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ANALYTIC HYPERCYCLIC OPERATORS

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We propose a simple method how to construct an analytic hypercyclic operators on Fréchet spaces and Banach spaces. Some examples are presented.

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Предложен простой способ построения аналитических гиперциклических операторов на пространствах Фреше и банаховых пространствах. Приведены примеры

Let X be a Fréchet linear space. An operator $T\colon X \to X$ is called *hypercyclic* if there is a vector $x \in X$ whose *orbit* under T

$$\operatorname{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$$

is dense in X. Every such vector x is called a *hypercyclic* vector for T. Many authors studied hypercyclic linear operators (see, e.g., the survey of Grosse-Erdmann [3]). Nonlinear hypercyclic operators were not studied in details. In [1] was shown that there is no n-homogeneous hypercyclic polynomial operator on any Banach space if $n > 1$. In [4] Peris constructed some examples of nonhomogeneous hypercyclic polynomial operators on Banach spaces using the Julia set theory.

In this paper we provide a simple method how to construct polynomial and analytic hypercyclic operators.

Let F be an analytic automorphism of X onto X and T be an hypercyclic operator on X. Then $T_F := FTF^{-1}$ (and $T_{F^{-1}} := F^{-1}TF$ as well) must be hypercyclic ([3]) and, in the general case, they are nonlinear. The following examples show that T_F are nonlinear for some well known hypercyclic operators T and simple analytic automorphisms F.

Example 1. Let $A(D)$ be the disk-algebra of all analytic functions on the unit disk D of $\mathbb{C}$ which are continuous on the closure $\overline{D}$. Denote $X_1 = \{\sum_{k=0}^{\infty} a_{2k+1}t^{2k+1} \in A(D)\}$ and $X_2 = \{\sum_{k=0}^{\infty} a_{2k}t^{2k} \in A(D)\}$. Clearly $A(D) = X_1 \oplus X_2$.

For every $f = f_1 + f_2$, $f_1 \in X_1$, $f_2 \in X_2$ we put

$$\begin{cases} F(f_1) & := & f_1, \\ F(f_2) & := & f_2 + f_1^2. \end{cases} \qquad \text{Then we have} \qquad \begin{cases} F^{-1}(f_1) & = & f_1, \\ F^{-1}(f_2) & = & f_2 - f_1^2. \end{cases}$$

Thus F is a polynomial automorphism of X. Let $T(f(t)) = f(\frac{t+1}{2})$. It is known that T is hypercyclic on $A(D)$ ([2, p. 4]).

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Let us show that $T_F = FTF^{-1}$ is nonlinear. It is enough to check that $T_F(\lambda f) \neq \lambda T_F(f)$ for some $\lambda \in \mathbb{C}$ and $f \in A(D)$. Let $f(t) = t + t^2 \in A(D)$. Then

$$\begin{aligned} T_F(\lambda f) &= F(T(F^{-1}(\lambda t + \lambda t^2))) = F(T(\lambda t + \lambda t^2 - \lambda^2 t^2)) \\ &= F(T(\lambda t + (\lambda - \lambda^2)t^2)) = F\Big(\lambda\Big(\frac{t+1}{2}\Big) + (\lambda - \lambda^2)\Big(\frac{t+1}{2}\Big)^2\Big) \\ &= \frac{(2\lambda - \lambda^2)t}{2} + \frac{(\lambda + 3\lambda^2 - 4\lambda^3 + \lambda^4)t^2}{4} + \frac{(3\lambda - \lambda^2)}{4} \end{aligned}$$

for any $\lambda \neq 0$, $\lambda \neq 1$. Thus $T_F(\lambda f) \neq \lambda T_F(f)$.

In a similar way, in the following example we consider the space of entire analytic functions $H(\mathbb{C})$ and $T(f) = f(x + a)$ to show that $T_{F^{-1}}$ is nonlinear, where F is defined as above.

Example 2. Let $f(t) = t + t^2 \in H(\mathbb{C})$ then $F(f) = t + 2t^2$, $F(\lambda f) = \lambda(t + t^2) + \lambda^2 t^2$. Thus

$$T(F(\lambda f)) = \lambda(t + a) + 2\lambda(1 + \lambda)at + (\lambda + \lambda^2)(t^2 + a^2)$$

for any $\lambda \neq 0$. Since $F^{-1}(f) = t - t^2$, we have

$$\begin{aligned} F^{-1}TF(\lambda f) &= \lambda(t + t^2) - 4\lambda^2 a^2(t + t^2) + \lambda(a + a^2 + 2t) + 4\lambda^2 at(t + a) \\ &\quad -4\lambda^3 at(t + a) - 4\lambda^3 a^2 t^2(2 + \lambda) \neq \lambda T_{F^{-1}}(f). \end{aligned}$$

Thus, the operator $T_{F^{-1}} = F^{-1}TF$ is nonlinear.

Example 3. Next we consider the Hilbert space ℓ_2. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in ℓ_2 and $x = \sum_{k=1}^{\infty} x_k e_k \in \ell_2$. We define an analytic automorphism $F\colon \ell_2 \to \ell_2$ by the formula

$$\begin{cases} F(x_{2k-1}e_{2k-1}) &= x_{2k-1}e_{2k-1} \\ F(x_{2k}e_{2k}) &= x_{2k}e^{-x_{2k-1}}e_{2k}, \qquad k = 1, 2, \ldots \end{cases}$$

Let T_μ be a weighted shift,

$$T_\mu(x) = (\mu x_2, \mu x_3, \ldots).$$

It is known that T_μ is a hypercyclic operator if $|\mu| > 1$ (see [5]). Then the operator $T_F = FT_\mu F^{-1}$ is hypercyclic. We will show that T_F is nonlinear.

Let $a \in \ell_2$, $a = (a_1, a_2, \ldots a_n, \ldots)$, $a = \sum_{k=1}^{\infty} a_k e_k$ and $\lambda \in \mathbb{C}$. We will show that $T_F(\lambda a) \neq \lambda T_F(a)$.

$$F^{-1}T_\mu F(\lambda a) = (\mu\lambda a_2 e^{-\lambda a_1}, \mu\lambda a_3 e^{\mu\lambda a_2 e^{-\lambda a_1}}, \mu\lambda a_4 e^{-\lambda a_3}, \mu\lambda a_5 e^{\mu\lambda a_4 e^{-\lambda a_3}}, \ldots).$$

Thus, $T_F(\lambda a) \neq \lambda T_F(a)$ and moreover, the map $\lambda \rightsquigarrow T_\mu(\lambda a)$ is not polynomial. Therefore T_F is an analytic (not polynomial) hypercyclic map.

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