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FRONTCZAK R.[1], GOY T.[2]

MERSENNE-HORADAM IDENTITIES USING GENERATING FUNCTIONS

The main object of the present paper is to reveal connections between Mersenne numbers $M_n = 2^n - 1$ and generalized Fibonacci (i.e., Horadam) numbers w_n defined by a second order linear recurrence $w_n = pw_{n-1} + qw_{n-2}$, $n \geq 2$, with $w_0 = a$ and $w_1 = b$, where a, b, $p > 0$ and $q \neq 0$ are integers. This is achieved by relating the respective (ordinary and exponential) generating functions to each other. Several explicit examples involving Fibonacci, Lucas, Pell, Jacobsthal and balancing numbers are stated to highlight the results.

Key words and phrases: Mersenne numbers, Horadam sequence, Fibonacci sequence, Lucas sequence, Pell sequence, generating function, binomial transform.

[1] Landesbank Baden-Württemberg, 70173, Stuttgart, Germany
[2] Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: robert.frontczak@lbbw.de (Frontczak R.), taras.goy@pnu.edu.ua (Goy T.)

INTRODUCTION

A *generalized Fibonacci sequence* $(w_n)_{n\geq 0} = (w_n(a, b; p, q))_{n\geq 0}$ is defined by a second order homogeneous linear recurrence

$$w_n = pw_{n-1} + qw_{n-2}, \qquad n \geq 2,$$

with $w_0 = a$ and $w_1 = b$, where a, b, p and q are integers with $p > 0$, $q \neq 0$. Since these numbers were first studied by A.F. Horadam (see, e.g., [11, 12]), they are often referred to as *Horadam numbers*. The Binet formula for w_n is given by [11]

$$w_n = \alpha r_1^n + \beta r_2^n, \qquad n \geq 0,$$

where $r_1 = \frac{p+\sqrt{p^2+4q}}{2}$ and $r_2 = \frac{p-\sqrt{p^2+4q}}{2}$ denote the distinct roots of the quadratic equation $x^2 - px - q = 0$,

$$\alpha = \frac{a}{2} + \frac{2b - ap}{2\sqrt{p^2 + 4q}} \quad \text{and} \quad \beta = \frac{a}{2} - \frac{2b - ap}{2\sqrt{p^2 + 4q}}.$$

It is worth noticing that an equivalent version of the Binet formula is given by

$$w_n = b\frac{r_1^n - r_2^n}{r_1 - r_2} + aq\frac{r_1^{n-1} - r_2^{n-1}}{r_1 - r_2}, \qquad n \geq 1.$$

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The sequence can be extended to negative subscripts according to

$$w_{-n} = -\frac{1}{q}(pw_{-n+1} - w_{-n+2}), \qquad n \geq 1.$$

Further results on Horadam sequences can be found in the survey paper [14]. In what follows, we will make frequent use of the generating functions of $(w_n)_{n\geq 0}$. We know [15] that the sequence w_n has the ordinary (non-exponential) generating function

$$w(z) = \sum_{n=0}^{\infty} w_n z^n = \frac{a + (b - ap)z}{1 - pz - qz^2}, \tag{1}$$

while for sequences w_{2n+1} and w_{2n}

$$w_1(z) = \sum_{n=0}^{\infty} w_{2n+1} z^n = \frac{a + (bp - qa - ap^2)z}{1 - (p^2 + 2q)z + q^2z^2}, \tag{2}$$

$$w_2(z) = \sum_{n=0}^{\infty} w_{2n} z^n = \frac{b + (apq - bq)z}{1 - (p^2 + 2q)z + q^2z^2}. \tag{3}$$

The Horadam sequence generalizes many other number and polynomial sequences, for instance, the *Fibonacci sequence* $F_n = w_n(0,1;1,1)$, the *Lucas sequence* $L_n = w_n(2,1;1,1)$, the *Pell sequence* $P_n = w_n(0,1;2,1)$, the *Jacobsthal sequence* $J_n = w_n(0,1;1,2)$, the *Mersenne sequence* $M_n = w_n(0,1;3,-2)$, the *balancing numbers* $B_n = w_n(0,1;6,-1)$, and so on. The first few terms of each sequence are stated below.

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89
L_n	2	1	3	4	7	11	18	29	47	76	123	199
P_n	0	1	2	5	12	29	70	169	408	985	2378	5741
J_n	0	1	1	3	5	11	21	43	85	171	341	683
M_n	0	1	3	7	15	31	63	127	255	511	1023	2047
B_n	0	1	6	35	204	1189	6930	40391	235416	1372105	7997214	46611179

The sequences $(F_n)_{n\geq 0}$, $(L_n)_{n\geq 0}$, $(P_n)_{n\geq 0}$, $(J_n)_{n\geq 0}$, $(M_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ are indexed in the On-Line Encyclopedia of Integer Sequences [19] (see entries A000045, A000032, A000129, A001045, A000225 and A001109, respectively).

In the present paper, we derive some connection formulas between Mersenne numbers and the Horadam sequence.

Recall that Mersenne numbers M_n belong to the Horadam sequence family. They are given by the explicit form

$$M_n = 2^n - 1, \quad n \geq 0.$$

Mersenne numbers are popular research objects because of their interesting properties. For instance, Mersenne numbers are numbers with the following representation in the binary system: $(1)_2, (11)_2, (111)_2, (1111)_2, (11111)_2, \ldots$. Also, the Mersenne number sequence contains primes, the so called *Mersenne primes* of the form $2^n - 1$. A simple calculation shows that if M_n is a prime number, then n is a prime number, though not all M_n are prime. Mersenne primes are also connected to perfect numbers. The search for Mersenne primes is an active field of research (see [18], among others). More information about Mersenne numbers and its generalizations can be taken from the papers [1, 3–6, 9, 10, 13, 16, 20] and references contained therein.

We conclude this section with some generating functions, which will be needed in the proofs. Using (1)–(3) we easily obtain non-exponential generating functions of the sequences M_n, M_{2n+1} and M_{2n} as follows

$$m(z) = \sum_{n=0}^{\infty} M_n z^n = \frac{z}{1-3z+2z^2}, \tag{4}$$

$$m_1(z) = \sum_{n=0}^{\infty} M_{2n+1} z^n = \frac{1+2z}{1-5z+4z^2}, \tag{5}$$

$$m_2(z) = \sum_{n=0}^{\infty} M_{2n} z^n = \frac{3z}{1-5z+4z^2}. \tag{6}$$

Finally, the exponential generating functions of the sequences M_n, M_{2n+1} and M_{2n} can be derived as

$$\mu(z) = \sum_{n=0}^{\infty} M_n \frac{z^n}{n!} = 2e^{\frac{3z}{2}} \sinh\left(\frac{z}{2}\right), \tag{7}$$

$$\mu_1(z) = \sum_{n=0}^{\infty} M_{2n+1} \frac{z^n}{n!} = 2e^{\frac{5z}{2}} \sinh\left(\frac{3z}{2}\right) + e^{4z}, \quad \mu_2(z) = \sum_{n=0}^{\infty} M_{2n} \frac{z^n}{n!} = 2e^{\frac{5z}{2}} \sinh\left(\frac{3z}{2}\right).$$

1 MERSENNE-HORADAM IDENTITIES USING ORDINARY GENERATING FUNCTIONS

Our first result provides a relation between Mersenne and Horadam numbers using its ordinary generating functions. The method of proof is the same as in [7] and [8]. We note that, in what follows, we will used the standard convention that $\sum_{k=0}^{n} a_k = 0$ for $n < 0$.

Theorem 1. *For $n \geq 0$, the following formula holds*

$$w_n = a + (b-a)M_n + \sum_{k=1}^{n-1} \big((p-3)w_{n-k} + (q+2)w_{n-k-1}\big) M_k.$$

Proof. By (1) and (4), we get

$$\begin{aligned} \frac{z}{m(z)} &= 1 - 3z + 2z^2 = (1 - pz - qz^2) + (pz + qz^2 - 3z + 2z^2) = \frac{a + (b-ap)z}{w(z)} \\ &\quad + (p-3)z + (q+2)z^2 = \frac{a + (b-ap)z + (p-3)zw(z) + (q+2)z^2 w(z)}{w(z)}, \end{aligned}$$

and thus $zw(z) = am(z) + (b-ap)zm(z) + (p-3)zw(z)m(z) + (q+2)z^2 w(z)m(z)$.

Expanding both sides of the last equation as a power series in z yields

$$\begin{aligned} z \sum_{n=0}^{\infty} w_n z^n &= a \sum_{n=0}^{\infty} M_n z^n + (b-ap) \sum_{n=0}^{\infty} M_n z^{n+1} \\ &\quad + (p-3)z \sum_{n=0}^{\infty} w_n z^n \sum_{n=0}^{\infty} M_n z^n + (q+2)z^2 \sum_{n=0}^{\infty} w_n z^n \sum_{n=0}^{\infty} M_n z^n. \end{aligned}$$

Using the formula for multiplication of two power series

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} z^n, \tag{8}$$

we then obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} w_n z^{n+1} &= a \sum_{n=0}^{\infty} M_n z^n + (b - ap) \sum_{n=1}^{\infty} M_{n-1} z^n \\
&\quad + (p-3) \sum_{n=0}^{\infty} \sum_{k=0}^{n} w_{n-k} M_k z^{n+1} + (q+2) \sum_{n=0}^{\infty} \sum_{k=0}^{n} w_{n-k} M_k z^{n+2}, \\
az + \sum_{n=2}^{\infty} w_{n-1} z^n &= az + a \sum_{n=2}^{\infty} M_n z^n + (b - ap) \sum_{n=2}^{\infty} M_{n-1} z^n \\
&\quad + (p-3) \sum_{n=2}^{\infty} \sum_{k=0}^{n-1} w_{n-k-1} M_k z^n + (q+2) \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} w_{n-k-2} M_k z^n.
\end{aligned}$$

Comparing the coefficients on both sides, we obtain

$$\begin{aligned}
w_n &= aM_{n+1} + (b - ap)M_n + (p-3) \sum_{k=0}^{n} w_{n-k} M_k + (q+2) \sum_{k=0}^{n-1} w_{n-k-1} M_k \\
&= a(2M_n + 1) + (b - ap)M_n + (p-3) \sum_{k=0}^{n-1} w_{n-k} M_k + (p-3)aM_n + (q+2) \sum_{k=0}^{n-1} w_{n-k-1} M_k \\
&= a + (b - a)M_n + \sum_{k=1}^{n-1} \big((p-3) w_{n-k} + (q+2) w_{n-k-1} \big) M_k,
\end{aligned}$$

as desired. □

Example 1. *By choosing suitable values on a, b, p and q, one can obtain the following identities valid for $n \geq 0$:*

$$\begin{aligned}
F_n &= M_n - \sum_{k=1}^{n-1} (2F_{n-k} - 3F_{n-k-1}) M_k, \quad L_n = 2 - M_n - \sum_{k=1}^{n-1} (2L_{n-k} - 3L_{n-k-1}) M_k, \\
P_n &= M_n - \sum_{k=1}^{n-1} (P_{n-k} - 3P_{n-k-1}) M_k, \quad J_n = M_n - 2 \sum_{k=1}^{n-1} (J_{n-k} - 2J_{n-k-1}) M_k, \\
B_n &= M_n + \sum_{k=1}^{n-1} (3B_{n-k} + B_{n-k-1}) M_k.
\end{aligned}$$

In a similar manner, we can use the generating functions (2), (5) and (3), (6), respectively, to prove two other relations between odd (even) indexed Horadam and Mersenne numbers. These relations are contained in the next two theorems, those proofs we leave to the reader.

Theorem 2. *For $n \geq 1$, the following formula hold*

$$\begin{aligned}
w_{2n+1} + 2w_{2n-1} &= 3b + (bp^2 + apq + bq - b) M_{2n-1} \\
&\quad + \sum_{k=1}^{n-1} \big((p^2 + 2q - 5) p w_{2(n-k)} + (q^2 + qp^2 - 5q + 4) w_{2(n-k)-1} \big) M_{2k-1}.
\end{aligned} \tag{9}$$

Example 2. *Formula* (9) *yields*

$$\begin{aligned}
F_{2n+1} + 2F_{2n-1} &= 3 + M_{2n-1} - \sum_{k=1}^{n-1} \Big(2F_{2(n-k)} - F_{2(n-k)-1} \Big) M_{2k-1}, \\
L_{2n+1} + 2L_{2n-1} &= 3 + 3M_{2n-1} - \sum_{k=1}^{n-1} \Big(2L_{2(n-k)} - L_{2(n-k)-1} \Big) M_{2k-1}, \\
P_{2n+1} + 2P_{2n-1} &= 3 + 4M_{2n-1} + 2\sum_{k=1}^{n-1} \Big(P_{2(n-k)} + 2P_{2(n-k)-1} \Big) M_{2k-1}, \\
J_{2n+1} + 2J_{2n-1} &= 2 + M_{2n}, \\
B_{2n+1} + 2B_{2n-1} &= 3 + 34M_{2n-1} + 2\sum_{k=1}^{n-1} \Big(87B_{2(n-k)} - 13B_{2(n-k)-1} \Big) M_{2k-1}.
\end{aligned}$$

Theorem 3. *For* $n \geq 0$*, the following formulas hold*

$$w_{2n} = a + \frac{pb + qa - a}{3} M_{2n} + \frac{1}{3} \sum_{k=1}^{n-1} \big((p^2 + 2q - 5) w_{2(n-k)} + (4 - q^2) w_{2(n-k-1)} \big) M_{2k}. \tag{10}$$

Example 3. *It follows from* (10) *that*

$$\begin{aligned}
F_{2n} &= \frac{1}{3} M_{2n} - \frac{1}{3} \sum_{k=1}^{n-1} \Big(2F_{2(n-k)} - 3F_{2(n-k-1)} \Big) M_{2k}, \\
L_{2n} &= 2 + \frac{1}{3} M_{2n} - \frac{1}{3} \sum_{k=1}^{n-1} \Big(2L_{2(n-k)} - 3L_{2(n-k-1)} \Big) M_{2k}, \\
P_{2n} &= \frac{2}{3} M_{2n} + \frac{1}{3} \sum_{k=1}^{n-1} \Big(P_{2(n-k)} + 3P_{2(n-k-1)} \Big) M_{2k}, \\
J_{2n} &= \frac{1}{3} M_{2n}, \qquad (11) \\
B_{2n} &= 2M_{2n} + \frac{1}{3} \sum_{k=1}^{n-1} \Big(29B_{2(n-k)} + 3B_{2(n-k-1)} \Big) M_{2k}.
\end{aligned}$$

Note that formula (11) is known (see [4]).

We finally remark, that Theorems 1, 2 and 3 can be generalized to sums of certain products of w_n and M_n; see [7] and [8] for details.

2 MERSENNE-HORADAM IDENTITIES VIA EXPONENTIAL GENERATING FUNCTIONS

Let us first consider the fundamental Fibonacci sequence $u_n = w_n(0, b; p, q)$. In this section, we derive connection formulas between u_n and Mersenne numbers M_n involving binomial coefficients.

Let $u(z)$, $u_1(z)$ and $u_2(z)$ be the exponential generating function of the sequences u_n, u_{2n+1} and u_{2n}. Then we have

$$\begin{aligned}
u(z) &= \sum_{n=0}^{\infty} u_n \frac{z^n}{n!} = \frac{2b}{\Delta} e^{\frac{pz}{2}} \sinh\left(\frac{\Delta z}{2}\right), \qquad (12)\\
u_1(z) &= \sum_{n=0}^{\infty} u_{2n+1} \frac{z^n}{n!} = \frac{2b(p^2+q)}{p\Delta} e^{\frac{p^2+q}{2}z} \sinh\left(\frac{p\Delta z}{2}\right)\\
&\quad - \frac{b}{2p\Delta}\left((p^2+2q-p\Delta)e^{\frac{p^2+2q+p\Delta}{2}z} - (p^2+2q+p\Delta)e^{\frac{p^2+2q-p\Delta}{2}z}\right),\\
u_2(z) &= \sum_{n=0}^{\infty} u_{2n} \frac{z^n}{n!} = \frac{2b}{\Delta} e^{\frac{p^2+2q}{2}z} \sinh\left(\frac{p\Delta z}{2}\right),
\end{aligned}$$

where $\Delta = \sqrt{p^2+4q}$; see [15].

Theorem 4. *For $n \geq 0$, the following identity holds*

$$u_n = \frac{b}{\Delta}\left(\frac{p-3\Delta}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2\Delta}{p-3\Delta}\right)^k M_k. \qquad (13)$$

Proof. Using (7) and (12), we have

$$u\left(\frac{z}{\Delta}\right) = \frac{b}{\Delta}\mu(z) e^{\left(\frac{p}{2\Delta}-\frac{3}{2}\right)z}.$$

From the formula above we now obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} u_n \frac{z^n}{\Delta^n n!} &= \frac{b}{\Delta}\sum_{n=0}^{\infty} M_n \frac{z^n}{n!} \cdot \sum_{n=0}^{\infty}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^n \frac{z^n}{n!} = \frac{b}{\Delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{M_k}{k!}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^{n-k} \frac{z^n}{(n-k)!}\\
&= \frac{b}{\Delta}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{p}{2\Delta}-\frac{3}{2}\right)^{n-k} M_k \frac{z^n}{n!},
\end{aligned}$$

and after simplification we have (13). □

Example 4. *Let $n \geq 0$. Then formula* (13) *gives*

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1-3\sqrt{5}}{2}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(-\frac{15+\sqrt{5}}{22}\right)^k M_k, \quad J_n = \frac{(-4)^n}{3}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{3}{4}\right)^k M_k,$$

$$P_n = \frac{(1-3\sqrt{2})^n}{2\sqrt{2}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{12+2\sqrt{2}}{17}\right)^k M_k, \quad B_n = \frac{(3-6\sqrt{2})^n}{4\sqrt{2}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{16+4\sqrt{2}}{21}\right)^k M_k.$$

Theorem 4 highlights the following issue. If we define the sequence a_n as

$$a_n = \left(\frac{2\Delta}{p-3\Delta}\right)^n M_n,$$

then the sequence

$$b_n = \frac{\Delta}{b}\left(\frac{p-3\Delta}{2}\right)^{-n} u_n,$$

is the binomial transform of a_n, where the binomial transform and its inverse transform are given by [2,17]

$$b_n = \sum_{k=0}^{n}\binom{n}{k} a_k \qquad \Leftrightarrow \qquad a_n = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k} b_k.$$

The inverse relation immediately gives the next identity.

Theorem 5. *For $n \geq 0$, we have*

$$M_n = \frac{\Delta^{1-n}}{b} \left(\frac{3\Delta - p}{2} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2}{3\Delta - p} \right)^k u_k. \tag{14}$$

Example 5. *Formula* (14) *yields*

$$M_n = \sqrt{5} \left(\frac{15 - \sqrt{5}}{10} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1 + 3\sqrt{5}}{22} \right)^k F_k, \tag{15}$$

$$M_n = 3 \left(\frac{4}{3} \right)^n \sum_{k=1}^{n} \binom{n}{k} \frac{J_k}{4^k}, \quad M_n = \sqrt{8} \left(\frac{6 - \sqrt{2}}{4} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1 + 3\sqrt{2}}{17} \right)^k P_k,$$

$$M_n = 4\sqrt{2} \left(\frac{12 - 3\sqrt{2}}{8} \right)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1 + 2\sqrt{2}}{21} \right)^k B_k.$$

Note that formula (15) may be rewritten in terms of the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$ as follows

$$M_n = \left(\frac{\varphi^2 + 2}{(2\varphi - 1)\varphi} \right)^n \sum_{k=1}^{n} \binom{n}{k} \frac{\varphi^{2k} - (-1)^k}{(\varphi^2 + 2)^k}.$$

We also have the following summation identity.

Theorem 6. *Let A and B be arbitrary complex numbers. Then for $n \geq 1$ it is true that*

$$\sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} u_k = \frac{b}{\Delta} \sum_{k=0}^{n} \binom{n}{k} (A\Delta)^k \left(\frac{Ap + 2B - 3\Delta A}{2} \right)^{n-k} M_k.$$

Proof. It is known [17] that if a_n is an arbitrary sequence of numbers with exponential generating function $F(z)$, then

$$S(z) = \sum_{n=0}^{\infty} S_n(A, B; a) \frac{z^n}{n!} = e^{Bz} F(Az),$$

where

$$S_n(A, B; a) = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k} a_k.$$

Hence,

$$S_u(z) = \sum_{n=0}^{\infty} S_n(A, B; u) \frac{z^n}{n!} = \frac{b}{\Delta} 2e^{\frac{Ap+2B}{2}z} \sinh\left(\frac{A\Delta z}{2} \right),$$

$$S_M(z) = \sum_{n=0}^{\infty} S_n(A, B; M) \frac{z^n}{n!} = 2e^{\frac{3A+2B}{2}z} \sinh\left(\frac{Az}{2} \right),$$

and finally

$$\sum_{n=0}^{\infty} S_n(A, B; u) \frac{z^n}{n!} = \frac{b}{\Delta} \sum_{n=0}^{\infty} S_n \left(A\Delta, \frac{Ap + 2B - 3\Delta A}{2}; M \right) \frac{z^n}{n!}.$$

□

We give some examples of the above summation identity. However, we restrict the list of examples to the pair (F_n, M_n). If $(A, B) = (1, 1)$, then

$$\sum_{k=0}^{n}\binom{n}{k}F_k = \frac{1}{\sqrt{5}}\sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k(1-\sqrt{5})^{n-k}\left(\frac{3}{2}\right)^{n-k}M_k.$$

Since $\sum\limits_{k=1}^{n}\binom{n}{k}F_k = F_{2n}$, we can restate the identity as $(\eta = -1/\varphi)$

$$F_{2n} = \frac{(3\eta)^n}{\varphi-\eta}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\varphi-\eta}{3\eta}\right)^k M_k.$$

If $(A, B) = (-1, 1)$, then

$$F_n = \frac{(2\varphi-\eta)^n}{\varphi-\eta}\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\left(\frac{\varphi-\eta}{2\varphi-\eta}\right)^k M_k.$$

This identity may be compared with the one from Example 8, where we have shown that

$$F_n = \frac{(2\eta-\varphi)^n}{\varphi-\eta}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\varphi-\eta}{2\eta-\varphi}\right)^k M_k.$$

Our last example is $(A, B) = (1, -1/2)$. In this case, we get the relation

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}2^{-(n-k)}F_k = 5^{\frac{n-1}{2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\frac{3}{2}\right)^{n-k}M_k$$

or

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^k F_k = 3^n 5^{\frac{n-1}{2}}\sum_{k=0}^{n}\binom{n}{k}\left(-\frac{2}{3}\right)^k M_k.$$

Theorem 7. *For $n \geq 0$ it holds that*

$$u_n = \frac{b}{\Delta}\left(\frac{3p-5\Delta}{6}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2\Delta}{3p-5\Delta}\right)^k M_{2k}$$

and

$$M_{2n} = \frac{\Delta}{b}\left(\frac{5\Delta-3p}{2\Delta}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{6}{5\Delta-3p}\right)^k u_k.$$

Proof. The first formula follows from the relation $u\left(\frac{3z}{\Delta}\right) = \frac{b}{\Delta}\mu_2(z)e^{\left(\frac{3p}{2\Delta}-\frac{5}{2}\right)z}$ and an application of formula (8). Moreover, the first formula shows that $u_n\frac{\Delta}{b}\left(\frac{6}{3p-5\Delta}\right)^n$ is the binomial transform of $M_{2n}\left(\frac{2\Delta}{3p-5\Delta}\right)^n$.

The second formula is a rearrangement of the inverse binomial transform relation. □

Theorem 8. *For $n \geq 0$ it holds that*

$$u_n = \frac{b}{\Delta}\left(\frac{3p-5\Delta}{6}\right)^n\sum_{k=0}^{n}\binom{n}{k}\left(\frac{2\Delta}{3p-5\Delta}\right)^k M_{2k+1} - \frac{b}{\Delta}\left(\frac{p+\Delta}{2}\right)^n,$$

and

$$M_{2n+1} = \frac{\Delta}{b}\left(\frac{5\Delta-3p}{2\Delta}\right)^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{5\Delta-3p}\right)^k u_k + 4^n.$$

Proof. To prove the first formula we use

$$u\left(\frac{3z}{\Delta}\right) = \frac{b}{\Delta}\left(\mu_1(z)e^{(\frac{3p}{2\Delta}-\frac{5}{2})z} - e^{(\frac{3p}{2\Delta}+\frac{3}{2})z}\right).$$

The second formula is once more an application of the inverse binomial transform, where we used that

$$\left(\frac{5\Delta-3p}{2\Delta}\right)^n \sum_{k=0}^{n}\binom{n}{k}\left(\frac{3\Delta+3p}{5\Delta-3p}\right)^k = \left(\frac{5\Delta-3p}{2\Delta}\right)^n\left(1+\frac{3\Delta+3p}{5\Delta-3p}\right)^n = 4^n.$$

□

Theorem 9. *For $n \geq 0$ we have*

$$u_{2n} = \frac{b}{\Delta}\left(\frac{3p^2+6q-5p\Delta}{6}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2p\Delta}{3p^2+6q-5p\Delta}\right)^k M_{2k}$$

and

$$M_{2n} = \frac{\Delta}{b}\left(\frac{5p\Delta-3p^2-6q}{2p\Delta}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{5p\Delta-3p^2-6q}\right)^k u_{2k}.$$

Proof. The first formula follows from the relation

$$u_2(z) = \frac{b}{\Delta}e^{(\frac{p^2+2q}{2}-\frac{5p\Delta}{b})z}\mu_2\left(\frac{p\Delta}{3}z\right)$$

and an application of formula (8). The second formula is a rearrangement of the inverse binomial transform relation. □

A proof comparable to the one given for Theorem 4 yields the following relation between numbers u_{2n+1} and M_{2n+1}. In this case we use the relations

$$2p\Delta\cdot u_1(z) = 2(p^2+q)e^{\frac{3p^2+6q-5p\Delta}{6}z}\mu_1\left(\frac{p\Delta z}{3}\right)$$
$$-\left(2(p^2+q)+b(p^2+2q-p\Delta)\right)e^{\frac{p^2+2q+p\Delta}{2}z} + b(p^2+2q+p\Delta)e^{\frac{p^2+2q-p\Delta}{2}z}$$

and

$$2(p^2+q)\mu_1(z) = 2p\Delta u_1\left(\frac{3z}{p\Delta}\right)e^{\frac{-3p^2-6q+5p\Delta}{2p\Delta}z}$$
$$+(2(p^2+q)+b(p^2+2q-p\Delta))e^{4z} - b(p^2+2q+p\Delta)e^{z}.$$

Theorem 10. *For $n \geq 0$*

$$u_{2n+1} = b\,\frac{p^2+q}{p\Delta}\left(\frac{3p^2+6q-5p\Delta}{6}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{2p\Delta}{3p^2+6q-5p\Delta}\right)^k M_{2k+1}$$
$$-b\,\frac{3p^2+4q-p\Delta}{2p\Delta}\left(\frac{p^2+2q+p\Delta}{2}\right)^n + b\,\frac{p^2+2q+p\Delta}{2p\Delta}\left(\frac{p^2+2q-p\Delta}{2}\right)^n$$

and

$$M_{2n+1} = \frac{(p\Delta)^{1-n}}{b(p^2+q)}\left(\frac{5p\Delta-3p^2-6q}{2}\right)^n \sum_{k=1}^{n}\binom{n}{k}\left(\frac{6}{5p\Delta-3p^2-6q}\right)^k u_{2k+1}$$
$$+\frac{p^2+2q-p\Delta}{2(p^2+q)}\cdot 4^n - \frac{p^2+2q+p\Delta}{2(p^2+q)} + 4^n.$$

3 MERSENNE-LUCAS IDENTITIES VIA EXPONENTIAL GENERATING FUNCTIONS

In this section we establish connections between the fundamental Lucas sequence $v_n = w_n(2, p; p, q)$ and Mersenne numbers M_n.

Theorem 11. *For $n \geq 0$*

$$M_n + 2 = \left(\frac{3\Delta - p}{2\Delta}\right)^n \sum_{k=0}^{n} \binom{n}{k} v_k \left(\frac{2}{3\Delta - p}\right)^k,$$

and

$$v_n = \left(\frac{p - 3\Delta}{2}\right)^n \sum_{k=0}^{n} \binom{n}{k} (M_k + 2) \left(\frac{2\Delta}{p - 3\Delta}\right)^k. \tag{16}$$

Proof. It is known that the exponential generating function of the sequence v_n can be given as

$$v(z) = \sum_{n=0}^{\infty} v_n \frac{z^n}{n!} = 2e^{\frac{p}{2}z} \cosh\left(\frac{\Delta}{2}z\right). \tag{17}$$

Using (7) and (17) we obtain

$$\begin{aligned} \mu'(z) &= \sum_{n=0}^{\infty} nM_n \frac{z^{n-1}}{n!} = \sum_{n=0}^{\infty} M_{n+1} \frac{z^n}{n!} = 3e^{\frac{3z}{2}} \sinh\left(\frac{z}{2}\right) + e^{\frac{3z}{2}} \cosh\left(\frac{z}{2}\right) \\ &= \frac{3}{2}\mu(z) + \frac{1}{2} v\left(\frac{z}{\Delta}\right) e^{\left(\frac{3}{2} - \frac{p}{2\Delta}\right)z}. \end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} (2M_{n+1} - 3M_n) \frac{z^n}{n!} = v\left(\frac{z}{\Delta}\right) e^{\left(\frac{3\Delta - p}{2\Delta}\right)z}.$$

To complete the first part, observe that $2M_{n+1} - 3M_n = M_n + 2$. To get (16) we may apply the argument of the inverse binomial transform. □

When $v_n = L_n$ is the Lucas sequence, then

$$M_n + 2 = \left(\frac{\varphi - 2\eta}{\varphi - \eta}\right)^n \sum_{k=0}^{n} \binom{n}{k} (\varphi - 2\eta)^{-k} L_k,$$

and

$$L_n = (2\eta - \varphi)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k (M_k + 2).$$

In view of

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k = \left(\frac{\eta}{2\eta - \varphi}\right)^n,$$

we observe that an equivalent version of the last identity is

$$L_n = 2\eta^n + (2\eta - \varphi)^n \sum_{k=1}^{n} \binom{n}{k} \left(\frac{\varphi - \eta}{2\eta - \varphi}\right)^k M_k. \tag{18}$$

We also have the following summation identity.

Theorem 12. *Let A and B be arbitrary complex numbers. Then for $n \geq 0$ it is true that*

$$\sum_{k=0}^{n}\binom{n}{k}A^kB^{n-k}v_k = \sum_{k=0}^{n}\binom{n}{k}(A\Delta)^k\left(\frac{Ap+2B-3\Delta A}{2}\right)^{n-k}(M_k+2).$$

Proof. The proof is very similar to that one given in the last section and omitted. □

As examples, we will state the companion results for $v_n = L_n$ from the previous section. If $(A,B) = (1,1)$, then

$$\sum_{k=0}^{n}\binom{n}{k}L_k = \sum_{k=0}^{n}\binom{n}{k}(\sqrt{5})^k(1-\sqrt{5})^{n-k}\left(\frac{3}{2}\right)^{n-k}(M_k+2).$$

This gives the identity

$$L_{2n} = 2\eta^{2n} + (3\eta)^n\sum_{k=1}^{n}\binom{n}{k}\left(\frac{\varphi-\eta}{3\eta}\right)^k M_k.$$

If $(A,B) = (-1,1)$, then the result is

$$L_n = 2\varphi^n + (2\varphi-\eta)^n\sum_{k=1}^{n}\binom{n}{k}(-1)^k\left(\frac{\varphi-\eta}{2\varphi-\eta}\right)^k M_k,$$

which should be compared with (18).

Finally, for $(A,B) = (1,-1/2)$ we get the relation

$$\sum_{k=0}^{n}\binom{n}{k}2^{-(n-k)}L_k = (-1)^n2^{1-n}5^{\frac{n}{2}} + 5^{\frac{n}{2}}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}\left(\frac{3}{2}\right)^{n-k}M_k$$

or

$$\sum_{k=0}^{n}\binom{n}{k}(-2)^kL_k = 2\cdot 5^{\frac{n}{2}} + 3^n5^{\frac{n}{2}}\sum_{k=1}^{n}\binom{n}{k}\left(-\frac{2}{3}\right)^k M_k.$$

The results of this section also highlight some other hidden relations, since ([4], Proposition 2.4)

$$M_n + 2 = \begin{cases} j_n, & \text{if } n \text{ is even},\\ 3J_n, & \text{if } n \text{ is odd},\end{cases}$$

where $(J_n)_{n\geq 0}$ is the Jacobsthal and $(j_n)_{n\geq 0}$ is the Jacobsthal-Lucas sequence.

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У роботі встановлені формули зв'язку між числами Мерсенна $M_n = 2^n - 1$ та узагальненими числами Фібоначчі (числами Горадама) w_n, які задовольняють лінійне рекурентне співвідношення другого порядку $w_n = pw_{n-1} + qw_{n-2}$, де $n \geq 2$, $w_0 = a$, $w_1 = b$, числа a, b, $p > 0$ і $q \neq 0$ є цілими. При цьому ми використовуємо відповідні співвідношення між звичайними та експоненційними генератрисами обох числових послідовностей. Зокрема, наведені приклади, які стосуються чисел Фібоначчі, Люка, Пелля, Якобсталя та збалансованих чисел.

Ключові слова і фрази: Числа Мерсенна, послідовність Горадама, послідовність Фібоначчі, послідовність Люка, послідовність Пелля, генератриса, біноміальне перетворення.