 1$ then

$$A^{(-)} \big/ \mathbb{F} \cdot 1 \;\cong\; [A, A] \big/ [A, A] \cap \mathbb{F} \cdot 1.$$

In this case, the Lie algebra $A^{(-)} \big/ \mathbb{F} \cdot 1$ is simple by I.N. Herstein's Theorem (see [6]). If $A = [A, A] + \mathbb{F} \cdot 1$ is a proper subspace of A then

$$[A, A] + \mathbb{F} \cdot 1 \big/ \mathbb{F} \cdot 1$$

is a proper ideal in the Lie algebra $A^{(-)} \big/ \mathbb{F} \cdot 1$. This completes the proof of Theorem 1. □

Consider the homomorphism

$$\varphi : A^{(-)} \to \mathrm{Inder}(A), \quad \varphi(a) = \mathrm{ad}(a), \quad a \in A.$$

Since $\mathrm{Ker}\,\varphi = Z(A) = \mathbb{F} \cdot 1$ it follows that $\mathrm{Inder}(A) \;\cong\; A^{(-)} \big/ \mathbb{F} \cdot 1$.

Corollary 1. (1) *The Lie algebra* $[\mathrm{Inder}(A), \mathrm{Inder}(A)]$ *is simple.*

(2) *The Lie algebra* $\mathrm{Inder}(A)$ *is simple if and only if* $\mathrm{char}\,\mathbb{F} = 0$ *or* $\mathrm{char}\,\mathbb{F} = p > 0$ *and* $\nu_p(\mathbf{st}(A)) = 0$ *or* ∞.

Proof. The Lie algebra

$$[\mathrm{Inder}(A), \mathrm{Inder}(A)] \;\cong\; [A, A] \big/ [A, A] \cap \mathbb{F} \cdot 1$$

is simple by I.N. Herstein's Theorem (see [6]). The part (2) immediately follows from Theorem 1. □

2 PROOF OF THE THEOREM 2

Lemma 1. *Let* A *be an infinite-dimensional locally matrix algebra. Let* $d \in \mathrm{Der}(A)$ *and suppose that* $d([A, A])$ *lies in the center of the algebra* A*. Then* $d = 0$.

Proof. Let Z be the center of A. If A is not unital then $Z = \{0\}$. If A is unital then $Z = \mathbb{F} \cdot 1$. Consider a subalgebra $A_1 \subset A$ such that $A_1 \cong M_n(\mathbb{F})$ for some $n \geq 4$, and let $\varphi : M_n(\mathbb{F}) \to A_1$ be an isomorphism. An arbitrary matrix unit e_{ij}, $1 \leq i \neq j \leq n$, lies in $[M_n(\mathbb{F}), M_n(\mathbb{F})]$. Choose distinct indices $1 \leq i, j, s, t \leq n$. Then $e_{ij} = [e_{is}, e_{sj}]$. Hence,

$$d\big(\varphi(e_{ij})\big) \in Z\varphi(e_{is}) + Z\varphi(e_{sj}).$$

On the other hand, $e_{ij} = [e_{it}, e_{tj}]$, which implies $d\big(\varphi(e_{ij})\big) \in Z\varphi(e_{it}) + Z\varphi(e_{tj})$. Hence, $d(\varphi(e_{ij})) = 0$. The algebra $M_n(\mathbb{F})$ is generated by matrix units e_{ij}, $1 \leq i \neq j \leq n$. So, $d(\varphi(M_n(\mathbb{F}))) = \{0\}$, and therefore $d(A) = \{0\}$. This completes the proof of the Lemma. □

In [1], we showed that for an arbitrary locally matrix algebra A the ideal $\mathrm{Inder}(A)$ is dense in the Lie algebra $\mathrm{Der}(A)$ in the Tykhonoff topology.

Proof of Theorem 2. (1) Let I be a nonzero closed ideal of the Lie algebra $[\mathrm{Der}(A), \mathrm{Der}(A)]$. Choose a nonzero element $d \in I$. For an arbitrary element $a \in [A, A]$ we have

$$[d, \mathrm{ad}(a)] = \mathrm{ad}\big(d(a)\big) \in [\mathrm{Inder}(A), \mathrm{Inder}(A)].$$

By Lemma 1, we can choose an element $a \in [A, A]$ so that $d(a) \neq 0$. Hence,

$$I \cap [\mathrm{Inder}(A), \mathrm{Inder}(A)] \neq \{0\}.$$

Since the Lie algebra $[\mathrm{Inder}(A), \mathrm{Inder}(A)]$ is simple it follows that

$$[\mathrm{Inder}(A), \mathrm{Inder}(A)] \subseteq I.$$

We have mentioned above that $\mathrm{Inder}(A)$ is dense in the Lie algebra $\mathrm{Der}(A)$ in the Tykhonoff topology; see [1]. Hence, $[\mathrm{Inder}(A), \mathrm{Inder}(A)]$ is dense in the Lie algebra $[\mathrm{Der}(A), \mathrm{Der}(A)]$. Since the ideal I is closed we conclude that $I = [\mathrm{Inder}(A), \mathrm{Inder}(A)]$.

(2) Let I be a nonzero closed ideal of the Lie algebra $\mathrm{Der}(A)$. Choose a nonzero derivation $d \in I$. By Lemma 1, there exists an element $a \in A$ such that $d(a)$ does not lie in $\mathbb{F} \cdot 1$, hence

$$0 \neq \mathrm{ad}\big(d(a)\big) = [d, \mathrm{ad}(a)] \in I \cap \mathrm{Inder}(A).$$

Suppose that $\mathrm{char}\,\mathbb{F} = 0$ or $\mathrm{char}\,\mathbb{F} = p > 0$ and $\nu_p(\mathbf{st}(A)) = 0$ or ∞. Then the Lie algebra $\mathrm{Inder}(A)$ is simple, and therefore $\mathrm{Inder}(A) \subseteq I$. Since $\mathrm{Inder}(A)$ is dense in $\mathrm{Der}(A)$ (see [1]) and the ideal I is closed it follows that $I = \mathrm{Der}(A)$.

Now suppose that $\nu_p(\mathbf{st}(A)) = k$, $1 \leq k < \infty$. There exists a subalgebra A_1 in A such that

$$1 \in A_1 \quad \text{and} \quad A_1 \cong M_{p^k}(\mathbb{F}).$$

Choose an element $a \in A_1$ such that $\mathrm{tr}_{A_1}(a) \neq 0$. We will show that the inner derivation $\mathrm{ad}(a)$ does not lie in the closure

$$\overline{[\mathrm{Der}(A), \mathrm{Der}(A)]} = \overline{[\mathrm{Inder}(A), \mathrm{Inder}(A)]},$$

and therefore $\overline{[\mathrm{Inder}(A), \mathrm{Inder}(A)]}$ is a proper closed ideal in the Lie algebra $\mathrm{Der}(A)$. If $\mathrm{ad}(a)$ lies in the closure of $[\mathrm{Inder}(A), \mathrm{Inder}(A)]$ then, by the definition of the Tykhonoff topology, there exist elements $a_i, b_i \in A$, $1 \leq i \leq n$, such that

$$\Big(\mathrm{ad}(a) - \sum_{i=1}^{n} \mathrm{ad}\big([a_i, b_i]\big)\Big)(A_1) = \{0\}.$$

There exists a subalgebra $A_2 \subset A$ such that

$$A_1 \subseteq A_2, \quad a, a_i, b_i \in A_2, \quad 1 \leq i \leq n, \quad \text{and} \quad A_2 \cong M_m(\mathbb{F}).$$

As above, we consider the centralizer C of the subalgebra A_1 in A_2 such that

$$A_2 = A_1 \otimes_{\mathbb{F}} C, \quad C \cong M_t(\mathbb{F}) \quad \text{and} \quad t = m/p^k \text{ is not a multiple of } p.$$

Consider the element

$$b \;=\; \sum_{i=1}^{n} [a_i, b_i] \;\in\; A_2.$$

The difference $a - b$ commutes with all elements from A_1, hence $a - b = c \in C$.

In the algebra A_2 we have

$$\mathrm{tr}_{A_2}(c) \;=\; \mathrm{tr}_{A_2}(1 \otimes c) \;=\; \mathrm{tr}_{A_1}(1) \cdot \mathrm{tr}_C(c) \;=\; 0.$$

Hence, $\mathrm{tr}_{A_2}(a) = \mathrm{tr}_{A_2}(b) + \mathrm{tr}_{A_2}(c) = 0$.

On the other hand, $\mathrm{tr}_{A_2}(a) = \mathrm{tr}_{A_2}(a \otimes 1) = \mathrm{tr}_{A_1}(a) \cdot t \neq 0$. This contradiction completes the proof of Theorem 2. □

REFERENCES

[1] Bezushchak O. *Derivations and automorphisms of locally matrix algebras*. arXiv:2007.15716v1 [math.RA]

[2] Bezushchak O., Oliynyk B. *Unital locally matrix algebras and Steinitz numbers*. J. Algebra Appl. 2020, **19** (09), 2050180. doi:10.1142/S0219498820501807

[3] Bezushchak O., Oliynyk B. *Primary decompositions of unital locally matrix algebras*. Bull. Math. Sci. 2020, **10** (1), 2050006 (7 pages). doi:10.1142/S166436072050006X

[4] Drozd Yu.A., Kirichenko V.V. Finite Dimensional Algebras. Springer-Verlag, Berlin–Heidelberg–New York, 1994.

[5] Glimm J.G. *On a certain class of operator algebras*. Trans. Amer. Math. Soc. 1960, **95** (2), 318–340.

[6] Herstein I.N. *On the Lie and Jordan rings of a simple associetive ring*. Amer. J. Math. 1955, **77** (2), 279–285.

[7] Jacobson N. Lie algebras. Dover Publications Inc., New Yourk, 1979.

[8] Jacobson N. Structure of rings. Colloquium Publications, **37**, 1956.

[9] Köthe G. *Schiefkörper unendlichen Ranges uber dem Zentrum*. Math. Ann. 1931, **105**, 15–39.

[10] Kurosh A. *Direct decompositions of simple rings*. Rec. Math. 1942, **11** (**53**) (3), 245–264.

[11] Seligman G.B. Modular Lie Algebras. Springer-Verlag, Berlin–Heidelberg, 1967.

[12] Steinitz E. *Algebraische Theorie der Körper*. J. Reine Angew. Math. 1910, **137**, 167–309.

[13] Willard S. General Topology. Dover Publications, Mineola, New York, 2004.

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Нехай A — унітальна локально матрична алгебра над полем $\mathbb{F}$ характеристики відмінної від 2. Знайдено необхідну і достатню умову того, щоб алгебра Лі $A/\mathbb{F}\cdot 1$ була простою, а алгебра Лі диференціювань $\mathrm{Der}(A)$ — топологічно простою. Сформульована умова залежить лише від числа Стейніца алгебри A.

Ключові слова і фрази: локально матрична алгебра, диференціювання.