ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2017, **9** (2), 145–153
doi:10.15330/cmp.9.2.145-153

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2017, Т.9, №2, С.145–153

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PARABOLIC BY SHILOV SYSTEMS WITH VARIABLE COEFFICIENTS

Because of the parabolic instability of the Shilov systems to change their coefficients, the definition parabolicity of Shilov for systems with time-dependent t coefficients, unlike the definition parabolicity of Petrovsky, is formulated by imposing conditions on the matricant of corresponding dual by Fourier system. For parabolic systems by Petrovsky with time-dependent coefficients, these conditions are the property of a matricant, which follows directly from the definition of parabolicity. In connection with this, the question of the wealth of the class Shilov systems with time-dependent coefficients is important.

A new class of linear parabolic systems with partial derivatives to the first order by the time t with time-dependent coefficients is considered in this work. It covers the class by Petrovsky systems with time-dependent younger coefficients. A main part of differential expression of each such system is parabolic (by Shilov) expression with constant coefficients. The fundamental solution of the Cauchy problem for systems of this class is constructed by the Fourier transform method. Also proved their parabolicity by Shilov. Only the structure of the system and the conditions on the eigenvalues of the matrix symbol were used. First of all, this class characterizes the wealth by Shilov class of systems with time-dependents coefficients.

Also it is given a general method for investigating a fundamental solution of the Cauchy problem for Shilov parabolic systems with positive genus, which is the development of the well-known method of Y.I. Zhitomirskii.

Key words and phrases: parabolic by Shilov system, fundamental solution, Cauchy problem.

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INTRODUCTION

In [1] G.E. Shilov formulated a new definition of parabolicity of systems of partial differential equations which generalizes the notion of parabolicity by I.G. Petrovsky [2] and leads to a significant expansion of the Petrovsky class of systems appearance

$$\partial_t u(t;x) = P(t;i\partial_x)u(t;x), \quad (t;x) \in \Pi_{(\tau;T]} := (\tau;T] \times \mathbb{R}^n, \tau \in [0;T). \tag{1}$$

Here i is imaginary unit, u is unknown vector function of m dimension, $P(t;i\partial_x)$ is matrix differential expression of $p \in \mathbb{N}$ order with t time-dependent coefficients.

If coefficients of system (1) are constants and $P(t;i\partial_x) \equiv P(i\partial_x)$, parabolicity by Shilov is defined like parabolicity by Petrovsky: by imposing conditions on the real part of the characteristic numbers $\lambda_j(\cdot)$ of matrix symbol $P(\sigma)$, $\sigma \in \mathbb{C}^n$, of differential expression of system (1): exists $h > 0$, exists $\delta_0 > 0$ and exists $\delta_1 \geq 0$ for all $\xi \in \mathbb{R}^n$ such that

$$\max_{j \in \mathbb{N}_m} \mathrm{Re}\lambda_j(\xi) \leq -\delta_0 \|\xi\|^h + \delta_1. \tag{2}$$

УДК 517.956.4
2010 *Mathematics Subject Classification:* 35A08, 35K40.

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Here h is index of parabolicity of system (1), $0 < h \le p$; $\mathbb{N}_m := \{1; 2; \ldots ; m\}$, $\| \cdot \| := (\cdot, \cdot)^{1/2}$, $(\cdot, \cdot)$ is scalar product in $\mathbb{R}^n$.

If the coefficients of system (1) depend on t (continuously), it has, unlike parabolicity by Petrovsky, parabolicity by Shilov for this system with the index of parabolicity h means performance for the matricant $\Theta_\tau^t(\cdot)$, $0 \le \tau < t \le T$, of corresponding dual by Fourier system following estimate [3]:

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, \quad (t; \xi) \in \Pi_{(\tau;T]} \tag{3}$$

(here $\gamma := (p-h)(m-1)$). Let's note that for parabolic by Petrovsky systems (1) condition (3) is characteristic property which is a direct consequence of the relevant condition of parabolicity of type (2). For parabolic systems (1) with dependent on t coefficients in the case of $p \ne h$ it is not possible to confirm this fact by means of classical theory of parabolic systems, generally speaking, due to the parabolic instability of such systems to changing of their coefficients [4]. So the information about the richness of Shilov class of systems with coefficients dependent on t, in particular about the examples of such systems which are not parabolic by Petrovsky is important.

In this paper a new class of systems of partial differential equations whose coefficients depend on t is defined; it is substantiated their parabolicity by Shilov and examples are given. This class of systems characterizes the richness of Shilov class of systems with depend on t coefficients. In addition, estimates of the derivatives of the fundamental solution of the Cauchy problem (FSCP) are established for parabolic by Shilov systems with coefficients dependent on t the genus of which is positive.

The study FSCP for Shilov-type parabolic systems with coefficients independent of t was carried out in the papers [3, 5–7] and scalar parabolic equations by Shilov, whose coefficients can depend on t was carried out in the papers [8–11].

1 PRELIMINARIES

Let $\mathbb{R}^n$ and $\mathbb{C}^n$ are respectively real and complex space of n dimension, $\mathbb{R} := \mathbb{R}^1$, $\mathbb{Z}_+^n$ is the set of all n -dimensional multi indices; $|x + iy| := (x^2 + y^2)^{1/2}$, $\{x, y\} \subset \mathbb{R}$, $|(a_{lj})_{l,j=1}^m| := \max_{\{l,j\} \subset \mathbb{N}_m} |a_{lj}|$; $z^l := z_1^{l_1} \ldots z_n^{l_n}$, $|z|_+^h := |z_1|^h + \ldots + |z_n|^h$, $|z|_+ := |z|_+^1$, if $z := (z_1; \ldots ; z_n) \in \mathbb{C}^n$, $l := (l_1; \ldots ; l_n) \in \mathbb{Z}_+^n$ and $h > 0$.

We shall consider the system (1) with matrix differential expression

$$P(t; i\partial_x) = \left(\sum_{|k|_+ \le p} a_k^{lj}(t) i^{|k|_+} \partial_x^k \right)_{l,j=1}^m$$

of p order coefficients $a_k^{lj}(\cdot)$ of which are continuous complex-valued functions on $[0; T]$. We shall suppose that this system is parabolic by Shilov on the set $\Pi_{(\tau;T]}$ with the index of parabolicity h, $0 < h \le p$, consolidated order p_0 and genus μ [3].

Let's remind now that matricant $\Theta_\tau^t(\cdot)$ of appropriate dual by Fourier to (1) system has the structure

$$\Theta_\tau^t(\xi) = E + \sum_{r=1}^{\infty} \int_\tau^t \int_\tau^{t_1} \ldots \int_\tau^{t_{r-1}} \left(\prod_{j=1}^{r} P(t_j; \xi) \right) dt_r \ldots dt_2 dt_1. \tag{4}$$

Here E is the identity matrix of m order. Hence, it abides bound

$$|P(t;\sigma)| \le c(1+\|\sigma\|^p),\ 0 \le t \le T, \sigma \in \mathbb{C}^n,$$

we obtain that

$$|\Theta_\tau^t(\sigma)| \le c_0 e^{\delta_0 (t-\tau)\|\sigma\|^p},\ 0 \le t \le T, \sigma \in \mathbb{C}^n \tag{5}$$

(here c_0 and δ_0 are positive constants which are not dependent on τ, t and σ).

The exact order of exponential increase of matricant $\Theta_\tau^t(\cdot)$ in complex space $\mathbb{C}^n$ is called the consolidated order p_0 of the system (1). Always $p \ge p_0 > 1$ for parabolic systems [3].

The genus of parabolic by Shilov system we shall call the maximum rate μ such that in the domain

$$\mathbb{K}_\mu = \{\xi + i\eta \in \mathbb{C}^n : \|\eta\| \le K(1+\|\xi\|)^\mu\}$$

with some $K > 0$ for matricant the following estimate holds

$$|\Theta_\tau^t(\xi + i\eta)| \le c(1+\|\xi\|^\gamma) e^{-\delta(t-\tau)\|\xi\|^h}, 0 \le \tau < t \le T. \tag{6}$$

In [3] it is established that $1-(p_0-h) \le \mu \le 1$.

2 ONE CLASS OF PARABOLIC SYSTEMS

Let's consider the system of equations

$$\partial_t u(t;x) = \{P_0(i\partial_x) + P_1(t;i\partial_x)\}u(t;x), \quad (t;x) \in \Pi_{(\tau;T]},\ \tau \in [0;T), \tag{7}$$

with $p \in \mathbb{N}$ order in which $u := \mathrm{col}(u_1, \ldots, u_m)$,

$$P_0(i\partial_x) := \left(\sum_{|k|_+ \le p} a_k^{lj} i^{|k|_+} \partial_x^k \right)_{l,j=1}^m, \quad P_1(t;i\partial_x) := \left(\sum_{|k|_+ \le p_1} a_k^{lj}(t) i^{|k|_+} \partial_x^k \right)_{l,j=1}^m.$$

We shall assume that corresponding system

$$\partial_t u(t;x) = P_0(i\partial_x)u(t;x), \quad (t;x) \in \Pi_{(\tau;T]}, \tag{8}$$

on the set $\Pi_{(\tau;T]}$ is parabolic by Shilov with constant coefficients and index of parabolicity h and coefficients of differential expression $P_1(t;i\partial_x)$ are continuous complex-valued functions defined on [0;T] with the values p, p_1 and h satisfying condition

$$0 \le p_1 + (p-h)(m-1) < h. \tag{A}$$

Examples of system (7) with condition (A).

I. Each parabolic by Petrovsky system (1) of $p = 2b$ order, $b \in \mathbb{N}$, with constant coefficients of group of senior members and dependent continuously on t coefficients of group of younger members is a system of kind (7) with condition (A). Because in this case $p = h = 2b$, $p_1 = 2b-1$ and respectively

$$0 < p_1 + (p-h)(m-1) = 2b-1 < 2b = h.$$

II. Let $n = 1, m = 2,\ a > 0$ i $c_j(\cdot),\ j \in \mathbb{N}_5$, are some continuous on [0;T] complex-valued functions. Then the system

$$\begin{cases} \partial_t u_1 &= \{-a\partial_x^4 + c_1(t)\partial_x^2\}u_1 + \{\partial_x^5 - \partial_x^3 + c_2(t)\partial_x\}u_2, \\ \partial_t u_2 &= \{(c_3(t)-1)\partial_x^3\}u_1 - \{a\partial_x^4 - c_4(t)\partial_x^3 - c_5(t)\}u_2, \end{cases}$$

is the system of kind (7) with condition (A). Indeed, putting

$$P_0(i\partial_x) = \begin{pmatrix} -a\partial_x^4 & \partial_x^5 - \partial_x^3 \\ -\partial_x^3 & -a\partial_x^4 \end{pmatrix},$$

$$P_1(t; i\partial_x) = \begin{pmatrix} c_1(t)\partial_x^2 & c_2(t)\partial_x \\ c_3(t)\partial_x^3 & c_4(t)\partial_x^3 + c_5(t) \end{pmatrix}$$

and solving the appropriate equation

$$\det(P_0(\sigma) - \lambda E) = 0, \quad \sigma \in \mathbb{C}^n,$$

we obtain that $\lambda_{1,2}(\sigma) = -a\sigma^4 \pm i\sqrt{\sigma^8 + \sigma^6}$, $p = 5$, $p_1 = 3$ i $h = 4$. For these values p, p_1 and h, obviously the condition (A) holds.

Theorem 1. *Let* (7) *is system with continuous coefficients for which the condition (A) holds. Then for matricant* $\Theta_\tau^t(\cdot)$ *of appropriate dual by Fourier system on the set* $\Pi_{(\tau;T]}$, $\tau \in [0;T)$, *the estimate* (3) *holds.*

Proof. Let's write down appropriate dual by Fourier system to (7):

$$\partial_t v(t;\xi) = \{P_0(\xi) + P_1(t;\xi)\}v(t;\xi), \quad (t;\xi) \in \Pi_{(\tau;T]}. \tag{9}$$

With the continuity of the coefficients matricant $\Theta_\tau^t(\cdot)$ is the only solution of the Cauchy problem for system (9) with the initial condition

$$v(t;\cdot)\ |_{t=\tau} = E. \tag{10}$$

Then the following equality holds

$$\partial_t \Theta_\tau^t(\xi) = P_0(\xi)\Theta_\tau^t(\xi) + Q(\tau, t; \xi). \tag{11}$$

Here $Q(\tau, t; \xi) := P_1(t;\xi)\Theta_\tau^t(\xi)$. Solving the Cauchy problem (11), (10), we obtain such representation:

$$\Theta_\tau^t(\xi) = e^{(t-\tau)P_0(\xi)} + \int_\tau^t e^{(t-\beta)P_0(\xi)}Q(\tau,\beta;\xi)d\beta,\ (t;\xi) \in \Pi_{(\tau;T]},\ \tau \in [0;T).$$

Hence, it abides performance of estimate (3) for $e^{(t-\tau)P_0(\cdot)}$ because $e^{(t-\tau)P_0(\cdot)}$ is matricant dual by Fourier system to (8) and inequality

$$|Q(\tau,t;\xi)| \le c_0(1 + \|\xi\|^{p_1})|\Theta_\tau^t(\xi)|,\ (t;\xi) \in \Pi_{(\tau;T]},\ \tau \in [0;T)$$

(here positive constant c_0 does not depend on τ, t i ξ), we get the estimate

$$|\Theta_\tau^t(\xi)| \le c(1 + \|\xi\|^\gamma)e^{-\delta(t-\tau)\|\xi\|^h} + c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\int_\tau^t e^{-\delta(t-\beta)\|\xi\|^h}|\Theta_\tau^\beta(\xi)|d\beta,$$

from which we come to correlation

$$\frac{|\Theta_\tau^t(\xi)|e^{\delta(t-\tau)\|\xi\|^h}}{(1+\|\xi\|^\gamma)} \le c + c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1}) \int\limits_\tau^t \frac{|\Theta_\tau^\beta(\xi)|e^{\delta(\beta-\tau)\|\xi\|^h}}{(1+\|\xi\|^\gamma)} d\beta.$$

Using now Lemma of Gronwall [12] we obtain

$$|\Theta_\tau^t(\xi)| \le c(1+\|\xi\|^\gamma)e^{-(t-\tau)\left(\delta\|\xi\|^h - c_1(1+\|\xi\|^\gamma)(1+\|\xi\|^{p_1})\right)}, \ (t;\xi) \in \Pi_{(\tau;T]}, \ \tau \in [0;T).$$

From here considering the condition (A) we come to existence of positive constants c and δ with which for all $(t;\xi) \in \Pi_{(\tau;T]}$, $\tau \in [0;T)$, bound (3) holds. Theorem is proved. □

Corollary 1. *System* (7) *with condition (A) is parabolic by Shilov system with coefficients dependent on t and index of parabolicity h.*

3 PROPERTIES OF FSCP

Let (1) is parabolic by Shilov system with continuous on $[0;T]$ coefficients. Solving this system by Fourier transform we obtain a representation of the fundamental solution of its Cauchy problem:

$$G(\tau,t;\cdot) = F^{-1}[\Theta_\tau^t(\xi)](\tau,t;\cdot), \quad 0 \le \tau < t \le T$$

(here $F^{-1}[\cdot]$ is inverse Fourier transform and $\Theta_\tau^t(\cdot)$ is appropriate matricant (4)).

The following statement holds.

Theorem 2. *Let the system* (1) *is parabolic by Shilov with dependent continuously on t coefficients and positive genus μ. Then its FSCP on the set $\mathbb{R}^n$ for spatial variable is infinitely differentiable function such that exists $\{c,B,\delta\} \subset (0;+\infty)$ for all $k \in \mathbb{Z}_+^n$, $\tau \in [0;T)$, $t \in (\tau;T]$, $x \in \mathbb{R}^n$ such that*

$$|\partial_x^k G(\tau,t;x)| \le c(t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} B^{|k|_+} k^{k\frac{1}{h}} e^{-\delta\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},$$

here $\alpha := \mu/p_0$.

Proof. Let's consider the matrix function

$$\varphi_{\tau,t}^k(x) := (t-\tau)^{\frac{\gamma+|k|_+}{h}} x^k \Theta_\tau^t(x), \ k \in \mathbb{Z}_+^n, \ x \in \mathbb{R}^n, \ 0 \le \tau < t \le T,$$

which obviously continues in a complex space $\mathbb{C}^n$ to an entire analytic function at each fixed k, t and τ.

Directly to the condition (3) we obtain that

$$\begin{aligned} |\varphi_{\tau,t}^k(x)| &\le c(t-\tau)^{\frac{\gamma+|k|_+}{h}} \|x\|^{|k|_+}(1+\|x\|^\gamma)e^{-\delta(t-\tau)\|x\|^h} \\ &= c\left(((t-\tau)\|x\|^h)^{\frac{|k|_+}{h}}(t-\tau)^{\frac{\gamma}{h}} + ((t-\tau)\|x\|^h)^{\frac{\gamma+|k|_+}{h}}\right)e^{-\delta(t-\tau)\|x\|^h} \\ &\le cT_0^{\frac{\gamma}{h}}\left(\sup_{\xi\ge 0}\left\{\xi^{\frac{|k|_+}{h}}e^{-\frac{\delta}{2}\xi}\right\} + \sup_{\xi\ge 0}\left\{\xi^{\frac{\gamma+|k|_+}{h}}e^{-\frac{\delta}{2}\xi}\right\}\right)e^{-\frac{\delta}{2}(t-\tau)\|x\|^h}, \text{ where } T_0 = \max\{1,T\}. \end{aligned}$$

Hence, taking the equality

$$\sup_{\xi\geq 0}\{\xi^{\beta}e^{-\delta\xi}\} = \left(\frac{\beta}{e\delta}\right)^{\beta},\ \ \beta>0,\ \ \delta>0, \tag{12}$$

into account we come to existence of positive constants c_1, B_1 and δ_1 such that for all $x \in \mathbb{R}^n,\ k\in\mathbb{Z}_+^n,\ \tau\in[0;T)$ and $t\in(\tau;T]$ inequality

$$|\varphi_{\tau,t}^{k}(x)| \leq c_1 B_1^{|k|+}k^{k\frac{1}{h}}e^{-\delta_1(t-\tau)\|x\|^{h}}$$

holds. Similarly way due to the definition of genus μ of parabolic system (1) we come to such an bound of matrix function $\varphi_{\tau,t}^{k}(\cdot)$ in the relevant domain $\mathbb{K}_\mu \subset \mathbb{C}^n$:

$$|\varphi_{\tau,t}^{k}(x+iy)| \leq c_2 B_2^{|k|+}k^{k\frac{1}{h}}e^{-\delta_2(t-\tau)\|x\|^{h}},\ \ k\in\mathbb{Z}_+^n,\ \ 0\leq\tau<t\leq T \tag{13}$$

(here positive constants c_2, B_2 and δ_2 do not depend on k, x, y, τ and t). In addition, using the estimate (5) and the equality (12) we obtain inequality

$$|\varphi_{\tau,t}^{k}(z)| \leq c_3 B_3^{|k|+}k^{k\frac{1}{h}}e^{\delta_3(t-\tau)\|z\|^{p_0}},\ \ k\in\mathbb{Z}_+^n,\ \ z\in\mathbb{C}^n,\ \ 0\leq\tau<t\leq T, \tag{14}$$

with an estimated constant not dependent on k, z, τ and t.

Note that when $\mu>0$ estimate (14) can be specified. Indeed, let $z = x+iy \in \mathbb{C}^n\setminus\mathbb{K}_\mu$, then inequality $\|y\|/K > \|x\|^{\mu}$ holds. From here, the estimates $\|z\|^{p_0} \leq c(\|x\|^{p_0}+\|y\|^{p_0}),\ z\in\mathbb{C}^n$ and (14) taking into account that $\mu\leq 1$ for all $z\in\mathbb{C}^n\setminus\mathbb{K}_\mu$, $\tau\in[0;T)$ and $t\in(\tau;T]$ we obtain

$$\begin{aligned}|\varphi_{\tau,t}^{k}(z)| &\leq c_3 B_3^{|k|+}k^{k\frac{1}{h}}e^{-\delta_2(t-\tau)\|x\|^{h}}e^{(t-\tau)(\delta_3\|z\|^{p_0}+\delta_2\|x\|^{h})}\\ &\leq c_4 B_3^{|k|+}k^{k\frac{1}{h}}e^{(t-\tau)\left(\delta_0\|y\|^{\frac{p_0}{\mu}}-\delta_2\|x\|^{h}\right)}\end{aligned}$$

(here estimated constants also do not depend on k, z, τ and t). If we now consider estimate (13) then we come to this statement: exists $\{c, B, \delta_1, \delta_2\}\subset(0;+\infty)$ for all $z = x+iy\in\mathbb{C}^n$, $k\in\mathbb{Z}_+^n$, $\tau\in[0;T)$ $t\in(\tau;T]$ such that

$$|\varphi_{\tau,t}^{k}(z)| \leq cB^{|k|+}k^{k\frac{1}{h}}e^{(t-\tau)\left(\delta_1\|y\|^{\frac{p_0}{\mu}}-\delta_2\|x\|^{h}\right)}. \tag{15}$$

Further, according to Cauchy integral formula we have

$$\partial_x^q\varphi_{\tau,t}^{k}(x) = \prod_{j=1}^{n}\frac{q_j!}{2\pi i}\int\limits_{\Gamma_{R_j}}\frac{\varphi_{\tau,t}^{k}(\sigma)d\sigma_j}{(\sigma_j-x_j)^{q_j+1}},\ \{k,q\}\subset\mathbb{Z}_+^n,\ x\in\mathbb{R}^n,\ 0\leq\tau<t\leq T, \tag{16}$$

here Γ_{R_j} is circle of radius R_j with center in the point x_j.

Let $\Gamma_R := \Gamma_{R_1}\times\ldots\times\Gamma_{R_n}$. Let us denote $\sigma^* = \xi^*+i\eta^*$ is the point from Γ_R such that

$$|\varphi_{\tau,t}^{k}(\sigma^*)| = \max_{\sigma\in\Gamma_R}|\varphi_{\tau,t}^{k}(\sigma)|.$$

Since the coordinates σ_j^* of the point σ^* are in Γ_{R_j} then the equality

$$(\xi_j^*-x_j)^2+\eta_j^{*2} = R_j^2,\ \ j\in\mathbb{N}_n,$$

holds and implies such correlations:

$$|\xi_j^* - x_j| \leq R_j, \ |\eta_j^*| \leq R_j, \ \ j \in \mathbb{N}_n. \tag{17}$$

Taking into consideration all above mentioned, estimations (15), inequality

$$n^{-1}|x|_+^r \leq \|x\|^r \leq n^{r/2}|x|_+^r, \ \ r > 0, x \in \mathbb{R}^n,$$

and (16) for $\mu > 0$ we obtain for all $R_j > 0$, $j \in \mathbb{N}_n$

$$|\partial_x^q \varphi_{\tau,t}^k(x)| \leq cB^{|k|_+} k^{k\frac{1}{h}} \prod_{j=1}^n \frac{q_j!}{R_j^{q_j}} e^{(t-\tau)(\hat{\delta}_1 R_j^{p_0/\mu} - \hat{\delta}_2|\xi_j^*|^h)} \tag{18}$$

(here $\hat{\delta}_1 =: \delta_1 n^{\frac{p_0}{2\mu}}$ and $\hat{\delta}_2 := \delta_2/n$).

Let us take radiuses R_j such that the ratio $e^{(t-\tau)\hat{\delta}_1 R_j^{p_0/\mu}}/R_j^{q_j}$ reaches a minimum. Then we put

$$R_j = \left(\frac{q_j \mu}{(t-\tau)\hat{\delta}_1 p_0}\right)^{\mu/p_0}, \ \ j \in \mathbb{N}_n.$$

Then bound (18) is reduced to

$$|\partial_x^q \varphi_{\tau,t}^k(x)| \leq cB^{|k|_+} \big((t-\tau) e p_0 \hat{\delta}_1/\mu\big)^{\mu|q|_+/p_0} k^{k\frac{1}{h}} q^{q(1-\frac{\mu}{p_0})} e^{-(t-\tau)\hat{\delta}_2|\xi^*|_+^h}. \tag{19}$$

Next, let's estimate the exponent $e^{-(t-\tau)\hat{\delta}_2|\xi_j^*|^h}$, $j \in \mathbb{N}_n$.

If $2|\xi_j^*| \geq |x_j|$ then we have

$$e^{-(t-\tau)\hat{\delta}_2|\xi_j^*|^h} \leq e^{-(t-\tau)\hat{\delta}_2(|x_j|/2)^h}.$$

If $|x_j| > 2|\xi_j^*|$ then according to (17) the following inequalities hold:

$$\begin{aligned} R_j^h &\geq |x_j - \xi_j^*|^h \geq ||x_j| - |\xi_j^*||^h = (|x_j|^h - |\xi_j^*|^h)\frac{||x_j| - |\xi_j^*||^h}{|x_j|^h - |\xi_j^*|^h} \\ &\geq (|x_j|^h - |\xi_j^*|^h)\big|1 - |\xi_j^*|/|x_j|\big|^h \geq (|x_j|^h - |\xi_j^*|^h)/2^h, \end{aligned}$$

and

$$|x_j|^h - |\xi_j^*|^h \leq (2R_j)^h.$$

Then $-|\xi_j^*|^h = -|x_j|^h + (|x_j|^h - |\xi_j^*|^h) \leq -|x_j|^h + (2R_j)^h$ and

$$e^{-(t-\tau)\hat{\delta}_2|\xi_j^*|^h} \leq e^{-(t-\tau)\hat{\delta}_2|x_j|^h + \hat{\delta}_0(t-\tau)R_j^h}, \ \ \hat{\delta}_0 := \hat{\delta}_2 2^h.$$

From here and the astimate (19), abides by that

$$\begin{aligned} (t-\tau)R_j^h &= (t-\tau)^{1-\mu h/p_0}\left(\frac{q_j\mu}{\hat{\delta}_1 p_0}\right)^{\mu h/p_0} \leq T_0^{1-\mu h/p_0}\left(\frac{q_j\mu}{\hat{\delta}_1 p_0}\right)^{\mu h/p_0} \\ &\equiv c q_j^{\mu h/p_0} \leq c q_j, \ \ j \in \mathbb{N}_n. \end{aligned}$$

If $\mu > 0$ we get the next statement: exists $\{c, A, B, \delta\} \subset (0; +\infty)$ for all $\{k, q\} \subset \mathbb{Z}^n_+$, $\tau \in [0; T)$, $t \in (\tau; T]$, $x \in \mathbb{R}^n$ such that

$$|\partial_x^q \varphi^k_{\tau,t}(x)| \leq c((t-\tau)^\alpha A)^{|q|_+} B^{|k|_+} k^{k\frac{1}{h}} q^{q(1-\alpha)} e^{-(t-\tau)\delta |x|^h_+}. \quad (20)$$

Directly from the estimate (20) and the definition of matrix function $\varphi^k_{\tau,t}(\cdot)$ and with the equality

$$(ix)^q \partial_x^k G(\tau, t; x) = (-i)^{|k|_+} (2\pi)^{-n} (t-\tau)^{-\frac{\gamma+|k|_+}{h}} \int\limits_{\mathbb{R}^n} \partial_\xi^q \varphi^k_{\tau,t}(\xi) e^{-i(x,\xi)} d\xi,$$

we obtain that

$$|\partial_x^k G(\tau, t; x)| \leq c_0 (t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} B^{|k|_+} k^{k\frac{1}{h}} \times \left(\prod_{j=1}^n \inf_{q_j} \left\{ ((t-\tau)^\alpha A)^{q_j} q_j^{q_j(1-\alpha)} |x_j|^{-q_j} \right\} \right)$$
$$\leq c(t-\tau)^{-\frac{n+\gamma+|k|_+}{h}} B^{|k|_+} k^{k\frac{1}{h}} e^{-\delta\left(\frac{\|x\|}{(t-\tau)^\alpha}\right)^{\frac{1}{1-\alpha}}},$$

for all $k \in \mathbb{Z}^n_+$, $x \in \mathbb{R}^n$ and $0 \leq \tau < t \leq T$, while estimated constants c, B and δ do not depend on t, τ, k and x. Theorem is proved. □

4 CONCLUSIONS

Parabolic systems of Shilov type are parabolically unstable systems to a change in their coefficients, in contrast to Petrovsky's parabolic systems. In this respect, information is important about parabolic systems with variable coefficients that significantly extend the Petrovsky class in the Shilov class and allow us to use the means of the classical theory of the Cauchy problem for their investigation. The class of systems defined in this article is such. The presence of this class, in particular, convinces that the class of Shilov vector equations with variable coefficients is not exhausted by the class of Petrine systems with time-dependent coefficients, but is much wider.

The obtained here estimates of the fundamental solution of the Cauchy problem for Shilov parabolic systems with coefficients that depend on t important to establish the correct solvability of the Cauchy problem in various functional spaces and, in the study of properties of solutions to this problem.

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Received 01.07.2017

Revised 28.12.2017

Літовченко В.А. *Параболічні за Шиловим системи із змінними коефіцієнтами* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 145–153.

Через параболічну нестійкість систем Шилова до зміни своїх коефіцієнтів, означення параболічності за Шиловим для систем із залежними від часу t коефіцієнтами, на відміну від параболічності за Петровським, формулюється шляхом накладання умов на матрицант відповідної двоїстої за Фур'є системи. Для параболічних за Петровським систем із залежними від часу коефіцієнтами ці умови є характерною властивістю матрицанта, які випливають безпосередньо із означення параболічності. У зв'язку з цим, набуває актуальності питання про багатство класу Шилова систем із змінними коефіцієнтами.

У даній роботі наведено новий клас лінійних параболічних систем рівнянь із частинними похідними першого порядку за t із залежними від часу коефіцієнтами, який охоплює клас Петровського систем із молодшими коефіцієнтами, залежними від t. Головна частина диференціального виразу кожної такої системи є параболічним за Шиловим виразом із сталими коефіцієнтами. Методом перетворення Фур'є побудовано фундаментальний розв'язок задачі Коші для систем цього класу та обгрунтовано їх параболічність за Шиловим. При цьому використано лише структуру системи та умови на власні числа її головного матричного символу. Цей клас, перед усім, характеризує багатство класу Шилова систем із змінними коефіцієнтами та невичерпність його системами Петровського.

Також наведено загальний метод дослідження фундаментального розв'язку задачі Коші для параболічних за Шиловим систем, який є розвиненням відомого методу Я.І. Житомирського.

Ключові слова і фрази: параболічна за Шиловим система, фундаментальний розв'язок, задача Коші.