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CLASSIFICATION OF GENERALIZED TERNARY QUADRATIC QUASIGROUP FUNCTIONAL EQUATIONS OF THE LENGTH THREE

A functional equation is called: *generalized* if all functional variables are pairwise different; *ternary* if all its functional variables are ternary; *quadratic* if each individual variable has exactly two appearances; *quasigroup* if its solutions are studied only on invertible functions. A *length* of a functional equation is the number of all its functional variables. A complete classification up to parastrophically primary equivalence of generalized quadratic quasigroup functional equations of the length three is given. Solution sets of a full family of representatives of the equivalence are found.

Key words and phrases: ternary quasigroup, quadratic equation, length of a functional equation, parastrophically primary equivalence.

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INTRODUCTION

We study functional equations which can be considered on an arbitrary set (a carrier) and therefore they have neither individual nor functional constants. Moreover, we focus our attention only on the solutions which are sequences of invertible functions (i.e., quasigroup functions) and such equations are called quasigroup equations. We do not pay attention to dependencies among functional variables. That is why, we consider generalized equations: all functional variables are pairwise different. The word 'ternary' means that every functional variable takes its value in the set Δ_3 of all ternary invertible operations defined on a carrier.

Every ternary invertible operation has three inverses: left, middle and right divisions and each of them is also invertible, etc. These operations are called parastrophes. Generally speaking, an arbitrary ternary invertible operation has $4! = 24$ parastrophes including itself and all of them are connected by some defining identities. These identities are true not only for all individual variables but for all functional variables provided they take their value in Δ_3. In other words, they are hyperidentities over the set Δ_3, and they are called primary. Renaming functional and individual variables and applying primary hyperidentities, one can transform one functional equation into some other equation. This relation between functional equations is an equivalence and is called a parastrophically primary equivalence. If two functional equations are parastrophically primarily equivalent, then there is an algorithm which transforms the solution set of the first equation into the solution set of the second one.

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The problem under consideration is *"Describe parastrophically primary equivalence on the set of all quasigroup functional equations, select all representatives (i.e., a maximal set of non-equivalent functional equations) and solve all of them"*.

This problem is discussed in A. Krapež [3], S. Krstić [15], A. Krapež and D. Živković [4], A. Ehsani, A. Krapež and Y. Movsisyan [5], F. Sokhatsky [8, 10], F. Sokhatsky and H. Krainichuk [6, 9], R. Koval' [14], H. Krainichuk [13] etc. for binary quasigroups. On ternary quasigroups, the parastrophically primary classification was carried out in the article [11], where a two-element transversal equivalence of the generalized non-trivial functional equations of the length one and the seven-element transversal of the equivalence of generalized non-trivial functional equations of the length two were singled out.

In this article, only quadratic generalized functional equations of the length three on invertible functions (i.e. quasigroup operations) are studied, that is, those equations in which each individual variable has exactly two appearances. If a quasigroup equation has one appearance of an individual variable, then it is trivial, i.e. it has solutions only on singletons.

In section 'Quasigroup solutions', general solutions of each element from a family of pairwise parastrophically primarily non-equivalent generalized quadratic functional equations of the length three on ternary quasigroups have been found in Theorems 2–5. In the next section 'Proof of Theorem 1', a full proof of the classification theorem is given.

1 PRELIMINARIES

1.1 Quasigroup

All operations considered in this article are defined on an arbitrary fixed set Q called a *carrier*. A binary operation is a mapping $g\colon Q^2 \to Q$, the set of all operations defined on Q is denoted by $\mathcal{O}_2$. A binary operation g is called *invertible*, if it is invertible in both monoids $(\mathcal{O}_2; \underset{1}{\oplus}, e_1)$ and $(\mathcal{O}_2; \underset{2}{\oplus}, e_2)$, where $e_1(x_1, x_2) := x_1$, $e_2(x_1, x_2) := x_2$ and

$$(g \underset{1}{\oplus} g_1)(x_1, x_2) := g(g_1(x_1, x_2), x_2), \qquad (g \underset{2}{\oplus} g_1)(x_1, x_2) := g(x_1, g_1(x_1, x_2)).$$

The operation g is the *main* one and its inverses in $(\mathcal{O}_2; \underset{1}{\oplus}, e_1)$ and $(\mathcal{O}_2; \underset{2}{\oplus}, e_2)$ are denoted by ${}^{\ell}g$ and ${}^{r}g$ and are called g's *left* and *right divisions* respectively. If an operation g is invertible, then the algebra $(Q; g, {}^{\ell}g, {}^{r}g)$ is called a *binary quasigroup* [10]. Usually, infix notations are used for binary operations. Therefore, an algebra $(Q; \circ, \overset{\ell}{\circ}, \overset{r}{\circ})$ is called a quasigroup if the identities

$$(x \overset{\ell}{\circ} y) \circ y = x, \quad (x \circ y) \overset{\ell}{\circ} y = x, \quad x \circ (x \overset{r}{\circ} y) = y, \quad x \overset{r}{\circ} (x \circ y) = y$$

hold.

Similarly, a *ternary operation* is a mapping $f\colon Q^3 \to Q$, the set of all ternary operations defined on Q is denoted by $\mathcal{O}_3$. A ternary operation f is called *invertible* if it is invertible in each of the monoids $(\mathcal{O}_3; \underset{i}{\oplus}, e_i)$, $i = 1, 2, 3$, where

$$(f \underset{1}{\oplus} f_1)(x_1, x_2, x_3) := f(f_1(x_1, x_2, x_3), x_2, x_3),$$
$$(f \underset{2}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, f_1(x_1, x_2, x_3), x_3),$$
$$(f \underset{3}{\oplus} f_1)(x_1, x_2, x_3) := f(x_1, x_2, f_1(x_1, x_2, x_3)),$$
$$e_i(x_1, x_2, x_3) := x_i, \; i = 1, 2, 3.$$

The operation f is the *main* one and its inverses in $(\mathcal{O}_3; \underset{1}{\oplus}, e_1)$, $(\mathcal{O}_3; \underset{2}{\oplus}, e_2)$, $(\mathcal{O}_3; \underset{3}{\oplus}, e_3)$ are denoted by ${}^{(14)}f$, ${}^{(24)}f$, ${}^{(34)}f$ and they are called f's *left*, *middle* and *right divisions* respectively. In other words, the operation f is invertible if the identities

$$f({}^{(14)}f(x,y,z),y,z) = x, \qquad (1) \qquad\qquad {}^{(14)}f(f(x,y,z),y,z) = x, \qquad (4)$$
$$f(x,{}^{(24)}f(x,y,z),z) = y, \qquad (2) \qquad\qquad {}^{(24)}f(x,f(x,y,z),z) = y, \qquad (5)$$
$$f(x,y,{}^{(34)}f(x,y,z)) = z, \qquad (3) \qquad\qquad {}^{(34)}f(x,y,f(x,y,z)) = z \qquad (6)$$

hold. If an operation f is invertible, then the algebra $(Q; f, {}^{(14)}f, {}^{(24)}f, {}^{(34)}f)$ (in brief, $(Q; f)$) is called a *ternary quasigroup* [10]. It is easy to verify that all divisions of an invertible operation are also invertible and so are their divisions.

A *σ-parastrophe* of an invertible operation f is called an operation ${}^{\sigma}f$ defined by

$${}^{\sigma}f(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}) = x_{4\sigma} \quad :\Leftrightarrow \quad f(x_1, x_2, x_3) = x_4, \quad \sigma \in S_4,$$

where S_4 denotes the group of all bijections of the set $\{1, 2, 3, 4\}$. Therefore in general, every invertible operation has 24 parastrophes. Some of them can coincide. If all parastrophes coincide, the quasigroup is called *totally symmetric*. Since parastrophes of a quasigroup satisfy the equalities

$${}^{\sigma}({}^{\tau}f) = {}^{\sigma\tau}f \quad \text{and} \quad {}^{\iota}f = f, \tag{7}$$

then the symmetric group S_4 defines an action on the set Δ_3 of all ternary invertible operations defined on the same carrier. In particular, the fact implies that the number of different parastrophes of an invertible operation is a factor of 24. More precisely, it is equal to $24/|\mathrm{Ps}(f)|$, where $\mathrm{Ps}(f)$ denotes a stabilizer group of f under this action which is called *parastrophic symmetry group* of the operation f. Consequently, a totally symmetric quasigroup is a quasigroup whose parastrophic symmetry group is S_4. If the parastrophic symmetry group of a ternary quasigroup is trivial, then the quasigroup has 24 different parastrophes and it is called *asymmetric*.

An element e of $(Q; f)$ is called *neutral* if for all x from Q the equalities

$$f(x,e,e) = x, \qquad f(e,x,e) = x, \qquad f(e,e,x) = x$$

hold. In contrast to the binary case, a neutral element is not necessarily unique in a ternary quasigroup. A quasigroup is called a *loop* if it has a neutral element. For example, let $(Q; +)$ be a group of the exponent two and an operation f be defined by

$$f(x,y,z) := x + y + z.$$

It is easy to see that every element of the quasigroup is neutral in the ternary quasigroup $(Q; f)$. Such a quasigroup will be called universally neutral. Namely, a ternary quasigroup $(Q; f)$ will be called a *left*, *middle*, *right universally neutral* if the respective identity holds:

$$f(x,y,y) = x, \qquad f(y,x,y) = x, \qquad f(y,y,x) = x.$$

It will be called *universally neutral* if all three identities take place. Note, that the given example of the ternary quasigroup is not only universally neutral, but it is totally symmetric. A quasigroup which is both universally neutral and totally symmetric is called a *Steiner quasigroup* [2, 12]. Thus, every ternary Steiner quasigroup is a loop. Moreover, each of its elements is neutral.

An invertible operation f is called *repetition-free decomposable* if there exist two binary invertible operations g, h and bijection $\sigma \in S_3$ such that

$$f(x_1, x_2, x_3) = g(h(x_{1\sigma}, x_{2\sigma}), x_{3\sigma}).$$

Theorem 1 from [16] implies the following result.

Corollary 1. *If a ternary Steiner quasigroup $(Q; f)$ is repetition-free decomposable, then there is a group $(Q; +)$ of the exponent two such that*

$$f(x, y, z) = x + y + z.$$

1.2 Functional equations

Throughout the article, we will use the notion 'functional equation' in the following sense. Let T_1 and T_2 are second order terms which have only individual and functional variables. A formula $T_1 = T_2$ is called a *functional equation*, if it is universally quantified on all individual variables and has at least one free functional variable. Moreover, we consider only generalized ternary quadratic functional equations of the length three on quasigroups, where the notion 'ternary quasigroup equation' means that all functional variables take their values only in the set of ternary invertible functions; the word 'generalized' means that the variables are pairwise different; the word 'quadratic' means that every individual variable has exactly two appearances or none; the notion 'length of a functional equation' is the number of functional variables including their repetitions (see [1, 10]).

A subterm of an equation is a subterm of its left or right sides. A subterm of a term T is called *proper* if it coincides neither with T nor an individual variable. Let $F(t_1, t_2, t_3)$ be a term, then the function variable F is called *main*.

Let $T_1 = T_2$ be a ternary functional equation of the length three, (F, G_i, G_j) be the lexicographical sequence of its functional variables, i.e., $i < j$. A sequence (f, g, h) of invertible ternary functions defined on a set Q is called a *solution* of $T_1 = T_2$ if substituting f for F, g for G_1 and h for G_2, we obtain a true proposition $t_1 = t_2$, i.e., $t_1 = t_2$ is an identity. A quasigroup functional equation is called *trivial* if it has a solution only on a singleton.

Consequently, in an arbitrary non-trivial quasigroup functional equation, every individual variable has at least two appearances. In this article, we consider the case when every individual variable has exactly two appearances, these equations are called *quadratic*.

Let Δ_3 be the set of all invertible ternary functions defined on a carrier Q. The relationships (1)–(6) and (7) are true for all functions from Δ_3. In other words, the following *hyperidentities* are true over the set Δ_3:

$$\left.\begin{array}{ll} {}^{\sigma}({}^{\tau}F) = {}^{\sigma\tau}F, \quad {}^{\iota}F = F, & {}^{(14)}F(F(x,y,z),y,z) = x; \\ {}^{(24)}F(x, F(x,y,z), z) = y; & {}^{(34)}F(x,y,F(x,y,z)) = z, \\ F(x_1, x_2, x_3) = {}^{\sigma}F(x_{1\sigma}, x_{2\sigma}, x_{3\sigma}), \quad \sigma \in S_3. & \end{array}\right\} \tag{8}$$

The hyperidentities are called *primary*.

Two quasigroup functional equations are called: *equivalent over a set Q* if they have the same solution set over the carrier; *equivalent* if they are equivalent over each set.

Definition 1 ([8]). *Two functional equations are called parastrophically primarily equivalent if one can be obtained from the other in a finite number of the following steps: 1) replacing of the equation sides; 2) renaming of the functional variables; 3) renaming of the individual variables; 4) applying the hyperidentities* (8).

A *lexicographical renaming of individual variables* is renaming all first appearances of these variables according to their lexicographical order.

Lemma 1. *Let $v = \omega$ and $v' = \omega'$ be generalized ternary functional equations of the length three. If they are parastrophically primarily equivalent, then there exists a bijection τ in S_3 and bijections σ_1, σ_2, σ_3 in S_4 such that for an arbitrary solution (f_1, f_2, f_3) of $v = \omega$ the sequence*

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is a solution of the equation $v' = \omega'$.

In this case, $(\tau, \sigma_1, \sigma_2, \sigma_3)$ is called a *defining bijection system* of the equations $v = \omega$ and $v' = \omega'$. This lemma implies a sufficient condition for parastrophically primary non-equivalence of ternary generalized functional equations of the length three. Namely, the following statement is valid.

Corollary 2. *If for every bijection τ in S_3 and bijections σ_1, σ_2, σ_3 in S_4 there exists a solution (f_1, f_2, f_3) of $v = \omega$ such that $({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$ is not a solution of $v' = \omega'$, then the functional equations $v = \omega$ and $v' = \omega'$ are not parastrophically primarily equivalent.*

A function f is called a *solution* of a functional equation if the sequence $(f, f, \dots, f)$ is solution of the equation.

Corollary 3. *If a totally symmetric function is a solution of a functional equation but it is not a solution of another functional equation, then the equations are not parastrophically primarily equivalent.*

2 QUASIGROUP SOLUTIONS

Theorem 1 gives a full classification of generalized quadratic ternary quasigroup functional equations of the length three up to parastrophically primary equivalence. Also, all quasigroup solutions of all representatives (9)–(12) of the classification are proved in Theorem 2–5.

Theorem 1. *Every generalized quadratic ternary quasigroup functional equation of the length three is parastrophically primarily equivalent to exactly one of the following equations:*

$$F_1(z, x, F_2(x, y, y)) = F_3(z, u, u), \tag{9}$$

$$F_1(F_2(x, y, y), z, z) = F_3(x, u, u), \tag{10}$$

$$F_1(F_2(x, y, z), u, u) = F_3(x, y, z), \tag{11}$$

$$F_1(F_2(x, y, z), x, u) = F_3(y, z, u). \tag{12}$$

Lemma 2. *Let α, f be the unary and ternary invertible operations respectively. Then the equality*

$$f(x, y, y) = \alpha x \tag{13}$$

is equivalent to the existence of a left-universally neutral invertible operation g such that

$$f(x, y, z) = g(\alpha x, y, z). \tag{14}$$

Proof. Define operation g, by

$$g(x,y,z) := f(\alpha^{-1}x,y,z). \tag{15}$$

Since f is invertible and g is an isotope of f, the operation g is invertible. Taking into account (13), we have $x = f(\alpha^{-1}x,y,y) = g(x,y,y)$. Hence, the operation g is left-universally neutral. Applying (15), we obtain (14).

Conversely, let g be a left-universally neutral invertible operation and let the relationship (14) holds. Then $f(x,y,y) = g(\alpha x,y,y) = \alpha x$. $\square$

Theorem 2. *A triplet (f_1,f_2,f_3) of ternary invertible operations is a solution of the equation (9) if and only if there exist left-universally neutral invertible operations h_1, h_2, h_3 and bijections α, β such that*

$$f_1(x,y,z) = h_1(\alpha x,y,\beta^{-1}z), \tag{16}$$
$$f_2(x,y,z) = h_2(\beta x,y,z), \tag{17}$$
$$f_3(x,y,z) = h_3(\alpha x,y,z). \tag{18}$$

Proof. Let a triplet (f_1,f_2,f_3) of ternary invertible operations defined on Q be a solution of the equation (9), i.e., for all x, y, z, u the identity

$$f_1(z,x,f_2(x,y,y)) = f_3(z,u,u) \tag{19}$$

holds. In particular, if $u = a \in Q$, we have

$$f_1(z,x,f_2(x,y,y)) = \alpha z, \tag{20}$$

where $\alpha z := f_3(z,a,a)$ is a bijection of Q because α is a left translation of the invertible operation f_3.

Also, from (20) and (19), we get the identity $f_3(z,u,u) = \alpha z$. According to Lemma 2, there exists a left-universally neutral invertible operation h_3 such that (18) holds.

Applying the definition of a parastrophe to the equality (20), we have

$$f_2(x,y,y) = {}^{(34)}f_1(z,x,\alpha z).$$

If $z = a \in Q$ and $\beta x :={}^{(34)} f_1(a,x,\alpha a)$, the equality is written as $f_2(x,y,y) = \beta x$. Note that β is bijective on Q since it is a translation of an invertible operation ${}^{(34)}f_1$. By Lemma 2, the latter relationship implies the existence of a left-universally neutral invertible operation h_2 such that (17) is true.

Replace $f_2(x,y,y)$ with βx in (20): $f_1(z,x,\beta x) = \alpha z$. Let $h_1(x,y,z) := f_1(\alpha^{-1}x,y,\beta z)$, then (16) holds and $h_1(x,y,y) = f_1(\alpha^{-1}z,x,\beta x) = \alpha\alpha^{-1}x = x$. Thus the operation h_1 is a left-universally neutral invertible.

Conversely, let the operations h_1, h_2, h_3 be left-universally neutral invertible operations and operations f_1, f_2, f_3 be defined by (16), (17), (18) for some bijections α, β of a set Q. Then

$$\begin{aligned} f_1(z,x,f_2(x,y,y)) &= h_1(\alpha z,x,\beta^{-1}h_2(\beta x,y,y) \\ &= h_1(\alpha z,x,\beta^{-1}\beta x) = h_1(\alpha z,x,x) = \alpha z \\ &= h_3(\alpha z,u,u) = f_3(z,u,u). \end{aligned}$$

Therefore, the triplet (f_1,f_2,f_3) is a solution of the equation (9). $\square$

Theorem 3. *A triplet of ternary invertible operations* (f_1, f_2, f_3) *is a solution of the equation (10) if and only if there exist left-universally neutral invertible operations* g_1*,* g_2*,* g_3 *and bijections* γ*,* δ *such that*

$$f_1(x,y,z) = g_1(\gamma x, y, z), \tag{21}$$

$$f_2(x,y,z) = g_2(\delta x, y, z), \tag{22}$$

$$f_3(x,y,z) = g_3(\gamma\delta x, y, z). \tag{23}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (10), i.e., the identity

$$f_1(f_2(x,y,y),z,z) = f_3(x,u,u) \tag{24}$$

holds. In particular, if $y = u = a \in Q$, we have $f_1(f_2(x,y,y),a,a) = f_3(x,a,a)$. Then $\alpha f_2(x,y,y) = \beta x$, where $\alpha x := f_1(x,a,a)$ and $\beta x := f_3(x,a,a)$ are bijective since α and β are translations of the invertible operations f_1 and f_3 respectively. Therefrom

$$f_2(x,y,y) = \alpha^{-1}\beta x.$$

Defining $\delta := \alpha^{-1}\beta$, we have $f_2(x,y,y) = \delta x$. According to Lemma 2, there exists a left-universally neutral invertible operation g_2 such that the equality (22) holds.

Let us substitute δx in (24) for $f_2(x,y,y)$:

$$f_1(\delta x, z, z) = f_3(x,u,u).$$

Replace x with $\delta^{-1}x$ in the equality: $f_1(x,z,z) = f_3(\delta^{-1}x,u,u)$ for all x, z, u. In particular, when $u = a \in Q$, we have $f_1(x,z,z) = \gamma x$, where $\gamma x := f_3(\delta^{-1}x,a,a)$ is a bijection of the carrier Q, because γ is the left translation of the invertible operation f_3. Therefore, the relationship (21) holds for some left-universally neutral operation g_1. Applying (21) and (22) to (24), we have

$$\gamma\delta x = f_3(x,u,u).$$

According to Lemma 2, there exists a left-universally neutral invertible operation g_3 such that the equality (23) holds.

Vise versa, let the relationships (21), (22), (23) be true for some left-universally neutral operations g_1, g_2, g_3 and bijections γ, δ, then

$$\begin{aligned} f_1(f_2(x,y,y),z,z) &= g_1(\gamma g_2(\delta x,y,y),z,z) \\ &= g_1(\gamma\delta x,z,z) = \gamma\delta x = g_3(\gamma\delta x,u,u) = f_3(x,u,u). \end{aligned}$$

Thus, the triplet (f_1, f_2, f_3) is a solution of the equation (10). □

Theorem 4. *A triplet* (f_1, f_2, f_3) *of ternary operations defined on a set* Q *is a quasigroup solution of the functional equation (11) if and only if the operation* f_2 *is invertible and there exists a bijection* μ *and a left-universally neutral operation* g *such that*

$$f_3(x,y,z) = \mu f_2(x,y,z), \qquad f_1(x,y,z) = g(\mu x, y, z). \tag{25}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations be a solution of the equation (11), i.e., for all x, y, z, u the identity

$$f_1(f_2(x,y,z), u, u) = f_3(x,y,z) \tag{26}$$

holds. In particular, when $u = a \in Q$ and $\mu x := f_1(x, a, a)$, we have the first identity from (25). Substituting μf_2 in (26) for f_3, we have

$$f_1(f_2(x,y,z), u, u) = \mu f_2(x,y,z).$$

Replacing $f_2(x,y,z)$ with x, we obtain $f_1(x,u,u) = \mu x$. According to Lemma 2, there exists a bijection μ and a left-universally neutral operation g such that the second relationship from (25) holds.

Conversely, let f_2 be invertible ternary operation and there exists a bijection μ and a left-universally neutral operation g such that the relationships (25) hold. Then

$$f_1(f_2(x,y,z), u, u) = g(\mu f_2(x,y,z), u, u) = g(f_3(x,y,z), u, u) = f_3(x,y,z).$$

Therefore, the triplet (f_1, f_2, f_3) is a quasigroup solution of the equation (11). ☐

Theorem 5. *A triplet (f_1, f_2, f_3) of ternary invertible operations defined on set Q is a solution of the functional equation (12) if and only if there exist binary invertible operations $\circ$, $*$, $\diamond$ on Q such that*

$$\begin{aligned} f_1(y,x,u) &= (x \diamond y) * u, \\ f_2(x,y,z) &= x \overset{r}{\diamond} (y \circ z), \\ f_3(y,z,u) &= (y \circ z) * u. \end{aligned} \tag{27}$$

Proof. Let a triplet (f_1, f_2, f_3) of ternary invertible operations is a solution of the equation (12), i.e., for all $x, y, z, u \in Q$ the identity:

$$f_1(f_2(x,y,z), x, u) = f_3(y,z,u) \tag{28}$$

holds. In particular, when $x = a \in Q$ and

$$y \circ z := f_2(a,y,z), \qquad t * u := f_1(t,a,u),$$

we have $(y \circ z) * u = f_3(y,z,u)$. Hence, we obtain the third relationship from (27). Note that $(\circ)$ and $(*)$ are invertible operations since they are retracts of ternary invertible operations f_2 and f_1. Applying the latter equality to (28), we get

$$f_1(f_2(x,y,z), x, u) = (y \circ z) * u. \tag{29}$$

Replace y with ${}^{(24)}f_2(x,y,z)$:

$$f_1(f_2(x, {}^{(24)}f_2(x,y,z), z), x, u) = ({}^{(24)}f_2(x,y,z) \circ z) * u.$$

Apply (2):

$$f_1(y,x,u) = ({}^{(24)}f_2(x,y,z) \circ z) * u.$$

Replacing z with a and denoting $x \diamond y := {}^{(24)}f_2(x, y, a) \circ a$, we obtain the first relationship from (27). Then (29) can be written as

$$(x \diamond f_2(x, y, z)) * u = (y \circ z) * u.$$

Since the operation $(*)$ is invertible, then

$$x \diamond f_2(x, y, z) = y \circ z.$$

Since the operation $(\diamond)$ is invertible, we can use the definition of the right division for binary operations. As a result, we obtain the second equality from (27).

Conversely, let $\circ$, $*$, $\diamond$ be invertible binary operations on Q. Then the ternary operations defined by the relationship (27) are invertible since they are repetition-free superpositions of binary invertible operations.

$$\begin{aligned} f_1(f_2(x, y, z), x, u) &= (x \diamond f_2(x, y, z)) * u \\ &= (x \diamond (x \overset{r}{\diamond} (y \circ z))) * u = (y \circ z) * u = f_3(y, z, u). \end{aligned}$$

Hence, for all x, y, z, u (28) holds. Therefore, the triplet (f_1, f_2, f_3) is a solution of (12). □

3 PROOF OF THEOREM 1

Proof. Let $v = \omega$ be a generalized quadratic ternary quasigroup functional equation of the length three. Changing its sides if necessary, we obtain an equation which has one of the following forms:

$$i)\ F_i(\ldots, F_j(\ldots), \ldots) = F_k(\ldots), \qquad ii)\ F_i(\ldots, F_j(\ldots, F_k(\ldots), \ldots), \ldots) = t,$$
$$iii)\ F_i(\ldots, F_j(\ldots), \ldots, F_k(\ldots), \ldots) = t,$$

where t is an individual variable and $(\ldots)$ denotes some sequence of variables or an empty sequence.

When the equation has the form *ii*) we substitute both sides of the equation for t' in the term ${}^{\sigma}F_i(\ldots, t', \ldots)$. As a result, we obtain

$${}^{\sigma}F_i(\ldots, F_i(\ldots, F_j(\ldots, F_k(\ldots), \ldots), \ldots), \ldots) = {}^{\sigma}F_i(\ldots, t, \ldots),$$

where ${}^{\sigma}F_i$ is a suitable division of F_i, i.e., σ is (14), (24) or (34). Applying the respective primary identity (1)–(6), we get

$$F_j(\ldots, F_k(\ldots) \ldots) = {}^{\sigma}F_i(\ldots, t, \ldots).$$

Therefore, every functional equation of the form *ii*) is parastrophically primarily equivalent to an equation of the form *i*).

If the functional equation has the form *iii*), we substitute both sides of the equation for v in the term ${}^{\tau}F_i(\ldots, F_j(\ldots), \ldots, v, \ldots)$:

$$\begin{aligned} &{}^{\tau}F_i(\ldots, F_j(\ldots), \ldots, F_i(\ldots, F_j(\ldots), \ldots, F_k(\ldots), \ldots), \ldots) \\ &\qquad = F_i'(\ldots, F_j(\ldots), \ldots, t, \ldots), \end{aligned}$$

where ${}^{\tau}F_i$ is a suitable division of F_i. Applying one of the primary identities (1)–(6), we have

$$F_i'(\ldots, F_j(\ldots), \ldots, t, \ldots) = F_k(\ldots).$$

Thus, every functional equation is parastrophically primarily equivalent to a functional equation of the form i).

Let a functional equation have the form i). Applying a suitable transformation to a parastrophe, we obtain an equation of the form

$$F_i(\ldots, F_j(\ldots), \ldots) = F_k(\ldots).$$

Renaming its functional and individual variables in lexicographical order, we obtain

$$F_1(F_2(x, t_2, t_3), t_4, t_5) = F_3(t_6, t_7, t_8), \tag{30}$$

where $t_i \in \{x, y, z, u\}$. Denote a lexicographical order of individual variables by $\preccurlyeq$. If $t_2 \not\preccurlyeq t_3$, we replace the subterm $F_2(x, t_2, t_3)$ with the subterm ${}^{(23)}F_2(x, t_3, t_2)$, mutually rename the individual variables t_2 and t_3 and rename ${}^{(23)}F_2$ by F_2. As a result, we obtain the functional equation of the form (30) in which $t_2 \preccurlyeq t_3$.

Analogically, we suppose that $t_4 \preccurlyeq t_5$ and $t_6 \preccurlyeq t_7 \preccurlyeq t_8$. At last, we can put in order the second appearances of x, t_2, t_3. Namely, we rename them in a lexicographical order, then we transform them to the corresponding parastrophe of F_2. The same transformation holds for the pair t_4, t_5.

Thus, we have proved that every quadratic functional equation is parastrophically primarily equivalent to the equation (30) in which: 1) the first appearances of individual variables have a lexicographical order; 2) $t_2 \preccurlyeq t_3$, $t_4 \preccurlyeq t_5$ and $t_6 \preccurlyeq t_7 \preccurlyeq t_8$; 3) the second appearances of x, t_2, t_3 as well as the second appearances of t_4, t_5 are in the lexicographical order.

Hence, the proper subterm is

$$1)\ F_2(x, x, y) \quad \text{or} \quad 2)\ F_2(x, y, z).$$

The case $F_2(x, y, y)$ is impossible because the second appearances of x and y should be in a lexicographical order.

Let the proper subterm be $F_2(x, x, y)$. If $y \in \{t_4, t_5\}$, then t_4 is y and t_5 is z thus, we have the equation

$$F_1(F_2(x, x, y), y, z) = F_3(z, u, u).$$

Transform F_1 and F_2 to (13)-parastrophes of F_1 and F_2 in the equation. We obtain

$${}^{(13)}F_1(y, z, {}^{(13)}F_2(y, x, x)) = F_3(z, u, u).$$

Mutually renaming x and y and renaming the functional variables in a lexicographical order, we obtain the functional equation (9).

If $y \notin \{t_4, t_5\}$, then there are two possibilities for the pair (t_4, t_5): (z, z) and (z, u). Therefore, we have two equations:

$$F_1(F_2(x, x, y), z, z) = F_3(y, u, u), \tag{31}$$
$$F_1(F_2(x, x, y), z, u) = F_3(y, z, u). \tag{32}$$

The equation (31) is parastrophically primarily equivalent to (10) by means of transforming to (13)-parastrophe of F_2, by mutually renaming x and y and replacing ${}^{(13)}F_2$ with F_2.

Apply the hyperidentity (4) to (32):

$$^{(14)}F_1(F_3(y,z,u),z,u) = F_2(x,x,y),$$

then apply the hyperidentity (3):

$$F_2(x,x,{}^{(34)}F_3(y,z,u)) = {}^{(14)}F_1(y,z,u).$$

Transform F_2 to (13)-parastrophe of F_2 and rename the functional variables in a lexicographical order:

$$F_1(F_2(y,z,u),x,x) = F_3(y,z,u).$$

Renaming the individual variables according to the cycle $(yxuz)$, we obtain the functional equation (11).

Let the proper subterm be $F_2(x,y,z)$. Since the second appearances are ordered, then t_4 is x and t_5 is y or u. Consequently, we have two equations: equation (12) and

$$F_1(F_2(x,y,z),x,y) = F_3(z,u,u).$$

Apply (1) to the last functional equation:

$$F_3({}^{(14)}F_2(x,y,z),u,u) = F_1(z,x,y).$$

To obtain equation (11), transform F_1 to (312)-parastrophe of F_1 and rename the functional variables.

It remains to prove that the equations (9)–(12) are pairwise parastrophically primarily non-equivalent. According to Corollary 2, we can prove that for every pair of these equations and for every bijection σ_1, σ_2, σ_3, τ of the set $\{1,2,3\}$ there is a solution (f_1,f_2,f_3) of one equation such that $({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$ is not a solution of the other one. Note that all parastrophes of a totally symmetric quasigroup and, in particular of a Steiner quasigroup, coincide.

It is easy to verify that an arbitrary Steiner quasigroup is a solution of each of the functional equations (9), (10), (11). Suppose, a Steiner quasigroup $(Q;f)$ is a solution of the equation (12). Theorem 5 implies that f is a repetition-free superposition of two binary quasigroups. According to the definition, every Steiner quasigroup is a loop. Therefore, by Corollary 1 there is a group $(Q;+)$ of exponent two such that $f(x,y,z) = x+y+z$. There is no group of exponent two of the order 10 but Steiner quadruple systems exist (see [7]) thus, there exists a Steiner quasigroup of the order 10, but it can not be a solution of (12). Hence, according to Corollary 1, the functional equation (12) is not parastrophically primarily equivalent to any of the equations (9), (10), (11).

Let (f_1,f_2,f_3) be an arbitrary triplet of Steiner quasigroup operations defined on the same carrier Q. These operations can be isomorphic but all of them are pairwise different. It is easy to see that (f_1,f_2,f_3) is the solution of both functional equations: (9) and (10). Suppose $(f_{1\tau},f_{2\tau},f_{3\tau})$ is a solution of the functional equation (11) for some $\tau \in S_3$, i.e., the identity

$$f_{1\tau}(f_{2\tau}(x,y,z),u,u) = f_{3\tau}(x,y,z)$$

holds. Since $f_{1\tau}$ is a Steiner quasigroup operation, then $f_{2\tau} = f_{3\tau}$. There is a contradiction to the assumption. Thus, the triplet $(f_{1\tau}, f_{2\tau}, f_{3\tau})$ is not a solution of (11) for all $\tau \in S_3$. Therefore, the functional equation (11) is parastrophically primarily equivalent to neither (9) nor (10).

Hence, it remains to prove the parastrophically primary non-equivalence of the equations (9) and (10).

To avoid repetition, we will prove the following assertion.

Assertion. *Let $(Q;\cdot,e)$ be an arbitrary non-commutative group, ρ is its non-identical automorphism and*

$$f(x,y,z) := \rho x \cdot y \cdot z^{-1}. \tag{33}$$

If for a bijection $\sigma \in S_4$ there exists a bijection ν such that for all x, y, z

$${}^{\sigma}\!f(x,y,z) = \nu x, \tag{34}$$

then $\nu = \rho$ or $\nu = \rho^{-1}$.

To prove Assertion, consider the following notations:

$$t_{1\sigma} := x, \quad t_{2\sigma} := y, \quad t_{3\sigma} := y, \quad t_{4\sigma} := \nu x.$$

Then (34) can be written as ${}^{\sigma}\!f(t_{1\sigma}, t_{2\sigma}, t_{3\sigma}) = t_{4\sigma}$. According to the definition of σ-parastrophe, the equality is equivalent to $f(t_1, t_2, t_3) = t_4$. Using (33), we obtain $\rho t_1 \cdot t_2 \cdot t_3^{-1} = t_4$, i.e.

$$\rho t_1 \cdot t_2 = t_4 \cdot t_3. \tag{35}$$

We will analyze the relationship taking into account that two of the terms t_1, t_2, t_3, t_4 coincide with y.

If $t_1 = y$, then (35) with $y = e$ implies one of the following equalities: $\nu x \cdot x = e$ or $x = \nu x$. Consequently, $\nu e = e$. That is why, (35) with $x = e$ implies $\rho y \cdot y = e$ or $\rho y = y$. Since $(\cdot)$ is not commutative and ρ is a non-identical automorphism of $(\cdot)$, then neither $\rho y = y^{-1}$ nor $\rho y = y$ is true.

If $t_1 = x$, $t_2 = \nu x$, then (35) with $x = e$ implies $\nu e = y^2$. Therefrom when $y = e$ we have $\nu e = e$, therefore $y^2 = e$. But the group of exponent two is commutative. As a result we have a contradiction to the assumption.

If $t_1 = x$ and $t_2 = y$, then (35) with $y = e$ implies $\rho x = \nu x$ that is $\nu = \rho$.

Finally, let $t_1 = \nu x$, then (35) with $y = e$ implies one of the equalities $\rho\nu x \cdot x = e$ or $\rho\nu x = x$. The first equality follows from (35) when $t_2 = x$. Therefore, $y^2 = e$ and consequently, the group is commutative. As a result we have a contradiction to the assumption. The second equality implies $\nu = \rho^{-1}$.

Thus, Assertion has been proved.

We provide a proof of parastrophically primary non-equivalence of (9) and (10) by contradiction. Suppose (9) and (10) are parastrophically primarily equivalent. Denote the corresponding defining bijection sequence by $(\tau, \sigma_1, \sigma_2, \sigma_3)$.

Let $(Q;\cdot,e)$ be an arbitrary non-commutative group and γ, δ, $\gamma\delta$ be different non-identical automorphisms of $(Q;\cdot,e)$. Then, according to Theorem 3, the triplet (f_1, f_2, f_3) of operations defined by

$$f_1(x,y,z) := \gamma x \cdot y \cdot z^{-1}, \quad f_2(x,y,z) := \delta x \cdot y \cdot z^{-1}, \quad f_3(x,y,z) := \gamma\delta x \cdot y \cdot z^{-1} \tag{36}$$

is a solution of the equation (10). Lemma 1 implies that the triplet

$$({}^{\sigma_1}f_{1\tau}, {}^{\sigma_2}f_{2\tau}, {}^{\sigma_3}f_{3\tau})$$

is a solution of the equation (9). By Theorem 2 there exist left-universally neutral operations h_1, h_2, h_3 and bijections α, β such that

$$\begin{aligned} {}^{\sigma_1}f_{1\tau}(x,y,z) &= h_1(\alpha x, y, \beta^{-1}z), \\ {}^{\sigma_2}f_{2\tau}(x,y,z) &= h_2(\beta x, y, z), \\ {}^{\sigma_3}f_{3\tau}(x,y,z) &= h_3(\alpha x, y, z). \end{aligned} \tag{37}$$

If $y = z$, the second and the third equations are

$${}^{\sigma_2}f_{2\tau}(x,y,y) = \beta x, \qquad {}^{\sigma_3}f_{3\tau}(x,y,y) = \alpha x.$$

Applying Assertion to these equalities, we have $\alpha, \beta \in \{\gamma, \gamma^{-1}, \delta, \delta^{-1}, \gamma\delta, \delta^{-1}\gamma^{-1}\}$. Replace z with βz in the first equality of (37): ${}^{\sigma_1}f_{1\tau}(x,y,\beta z) = h_1(\alpha x, y, z)$. If $y = z$, then

$${}^{\sigma_1}f_{1\tau}(x,y,\beta y) = \alpha x. \tag{38}$$

Introduce the notations: $t_{1\sigma_1} := x$, $t_{2\sigma_1} := y$, $t_{3\sigma_1} := \beta y$, $t_{4\sigma_1} := \alpha x$. Thus, (38) can be written as ${}^{\sigma_1}f_{1\tau}(t_{1\sigma_1}, t_{2\sigma_1}, t_{3\sigma_1}) = t_{4\sigma_1}$. Using the definition of a parastrophe, we have $f_{1\tau}(t_1, t_2, t_3) = t_4$. But $f_{1\tau}$ is one of the operations f_1, f_2, f_3, that is why we can apply the relationship (36): $\theta t_1 \cdot t_2 \cdot t_3^{-1} = t_4$, i.e.,

$$\theta t_1 \cdot t_2 = t_4 \cdot t_3,$$

where $\theta \in \{\gamma, \delta, \gamma\delta\}$.

If x has an appearance in θt_1, then we put $x = 0$. As a result, we obtain one of the equalities $y = \beta y$ or $0 = y \cdot \beta y$. The first equality is impossible, since the automorphisms γ, δ, $\gamma\delta$ are not identical. The second identity is impossible because the group is not commutative. If x has no appearance in θt_1, then we put $y = 0$ and obtain the same contradictions.

Thus, our assumption is not true, therefore, the equations (9) and (10) are not parastrophically primarily equivalent. Theorem 1 has been proved. $\square$

4 CONCLUSION

There exist exactly four classes of generalized quadratic functional equations of the length three on invertible functions (i.e. quasigroup operations) concerning the parastrophically primary equivalence, (9)–(12) are their representatives whose solution sets are found in Theorems 2–5.

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Сохацький Ф.М., Тарасевич А.В. *Класифікація узагальнених тернарних квадратичних функційних рівнянь довжини три* // Карпатські матем. публ. — 2019. — Т.11, №1. — С. 179–192.

Функційне рівняння називається: *узагальненим*, якщо всі функційні змінні попарно різні; *тернарним*, якщо всі його функційні змінні є тернарними; *квадратичним*, якщо кожна предметна змінна має точно дві появи; *квазігруповим*, якщо його розв'язки вивчають лише на оборотних функціях. *Довжиною* функційного рівняння є кількість всіх його функційних змінних. Здійснено повну класифікацію з точністю до парастрофно-первинної рівносильності узагальнених квадратичних квазігрупових функційних рівнянь довжини три. Знайдено множини розв'язків повного набору представників.

Ключові слова і фрази: тернарна квазігрупа, квадратичне рівняння, довжина функційного рівняння, парастрофно-первинна рівносильність.