On the third initial-boundary value problem for some class of pseudo-differential equations related to a symmetric α-stable process

M. M. Osypchuk and M. I. Portenko

Abstract. A fundamental solution of the so-called third initial-boundary value problem for one class of pseudo-differential equations is constructed. Those equations are related to a symmetric α-stable stochastic process and our constructions are inspired by some probabilistic ideas. However, we expound our results in a way completely independent of any probabilistic notion. Only the last section of the paper is based on the notion of a stochastic process and also a pseudo-process and it gives some interpretation of our results in terms of stochastic analysis.

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1. Introduction

One of the most important notions in the theory of partial differential equations of parabolic and elliptic types is the notion of a single-layer potential. The theorem on the jump of the (co-)normal derivative of such a potential is a significant result of classical analysis. In particular, just this theorem allows one to construct a solution to the second initial-boundary value problem for the corresponding equation (see [4], Chapter V and also bibliographical remarks to it).

In the paper [8] some analog to the theory of single-layer potentials was constructed in the situation when, instead of differential, some class of pseudo-differential equations was considered. The main theorem of [8] is analogical to the classical one mentioned above and it is applied there for solving the *second* initial-boundary value problem for the equations under consideration.

The aim of this paper is to construct a solution to the *third*[1] initial-boundary value problem for pseudo-differential equations of the same type as in [8]. Those equations are related to stochastic processes known as symmetric (more precisely — rotationally invariant) α-stable processes. It is natural that our constructions were inspired with some probabilistic ideas. Nevertheless, we expound our results in a form completely independent of any probabilistic notions. Only the last section of this article can be considered as a probabilistic interpretation of the matter of prior sections and in order to understand it, one should be familiar with some notions of stochastic analysis such as Markov processes, the Feynman-Kac formula, W-functionals, local times etc.

We consider here a model problem: the surface where the boundary value conditions are to be given is supposed to be a hyperplane in a Euclidean space. We formulate the main problem (Section 2) in such a way that there are two versions of it called symmetric and asymmetric ones. Sections 4 and 5 are devoted to those versions, respectively. In Section 3 some known facts are exposed that are necessary for understanding the subsequent considerations. Finally, Section 6, as mentioned above, contains the probabilistic aspects of the results obtained in Sections 4 and 5.

2. The main problem

2.1. The operator A

For given parameters α and c, $1 < \alpha < 2$, $c > 0$, let $\mathbf{A}$ denote a pseudo-differential operator whose symbol is given by $(-c|\xi|^\alpha)_{\xi\in\mathbb{R}^d}$ (a d-dimensional Euclidean space is denoted by $\mathbb{R}^d$). This operator acts on a function $(\varphi(x))_{x\in\mathbb{R}^d}$ being smooth enough and bounded along with its derivatives according to the formula

$$\mathbf{A}\varphi(x) = \frac{c}{\varkappa}\int_{\mathbb{R}^d}[\varphi(x+y) - \varphi(x) - (\nabla\varphi(x), y)]|y|^{-d-\alpha}\,dy,\ x\in\mathbb{R}^d, \qquad (1)$$

where $\varkappa$ is the constant given by

$$\varkappa = \frac{-2\pi^{\frac{d-1}{2}}\Gamma(2-\alpha)\Gamma((\alpha+1)/2)\cos(\pi\alpha/2)}{\alpha(\alpha-1)\Gamma((d+\alpha)/2)}.$$

In the limiting case of $\alpha = 2$ the operator $\mathbf{A}$ coincides with $c\cdot\Delta$, where Δ is the Laplace operator.

[1]The terms: "the first", "the second" and "the third" (initial-)boundary value problems are as widely used in the theory of *differential* equations as their synonyms: "the Dirichlet", "the Neumann" and "the Mixed" problems, respectively. One should be careful, when making use of these terms in the theory of *pseudo-differential* equations. In our opinion, there is a sufficient reason to consider the main problem of this article (see Section 2) as some analogy to the third (or the Mixed) problem.

2.2. The operator B

Let $\mathbf{B}$ denote a pseudo-differential operator whose symbol is given by the $\mathbb{R}^d$-valued function $(2ic|\xi|^{\alpha-2}\xi)_{\xi\in\mathbb{R}^d}$. The role of this operator in our theory is similar to that of the gradient in classical theory. In particular, for a given unit vector $\nu \in \mathbb{R}^d$, the function $(2ic|\xi|^{\alpha-2}(\xi,\nu))_{\xi\in\mathbb{R}^d}$ is the symbol of the pseudo-differential operator (denoted by $\mathbf{B}_\nu$) that is analogical to the partial derivative in the direction ν.

If a function $(\varphi(x))_{x\in\mathbb{R}^d}$ is bounded and satisfies the Lipschiz condition, then

$$\mathbf{B}\varphi(x) = \frac{2c}{\alpha\varkappa}\int_{\mathbb{R}^d}[\varphi(x+y)-\varphi(x)]|y|^{-d-\alpha}y\,dy, \quad x\in\mathbb{R}^d, \tag{2}$$

where $\varkappa$ is the constant defined above. We put $g_0^{(\nu)}(t,x,y) = \mathbf{B}_\nu g_0(t,\cdot,y)(x)$, $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$. A very simple calculation (see [7]) leads us to the equality

$$g_0^{(\nu)}(t,x,y) = \frac{2(y-x,\nu)}{\alpha t}g_0(t,x,y), \quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d. \tag{3}$$

Notice, that $\mathbf{A} = \frac{1}{2}\mathbf{div}\,\mathbf{B}$, so the role of the operator $\mathbf{A}$ is similar to that of the Laplacian in the classical theory of potentials.

2.3. Formulating the main problem

Denote by $\mathbb{C}_b(\mathbb{R}^d)$ the Banach space of all real-valued continuous bounded functions $(\varphi(x))_{x\in\mathbb{R}^d}$ with the norm $\|\varphi\| = \sup\limits_{x\in\mathbb{R}^d}|\varphi(x)|$ and by $\mathbb{C}_0(\mathbb{R}^d)$ the subspace of $\mathbb{C}_b(\mathbb{R}^d)$ being the collection of all $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ such that the set $\{x\in\mathbb{R}^d : |\varphi(x)|\geq\varepsilon\}$ is a compact in $\mathbb{R}^d$ for each $\varepsilon>0$.

Let S be a hyperplane in $\mathbb{R}^d$ that is orthogonal to a fixed unit vector $\nu\in\mathbb{R}^d$ and let two continuous bounded functions $(q(x))_{x\in S}$ and $(r(x))_{x\in S}$ with real and non-negative values, respectively, be given. As was mentioned above, our aim is to construct the solution of the following initial-boundary value problem.

For a given $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, a continuous function U of the arguments $t>0$ and $x\in\mathbb{R}^d$ is being looked for such that it satisfies

(i) the equation
$$\frac{\partial U}{\partial t} = \mathbf{A}U$$
in the region $t>0$ and $x\notin S$;

(ii) the initial condition
$$U(0+,x) = \varphi(x)$$
for all $x\in\mathbb{R}^d$;

(iii) the boundary value condition
$$\frac{1+q(x)}{2}\mathbf{B}_\nu U(t,\cdot)(x+) - \frac{1-q(x)}{2}\mathbf{B}_\nu U(t,\cdot)(x-) = r(x)U(t,x)$$
for all $t>0$ and $x\in S$.

Notice, that the boundary value condition is formulated in a normalized form: the sum of the coefficients on the left-hand side of (iii) is identically equal to 1.

In the case of $q(x) \equiv 0$, we call the corresponding problem *a symmetric* one and in accordance to that, the general problem is called *an asymmetric* one. We construct fundamental solutions to them in Sections 4 and 5.

Another name of the general problem is the *third* initial-boundary value problem. If $r(x) \equiv 0$, then it is nothing else but the *second* initial-boundary value problem. Its solution was constructed in [8] for a general surface S. In the case of S being a hyperplane, the fundamental solution to the corresponding problem was explicitly constructed in [7]. We briefly expose the results from [7] in Subsection 3.5.

Finally, the problem (i) — (iii) with $q(x) \equiv 0$ and $r(x) \equiv 0$ coincides with the Cauchy problem considered in Section 3.1.

3. Preliminaries

3.1. The case of $q(x) \equiv 0$ and $r(x) \equiv 0$

Consider the following Cauchy problem: for a given $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, a continuous function $(u_0(t,x))_{t>0, x\in\mathbb{R}^d}$ is being looked for such that it satisfies the equation

$$\frac{\partial u_0}{\partial t} = \mathbf{A} u_0 \tag{4}$$

in the region $(t,x) \in (0,+\infty) \times \mathbb{R}^d$ and the initial condition

$$u_0(0+,x) = \varphi(x) \tag{5}$$

for all $x \in \mathbb{R}^d$.

It can be easily verified that the fundamental solution to this problem is given by ($t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$)

$$g_0(t,x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\{i(x-y,\xi) - ct|\xi|^\alpha\}\, d\xi. \tag{6}$$

This means that the function g_0 as a function of the arguments $(t,x) \in (0,+\infty) \times \mathbb{R}^d$ for fixed $y \in \mathbb{R}^d$ satisfies equation (4) and for any $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ the function

$$u_0(t,x,\varphi) = \int_{\mathbb{R}^d} g_0(t,x,y)\varphi(y)\, dy, \quad t > 0,\ x \in \mathbb{R}^d, \tag{7}$$

solves the problem (4), (5).

The maximum principle for equation (4) (see [3], Lemma 4.7) implies the uniqueness of the solution in the class $\mathbb{C}_0(\mathbb{R}^d)$. One can easily observe that for any $t > 0$, the function (7) belongs to $\mathbb{C}_0(\mathbb{R}^d)$ if only so does φ.

As a consequence of these facts, we have the following properties of the function g_0.

3.1.A. The values of the function g_0 are positive.

3.1.B. For all $s>0$, $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$ the equality

$$g_0(s+t,x,y)=\int_{\mathbb{R}^d} g_0(s,x,z)g_0(t,z,y)\,dz$$

holds true.

3.1.C. The equality $\int_{\mathbb{R}^d} g_0(t,x,y)\,dy=1$ is valid for all $t>0$ and $x\in\mathbb{R}^d$.

Besides, the function g_0 satisfies the inequality

$$g_0(t,x,y)\le N\frac{t}{(t^{1/\alpha}+|y-x|)^{d+\alpha}},\quad t>0,\ x\in\mathbb{R}^d,\ y\in\mathbb{R}^d, \tag{8}$$

where N is a positive constant (see [3], Chapter IV).

Let S be a hyperplane in $\mathbb{R}^d$ that is orthogonal to a fixed unit vector $\nu\in\mathbb{R}^d$. We will have an opportunity to make use of the equality (the integral on the left-hand side is a surface one)

$$\int_S e^{i(\xi,y)}g_0(t,x,y)\,d\sigma_y=\frac{1}{\pi}\int_0^\infty e^{-ct(|\xi|^2+\rho^2)^{\alpha/2}}\cos(\rho(x,\nu))\,d\rho \tag{9}$$

valid for $t>0$, $x\in\mathbb{R}^d$ and $\xi\in S$ (see [8]). Combining (8) and (9), we have the estimate

$$\int_S g_0(t,x,y)\,d\sigma_y\le N\frac{t}{(t^{1/\alpha}+|(x,\nu)|)^{1+\alpha}} \tag{10}$$

held true for $t>0$ and $x\in\mathbb{R}^d$ with some constant $N>0$.

3.2. Single-layer potentials

Let S be the same as above and let a continuous function $(v(t,x))_{t>0,x\in S}$ with real values be given such that the inequality $|v(t,x)|\le Ct^{-\beta}$ holds true for all $t>0$ and $x\in S$ with some constants $C>0$ and $\beta<1$. Define a function V_0 of the arguments $t>0$ and $x\in\mathbb{R}^d$ by setting

$$V_0(t,x)=\int_0^t d\tau\int_S g_0(t-\tau,x,y)v(\tau,y)\,d\sigma_y. \tag{11}$$

Inequality (10) implies the following estimate for V_0

$$|V_0(t,x)|\le CN\frac{\Gamma(1-\beta)\Gamma\left(1-\frac{1}{\alpha}\right)}{\Gamma\left(2-\frac{1}{\alpha}-\beta\right)}t^{1-\beta-\frac{1}{\alpha}},\quad t>0,\ x\in\mathbb{R}^d.$$

It shows that the function V_0 is not only well-defined, but also is continuous with respect to the arguments $t>0$ and $x\in\mathbb{R}^d$. This function is called the single-layer potential (with “the mass v distributed on $(0,+\infty)\times S$”).

3.3. The function $\mathbf{B}_\nu V_0$

Let a hyperplane S and a continuous function $(v(t,x))_{t>0,x\in S}$ be the same as in Section 3.2. We now prove the relation

$$\mathbf{B}_\nu V_0(t,\cdot)(x) = \int_0^t d\tau \int_S g_0^{(\nu)}(t-\tau,x,y)v(\tau,y)\,d\sigma_y \tag{12}$$

valid for $t>0$ and $x\notin S$.

First, we observe that the function on the right-hand side of (12) (denote it by $V_0^{(\nu)}(t,x)$ for $t>0$ and $x\in\mathbb{R}^d$) is well-defined. It is clear that $V_0^{(\nu)}(t,x)=0$ for $t>0$ and $x\in S$, since $g_0^{(\nu)}(t-\tau,x,y)=0$ for $x\in S$ and $y\in S$ in accordance to (3). If $x\notin S$, then (3) and (8) imply the inequality

$$|g_0^{(\nu)}(t-\tau,x,y)| \le \frac{2}{\alpha}N\frac{|(x,\nu)|}{[(t-\tau)^{1/\alpha}+((x,\nu)^2+|y-\tilde{x}|^2)^{1/2}]^{d+\alpha}},\quad y\in S,$$

where $\tilde{x}=x-\nu(x,\nu)$ ($\tilde{x}$ is the orthogonal projection of x on S). Therefore

$$\begin{aligned}|V_0^{(\nu)}(t,x)| &\le \\ &\le \frac{2}{\alpha}CN\int_0^t \tau^{-\beta}\,d\tau\int_S \frac{d\sigma_y}{[(t-\tau)^{1/\alpha}+((x,\nu)^2+|y-\tilde{x}|^2)^{1/2}]^{d+\alpha-1}} \le \\ &\le \frac{2}{\alpha(1-\beta)}CN|(x,\nu)|^{-\alpha}t^{1-\beta}\int_{\mathbb{R}^{d-1}}\frac{dz}{(1+|z|^2)^{(d+\alpha-1)/2}}\end{aligned}$$

and the function $V_0^{(\nu)}$ is indeed well-defined.

It remains to show that the function $\left(\frac{V_0(t,x+y)-V_0(t,x)}{|y|^{d+\alpha}}(y,\nu)\right)_{y\in\mathbb{R}^d}$ is integrable over $\mathbb{R}^d$ and that $\mathbf{B}_\nu V_0(t,\cdot)(x)=V_0^{(\nu)}(t,x)$ for $t>0$ and $x\notin S$. Taking into account the estimate for V_0, we arrive at the conclusion that it is sufficient to verify that the integral (we use the notation $B_\delta=\{y\in\mathbb{R}^d: |y|\le\delta\}$)

$$\int_{B_\delta}|V_0(t,x+y)-V_0(t,x)||y|^{-d-\alpha+1}dy,\quad x\notin S, \tag{13}$$

is finite for a positive δ being small enough.

We choose $0<\delta<\frac{1}{2}|(x,\nu)|$ ($x\notin S$ is a fixed point). The well-known theorem of analysis allows one to write down the equality

$$g_0(t-\tau,x+y,z)-g_0(t-\tau,x,z)=(\nabla g_0(t-\tau,\cdot,z)(x^*),y),$$

where $x^*=x+\theta y$ for some $\theta\in(0,1)$. According to Kochubei's inequality (see [3], Lemma 4.1)

$$|\nabla g_0(t-\tau,\cdot,z)(x^*)|\le N\frac{t-\tau}{[(t-\tau)^{1/\alpha}+|z-x^*|]^{d+\alpha+1}}.$$

Since $|z-x^*|>\frac{1}{2}|z-x|=\frac{1}{2}\left(|z-\tilde{x}|^2+(x,\nu)^2\right)^{1/2}$ for $z\in S$ and

$$\begin{aligned}\int_S|\nabla g_0(t-\tau,\cdot,z)(x^*)|\,d\sigma_z \le& N(t-\tau)\int_{\mathbb{R}^{d-1}}\frac{2^{d+\alpha+1}\,dz}{(|z|^2+(x,\nu)^2)^{(d+\alpha+1)/2}} = \\ =& const\cdot(t-\tau)|(x,\nu)|^{-\alpha-2},\end{aligned}$$

the following estimate

$$\int_{B_\delta} |V_0(t,x+y)-V_0(t,x)||y|^{-d-\alpha}|(y,\nu)|\,dy \le$$
$$\le const\cdot|(x,\nu)|^{-\alpha-2}\int_0^t \tau^{-\beta}(t-\tau)\,d\tau\int_{B_\delta}|y|^{-d-\alpha+2}dy$$

is held. This estimate shows that the integral (13) is finite. On the other hand, it shows that the order of integrating in $\mathbf{B}_\nu V_0(t,\cdot)(x)$ can be changed and this completes the proof of (12).

Similar arguments allow one to assert that the function $(V_0(t,x))_{t>0,x\in\mathbb{R}^d}$ satisfies equation (4) in the region $t>0$ and $x\notin S$ (see [8] for details).

3.4. The jump of the function $\mathbf{B}_\nu V_0$

The function $V_0^{(\nu)}$ defined by (12) is continuous with respect to $t>0$ and $x\notin S$ and it has jumps at the points of S described by the following particular case of the general theorem (see [8]):

the relations

$$\lim_{z\to x\pm}\int_0^t d\tau\int_S g_0^{(\nu)}(t-\tau,z,y)v(\tau,y)\,d\sigma_y=\mp v(t,x) \tag{14}$$

hold true for all $t>0$ and $x\in S$, where $z\to x+$ (respectively, $z\to x-$) means that z approaches x along any curve lying in a finite closed cone $\mathcal{K}$ in $\mathbb{R}^d$ with vertex at x such that $\mathcal{K}\subset\{z\in\mathbb{R}^d:(z,\nu)>0\}\cup\{x\}$ (respectively, $\mathcal{K}\subset\{z\in\mathbb{R}^d:(z,\nu)<0\}\cup\{x\}$). The so-called *direct* value of $\mathbf{B}_\nu V_0(t,\cdot)(x)$ for $t>0$ and $x\in S$ vanishes in (14) because of $g_0^{(\nu)}(t,x,y)=0$ for $t>0$, $x\in S$ and $y\in S$ (see (3)). The dual formula for (14) is as follows

$$\lim_{z\to y\pm}\int_0^t d\tau\int_S v(t-\tau,x)g_0^{(\nu)}(\tau,x,z)\,d\sigma_x=\pm v(t,y) \tag{15}$$

for $t>0$ and $y\in S$.

The proof of (14) is based on the following reason. Consider the integral

$$I(t,z)=\int_0^t d\tau\int_S g_0^{(\nu)}(\tau,z,y)\,d\sigma_y,\quad t>0,\ z\notin S.$$

According to (9), we have

$$I(t,z)=-\frac{2(z,\nu)}{\pi\alpha}\int_0^t\frac{d\tau}{\tau}\int_0^\infty e^{-c\tau\rho^\alpha}\cos(\rho(z,\nu))\,d\rho.$$

Integrating by parts, we obtain for $t>0$ and $z\notin S$

$$I(t,z)=-\frac{2c}{\pi}\int_0^t d\tau\int_0^\infty \rho^\alpha e^{-c\tau\rho^\alpha}\frac{\sin(\rho(z,\nu))}{\rho}\,d\rho=$$
$$=-\lim_{\delta\to0+}\frac{2}{\pi}\int_0^\infty e^{-c\delta\rho^\alpha}\frac{\sin(\rho(z,\nu))}{\rho}\,d\rho+\frac{2}{\pi}\int_0^\infty e^{-ct\rho^\alpha}\frac{\sin(\rho(z,\nu))}{\rho}\,d\rho.$$

Hence, the following formula

$$I(t,z) = -\operatorname{sign}(z,\nu) + \frac{2}{\pi}\int_0^\infty e^{-ct\rho^\alpha}\,\frac{\sin(\rho(z,\nu))}{\rho}\,d\rho$$

holds true for $t>0$ and $z\notin S$. Equality (14) is a simple consequence of this formula (see [7], [8] for details).

3.5. The fundamental solution in the case of $r(x)\equiv 0$

We put for $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$

$$G_0(t,x,y) = g_0(t,x,y) + \int_0^t d\tau\int_S g_0(t-\tau,x,z)g_0^{(\nu)}(\tau,z,y)q(z)\,d\sigma_z \tag{16}$$

and first of all show that the integrals in (16) are well-defined. It is evident for $y\in S$ because of the equality $g_0^{(\nu)}(\tau,z,y)=0$ for $z\in S$ and $y\in S$ (see (3)); so, we have $G_0(t,x,y)=g_0(t,x,y)$ for $t>0$, $x\in\mathbb{R}^d$ and $y\in S$. Further, as follows from (3) and (8), the inequality

$$|g_0^{(\nu)}(\tau,z,y)| \le \frac{2}{\alpha}N\frac{|(y,\nu)|}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha}}$$

is held for $\tau>0$, $z\in S$ and $y\in\mathbb{R}^d$. Since $|y-z| = (|\tilde{y}-z|^2+(y,\nu)^2)^{1/2}\ge |(y,\nu)|$ for $z\in S$ and $y\in\mathbb{R}^d$ (we remind that $\tilde{y}=y-\nu(y,\nu)$ for $y\in\mathbb{R}^d$), the estimate $|g_0^{(\nu)}(\tau,z,y)|\le\frac{2}{\alpha}N|(y,\nu)|^{-d-\alpha+1}$ is valid for $\tau>0$, $z\in S$ and $y\in\mathbb{R}^d\setminus S$. It implies the inequalities

$$\begin{aligned}\left|\int_S g_0(t-\tau,x,z)g_0^{(\nu)}(\tau,z,y)q(z)\,d\sigma_z\right| &\le\\ \le \frac{2}{\alpha}\|q\|N|(y,\nu)|^{-d-\alpha+1}\int_S g_0(t-\tau,x,z)\,d\sigma_z &\le\\ \le \frac{2}{\alpha}\|q\|N^2|(y,\nu)|^{-d-\alpha+1}(t-\tau)^{-1/\alpha}&\end{aligned}$$

that are fulfilled for $0<\tau<t$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d\setminus S$, where $\|q\|=\sup\limits_{x\in S}|q(x)|$ and the evident consequence of (10)

$$\int_S g_0(t-\tau,x,z)\,d\sigma_z \le N(t-\tau)^{-1/\alpha}$$

has been used.

We have just proved that the function G_0 is defined correctly. Moreover, it is a continuous function of the arguments $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d\setminus S$. As the relations (15) show, G_0 has jumps at the points $y\in S$ and they are described as follows

$$G_0(t,x,y\pm) = (1\pm q(y))g_0(t,x,y),\quad t>0,\ x\in\mathbb{R}^d,\ y\in S. \tag{17}$$

Let us now show that, as a function of the third argument, G_0 is absolutely integrable over $\mathbb{R}^d$. Some very simple calculations allow us to write down the formula

$$\int_{\mathbb{R}^d} |(y,\nu)| g_0(\tau,z,y)\,dy = \frac{2c^{1/\alpha}}{\pi}\Gamma\left(1-\frac{1}{\alpha}\right)\tau^{1/\alpha}, \quad \tau>0,\ z\in S,$$

and the inequality (more precise than written above)

$$\int_S g_0(t-\tau,x,z)\,d\sigma_z \le \frac{\Gamma(1/\alpha)}{\pi\alpha c^{1/\alpha}}(t-\tau)^{-1/\alpha}, \quad 0<\tau<t,\ x\in\mathbb{R}^d.$$

Hence,

$$\int_{\mathbb{R}^d} |G_0(t,x,y)|\,dy \le$$

$$\le 1+\frac{4c^{1/\alpha}\|q\|}{\pi\alpha}\Gamma\left(1-\frac{1}{\alpha}\right)\int_0^t \tau^{\frac{1}{\alpha}-1}\,d\tau\int_S g_0(t-\tau,x,z)\,d\sigma_z \le 1+\frac{4\|q\|}{\alpha^2\sin^2\frac{\pi}{\alpha}}$$

which means the integrability desired.

As a consequence, we have the inequality

$$\left|\int_{\mathbb{R}^d} G_0(t,x,y)\varphi(y)\,dy\right| \le \left(1+\frac{4\|q\|}{\alpha^2\sin^2\frac{\pi}{\alpha}}\right)\|\varphi\| \tag{18}$$

valid for all $t>0$, $x\in\mathbb{R}^d$ and $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$.

Since $\int_{\mathbb{R}^d} g_0^{(\nu)}(t,x,y)\,dy\equiv 0$, we have the identity

$$\int_{\mathbb{R}^d} G_0(t,x,y)\,dy \equiv 1. \tag{19}$$

Besides, the function G_0 satisfies the equation (compare with 3.1.B above)

$$G_0(s+t,x,y)=\int_{\mathbb{R}^d} G_0(s,x,z)G_0(t,z,y)\,dz \tag{20}$$

for all $s>0$, $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$. The proof of this equality consists of very simple calculations (we omit them) based on the formula ($s>0$, $t>0$, $x\in\mathbb{R}^d$ and $y\in\mathbb{R}^d$)

$$\int_{\mathbb{R}^d} g_0^{(\nu)}(s,x,z)g_0(t,z,y)\,dz = g_0^{(\nu)}(s+t,x,y),$$

the validity of which can be verified immediately.

In one-dimensional case the function G_0 is given by

$$G_0(t,x,y)=g_0(t,x,y)+\frac{2qy}{\alpha}\int_0^t g_0(t-\tau,x,0)g_0(\tau,0,y)\frac{d\tau}{\tau} \tag{21}$$

for $t>0$, $x\in\mathbb{R}^1$ and $y\in\mathbb{R}^1$. As was shown in [6], this function takes on not only non-negative values but also negative ones (if $q\neq 0$). The same concerns the function G_0 in the case of $d\ge 2$ (and $q(x)\not\equiv 0$).

It can be now verified that the function G_0 is the fundamental solution to the problem (i) — (iii) in the case of $r(x)\equiv 0$. First, for fixed $y\notin S$ it satisfies equation (4) in the region $(t,x)\in(0,+\infty)\times(\mathbb{R}^d\setminus S)$. If $y\in S$, then $G_0(t,x,y)=g_0(t,x,y)$ and consequently, that equation is satisfied by the function $(G_0(t,x,y))_{t>0,x\in\mathbb{R}^d}$ in the whole region $(t,x)\in(0,+\infty)\times\mathbb{R}^d$. Second, for a given $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$, we put

$$U_0(t,x,\varphi)=\int_{\mathbb{R}^d}\varphi(y)G_0(t,x,y)\,dy,\quad t>0,\ x\in\mathbb{R}^d.$$

This function can be written as follows (the function u_0 is defined by (7))

$$U_0(t,x,\varphi)=u_0(t,x,\varphi)+\int_0^t d\tau\int_S g_0(t-\tau,x,y)u_0^{(\nu)}(\tau,y,\varphi)q(y)\,d\sigma_y, \quad (22)$$

where

$$u_0^{(\nu)}(\tau,y,\varphi)=\int_{\mathbb{R}^d}g_0^{(\nu)}(\tau,y,z)\varphi(z)\,dz=\mathbf{B}_\nu u_0(\tau,\cdot,\varphi)(y),\quad \tau>0,\ y\in S.$$

As follows from [7], the function U_0 satisfies the conditions (i) and (ii). Applying now the operator $\mathbf{B}_\nu$ to both sides of (22) and using relations (14), we arrive at the following equalities

$$\mathbf{B}_\nu U_0(t,\cdot,\varphi)(x\pm)=(1\mp q(x))u_0^{(\nu)}(t,x,\varphi).$$

Hence, the function U_0 satisfies the boundary value condition (iii) for $r(x)\equiv 0$.

4. The fundamental solution of the symmetric problem

In this section the function q is supposed to be identically equal to zero and the bounded continuous function $(r(x))_{x\in S}$ with non-negative values remains to be given. Our aim is to construct the fundamental solution to the problem (i) — (iii) in this case.

4.1. The equations of perturbations

Notice that for any $\varphi\in\mathbb{C}_b(\mathbb{R}^d)$ the function $(u_0(t,x,\varphi))_{t>0,x\in\mathbb{R}^d}$ defined by (7) constitutes a semigroup

$$u_0(s+t,x,\varphi)=u_0(s,x,u_0(t,\cdot,\varphi)),\quad s>0,\ t>0,\ x\in\mathbb{R}^d,$$

with the operator $\mathbf{A}$ serving as the generator of this semigroup. It is not difficult to guess that in order to obtain the semigroup connected with the problem (i) — (iii) (for $q(x)\equiv 0$), one should additively perturb the generator $\mathbf{A}$ by an operator whose action on a given function consists in multiplying it by the function $(r(x)\delta_S(x))_{x\in\mathbb{R}^d}$, where δ_S is a generalized function on $\mathbb{R}^d$ determined by the relation

$$\langle\delta_S,\psi\rangle=\int_S\psi(x)\,d\sigma \quad (23)$$

valid for an arbitrary test function $(\psi(x))_{x\in\mathbb{R}^d}$. According to the perturbations theory (see [5]), such a perturbed semigroup must be determined by the kernel $g(t,x,y)$, $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$, satisfying each one of the following pair of equations

$$\begin{aligned} g(t,x,y) &= g_0(t,x,y) - \int_0^t d\tau \int_S g_0(t-\tau,x,z)g(\tau,z,y)r(z)\,d\sigma_z,\\ g(t,x,y) &= g_0(t,x,y) - \int_0^t d\tau \int_S g(t-\tau,x,z)g_0(\tau,z,y)r(z)\,d\sigma_z. \end{aligned} \tag{24}$$

We now construct the solution to these equations. Some approximating procedure will be described after that.

4.2. Solving equations (24)

The method of successive approximations will be used for consructing a solution to (24). For $t>0$, $x\in\mathbb{R}^d$, $y\in\mathbb{R}^d$ and $k=1,2,\dots$, we put

$$g_k(t,x,y) = \int_0^t d\tau \int_S g_0(t-\tau,x,z)g_{k-1}(\tau,z,y)r(z)\,d\sigma_z.$$

By induction on k one can verify that

$$g_k(t,x,y) = \int_0^t d\tau \int_S g_{k-1}(t-\tau,x,z)g_0(\tau,z,y)r(z)\,d\sigma_z.$$

We need some proper estimates for g_k, in order to assert that the sum of the series

$$g(t,x,y) = \sum_{k=0}^{\infty}(-1)^k g_k(t,x,y) \tag{25}$$

solves each one of equations (24).

Our plan is as follows. We will establish the estimates desired in the region $t>0$, $x\in S$ and $y\in S$ and therefore, we will have got the solution to equations (24) in that region. As follows from those equations, the function g is uniquely determined by its values in the region $t>0$, $x\in S$ and $y\in S$ (even only in the region $t>0$, $x\in S_+$ and $y\in S_+$, where $S_+=\{z\in S: r(x)>0\}$).

To avoid some trivial remarks, we consider the case of $d=1$ separately. In this case equations (24) can be rewritten as follows

$$\begin{aligned} g(t,x,y) &= g_0(t,x,y) - r\int_0^t g_0(t-\tau,x,0)g(\tau,0,y)\,d\tau,\\ g(t,x,y) &= g_0(t,x,y) - r\int_0^t g(t-\tau,x,0)g_0(\tau,0,y)\,d\tau. \end{aligned} \tag{26}$$

where r is a non-negative number. By induction on k, we obtain ($\theta = 1-\frac{1}{\alpha}$)

$$g_k(t,0,0) = \frac{(\alpha c^{1/\alpha}\sin\frac{\pi}{\alpha})^{-k-1}}{\Gamma((k+1)\theta)}t^{-\frac{1}{\alpha}+k\theta}, \quad t>0,\ k=0,1,\dots$$

Therefore,

$$g(t,0,0) = \sum_{k=0}^{\infty}(-r)^k\frac{(\alpha c^{1/\alpha}\sin\frac{\pi}{\alpha})^{-k-1}}{\Gamma((k+1)\theta)}t^{-\frac{1}{\alpha}+k\theta}.$$

Having already had $g(t,0,0)$, we find out $g(t,x,0)$ from the first equation in (26)

$$g(t,x,0) = g_0(t,x,0) - r\int_0^t g_0(t-\tau,x,0)g(\tau,0,0)\,d\tau$$

and after that, the function $g(t,x,y)$ is expressed from the second equation in (26)

$$g(t,x,y) = g_0(t,x,y) - r\int_0^t g(t-\tau,x,0)g_0(\tau,0,y)\,d\tau.$$

If we put

$$\tilde{g}_0(\lambda,x,y) = \int_0^\infty e^{-\lambda t} g_0(t,x,y)\,dt, \quad \tilde{g}(\lambda,x,y) = \int_0^\infty e^{-\lambda t} g(t,x,y)\,dt$$

for $\lambda > 0$, $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, then we arrive at the formula

$$\begin{aligned}\tilde{g}(\lambda,x,y) = \tilde{g}_0(\lambda,x,y) - \frac{\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{\tilde{g}(\lambda,0,0)} + \\ + \frac{1}{1+r\tilde{g}_0(\lambda,0,0)}\frac{\tilde{g}_0(\lambda,x,0)\tilde{g}_0(\lambda,0,y)}{\tilde{g}(\lambda,0,0)}.\end{aligned} \tag{27}$$

Some interesting consequences of this formula are discussed in our paper "On some Markov processes related to a symmetric α-stable process" to be published soon.

We now return to equation (24) supposing $d \geq 2$.

Lemma 1. *If $d \geq 2$ then the following estimate*

$$g_k(t,x,y) \leq \frac{(4C)^k(\Gamma(\theta))^k N^{k+1}\|r\|^k}{\Gamma(2+k\theta)} \frac{t^{k\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \tag{28}$$

holds true for all $t > 0$, $x \in S$, $y \in S$ and $k = 0,1,2,\ldots$, where N is the constant from (8), $\theta = 1 - \frac{1}{\alpha}$, $\|r\| = \sup_{x\in S} r(x)$, $C = 2^{d+\alpha}\int_{\mathbb{R}^{d-1}} \frac{dz}{(1+|z|)^{d+\alpha}}$.

Proof. We use the method of mathematical induction on k. Inequality (28) for $k=0$ coincides with (8) and is true. Suppose now that it is true for some $k \geq 0$. We have to estimate the following integral for $t > 0$, $x \in S$ and $y \in S$

$$I = \int_0^t d\tau \int_S \frac{t-\tau}{[(t-\tau)^{1/\alpha}+|z-x|]^{d+\alpha}} \frac{\tau^{k\theta+1}}{[\tau^{1/\alpha}+|y-z|]^{d+\alpha}}\,d\sigma_z. \tag{29}$$

Our reasoning is similar to that given by A. N. Kochubei in [3] (see also [9]). We put for $t > 0$, $x \in S$ and $y \in S$

$$\Pi_1 = \{(\tau,z) : \tau \in (0,t/2), z \in S\}, \quad \Pi_2 = \{(\tau,z) : \tau \in (t/2,t), z \in S\},$$

$$\Pi_{11} = \{(\tau,z) \in \Pi_1 : \tau^{1/\alpha} + |y-z| \leq \frac{1}{2}(t^{1/\alpha}+|y-x|)\}, \quad \Pi_{12} = \Pi_1 \setminus \Pi_{11},$$

$$\Pi_{21} = \{(\tau,z) \in \Pi_2 : (t-\tau)^{1/\alpha} + |z-x| \leq \frac{1}{2}(t^{1/\alpha}+|y-x|)\},$$

$$\Pi_{22} = \Pi_2 \setminus \Pi_{21}.$$

For $l \in \{1,2\}$ and $m \in \{1,2\}$, denote by I_{lm} the integral with the same integrand as in (29) and the domain of integration $(\tau,z) \in \Pi_{lm}$. Then $I = \sum_{l=1}^{2}\sum_{m=1}^{2} I_{lm}$.

For $(\tau,z) \in \Pi_{11}$, we have

$$(t-\tau)^{1/\alpha} + |z-x| \geq t^{1/\alpha} - \tau^{1/\alpha} + |y-x| - |z-y| \geq \frac{1}{2}(t^{1/\alpha} + |y-x|) \quad (30)$$

since the inequality $(u-v)^\rho \geq u^\rho - v^\rho$ is true for $0 < v < u/2$ and $\rho < 1$. Consequently,

$$I_{11} \leq \frac{2^{d+\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \int_0^t (t-\tau)\tau^{k\theta+1}\, d\tau \int_S \frac{d\sigma_z}{(\tau^{1/\alpha}+|y-z|)^{d+\alpha}} =$$
$$= \frac{Ct^{(k+1)\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} B((k+1)\theta, 2),$$

where C is defined above and $B(\cdot,\cdot)$ is Euler's beta-function.

If $(\tau,z) \in \Pi_{12}$, then the inequality $\tau^{1/\alpha} + |y-z| > \frac{1}{2}(t^{1/\alpha} + |y-x|)$ is fulfilled and it implies the estimate

$$I_{12} \leq \frac{2^{d+\alpha}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} \int_0^t (t-\tau)\tau^{k\theta+1}\, d\tau \int_S \frac{d\sigma_z}{((t-\tau)^{1/\alpha}+|z-x|)^{d+\alpha}} =$$
$$= \frac{Ct^{(k+1)\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} B(\theta, 2+k\theta).$$

Now in the region $(\tau,z) \in \Pi_{21}$ the inequality (see (30))

$$\tau^{1/\alpha} + |y-z| \geq t^{1/\alpha} - (t-\tau)^{1/\alpha} + |y-x| - |z-x| \geq \frac{1}{2}(t^{1/\alpha} + |y-x|)$$

is valid and we have

$$I_{21} \leq \frac{Ct^{(k+1)\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} B(\theta, 2+k\theta).$$

Finally, if $(\tau,z) \in \Pi_{22}$, then $(t-\tau)^{1/\alpha} + |z-x| > \frac{1}{2}(t^{1/\alpha} + |y-x|)$, and we have

$$I_{22} \leq \frac{Ct^{(k+1)\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} B((k+1)\theta, 2).$$

Taking into account the evident inequality $B(\theta, 2+k\theta) \geq B((k+1)\theta, 2)$ valid for all $k = 0,1,2,\ldots$ and $0 < \theta < 1$, we arrive at the estimate

$$I \leq \frac{4Ct^{(k+1)\theta+1}}{(t^{1/\alpha}+|y-x|)^{d+\alpha}} B(2+k\theta, \theta),$$

that implies (28). The lemma has been proved. $\square$

As follows from the lemma, the sum of the series (25) in the region $t > 0$, $x \in S$ and $y \in S$ is a continuous function $g(t,x,y)$ satisfying equation (24) in that region. Moreover, for any $T < +\infty$ there exists a positive constant L_T such that the inequality

$$|g(t,x,y)| \le L_T \frac{t}{(t^{1/\alpha} + |y-x|)^{d+\alpha}} \tag{31}$$

is held for all $t \in (0,T]$, $x \in S$ and $y \in S$. Using this result and the second equation in (24), we can extend the function g to the region $t > 0$, $x \in S$ and $y \in \mathbb{R}^d$. The fact that this extended function possesses the property (31) follows from the estimates similar to those proved Lemma 1 (for $k = 0$). After that, with the help of the first equation in (24), the function g can be extended to the whole region $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ and this extended function g will satisfy inequality (31) for all $t \in (0,T]$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ with some positive constant L_T being finite for $T < +\infty$.

Notice that the solution to each one of equations (24) possessing the property (31) is unique. This assertion follows immediately from Lemma 1, since the estimate (28) is fulfilled for the difference between any two solutions of the kind.

4.3. The approximating procedure

We now approximate the generalized function $(r(x)\delta_S(x))_{x\in\mathbb{R}^d}$ by the regular functions $(v_h(x))_{x\in\mathbb{R}^d}$, as $h \to 0+$, where

$$v_h(x) = \int_S g_0(h,x,y) r(y)\, d\sigma_y, \quad x \in \mathbb{R}^d,\ h > 0.$$

It is an easy exercise to verify that the relation

$$\lim_{h\to 0+} \int_{\mathbb{R}^d} v_h(x)\varphi(x)\,dx = \int_S r(x)\varphi(x)\,d\sigma \tag{32}$$

is true for any continuous compactly supported function φ. In other words $\lim\limits_{h\to 0+} v_h(x) = r(x)\delta_S(x)$.

Let $u^{(h)}(t,x,\varphi)$, $h > 0$, $t > 0$, $x \in \mathbb{R}^d$, $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, be the solution of the following problem

$$\begin{cases} \dfrac{\partial u^{(h)}}{\partial t} = \mathbf{A}u^{(h)} - v_h(x)u^{(h)}, \quad t > 0,\ x \in \mathbb{R}^d \\ u^{(h)}(0+,x,\varphi) = \varphi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{33}$$

As follows from the perturbations theory, the solution of this problem must solve the following integral equation

$$u^{(h)}(t,x,\varphi) = u_0(t,x,\varphi) - \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t-\tau,x,y) u^{(h)}(\tau,y,\varphi) v_h(y)\,dy. \tag{34}$$

The method of successive approximations allows one to construct a solution to this equation. We put $u_0^{(h)}(t,x,\varphi) = u_0(t,x,\varphi)$ (see (7)) and for $k \ge 1$

$$u_k^{(h)}(t,x,\varphi) = \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t-\tau,x,y) u_{k-1}^{(h)}(\tau,y,\varphi) v_h(y)\,dy.$$

As follows from (10) and 3.1.B,

$$\int_{\mathbb{R}^d} g_0(t-\tau,x,y)v_h(y)\,dy = \int_S g_0(t-\tau+h,x,z)r(z)\,d\sigma_z \le \\ \le \|r\|N(t-\tau+h)^{-1/\alpha} \le \|r\|N(t-\tau)^{-1/\alpha} \tag{35}$$

Making use of this estimate, by induction on k, one can easily arrive at the following inequality

$$|u_k^{(h)}(t,x,\varphi)| \le \frac{\|\varphi\|(N\|r\|\Gamma(\theta))^k}{\Gamma(k\theta+1)}t^{k\theta}, \quad t>0,\ x\in\mathbb{R}^d,\ h>0,\ k=0,1,\dots,$$

where $\theta = 1-\frac{1}{\alpha}$, as above. Therefore, the sum

$$u^{(h)}(t,x,\varphi) = \sum_{k=0}^{\infty}(-1)^k u_k^{(h)}(t,x,\varphi)$$

is a solution to equation (34) satisfying the condition

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |u^{(h)}(t,x,\varphi)| < +\infty$$

for any $T<+\infty$. Such a solution is unique.

The maximum principle for the equation in (33) allows one to assert that the values of the function $u^{(h)}$ are non-negative if only the values of φ are so.

Now, we are going to pass to the limit, as $h\to 0+$, in equation (34). In order to do this, we make use of the following auxiliary result.

Let a measurable complex-valued function $(\psi(t,x))_{t\ge0,x\in\mathbb{R}^d}$ be such that $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\psi(t,x)| < +\infty$ for any $T<+\infty$. Consider its transformation ψ_h for $h>0$ given by

$$\psi_h(t,x) = \int_0^t d\tau \int_{\mathbb{R}^d} g_0(t-\tau,x,y)\psi(\tau,y)v_h(y)\,dy, \quad t>0,\ x\in\mathbb{R}^d.$$

We assert that this transformation is compact in the following sense (as above, we use the notation $B_R=\{y\in\mathbb{R}^d : |y|\le R\}$ for $R>0$ and $B_R^c=\mathbb{R}^d\setminus B_R$).

Lemma 2. *For given numbers $\varepsilon>0$, $L>0$, $T>0$ and $R>0$, there exists a number $\delta>0$ such that the inequality*

$$|\psi_h(t',x')-\psi_h(t,x)|<\varepsilon$$

is held for all $h>0$, $t\in[0,T]$, $t'\in[0,T]$, $x\in B_R$, $x'\in B_R$ and all measurable function $(\psi(t,x))_{t\ge0,x\in\mathbb{R}^d}$ with the property $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\psi(t,x)|\le L$, if only the inequality $|t'-t|+|x'-x|<\delta$ is fulfilled.

Proof. For $t<t'$, $x\in\mathbb{R}^d$ and $x'\in\mathbb{R}^d$, we represent the difference $\psi_h(t',x')-\psi_h(t,x)$ as the sum of two terms I_1 and I_2, where

$$I_1 = \int_0^t d\tau \int_{\mathbb{R}^d}[g_0(t'-\tau,x',y)-g_0(t-\tau,x,y)]\psi(\tau,y)v_h(y)\,dy,$$

$$I_2 = \int_t^{t'} d\tau \int_{\mathbb{R}^d} g_0(t'-\tau, x', y)\psi(\tau, y)v_h(y)\, dy.$$

Inequality (35) implies the estimates

$$|I_2| \le L \int_t^{t'} d\tau \int_{\mathbb{R}^d} g_0(t'-\tau, x', y)v_h(y)\, dy \le \frac{L\|r\|N}{\theta}(t'-t)^\theta$$

and therefore $I_2 \to 0$, as $t'-t \to 0$ uniformly with respect to $h > 0$ and $x' \in \mathbb{R}^d$. The same reason is applicable to estimating the integral (for $0 < \gamma < t < t' \le T$ and $x' \in \mathbb{R}^d$)

$$\begin{aligned} &\left| \int_{t-\gamma}^{t} d\tau \int_{\mathbb{R}^d} g_0(t'-\tau, x', y)\psi(\tau, y)v_h(y)\, dy \right| \le \\ &\le L\|r\| \int_0^\gamma d\tau \int_S g_0(t'-t+\tau+h, x', y)\, d\sigma_y \le L\|r\|N\frac{\gamma^\theta}{\theta}. \end{aligned}$$

This means that $I_1 = I_1^{'} + I_2^{''}$, where

$$I_1^{'} = 1\!\!1_{\{t>\gamma\}} \int_0^{t-\gamma} d\tau \int_{\mathbb{R}^d} [g_0(t'-\tau, x', y) - g_0(t-\tau, x, y)]\psi(\tau, y)v_h(y)\, dy,$$

$$|I_1^{''}| \le const[(t'-t)^\theta + \gamma^\theta]$$

(the *const* depends only on L, N, $\|r\|$, c and α). So, the quantity $I_1^{''}$ becomes small enough if $t'-t$ and $\gamma > 0$ are chosen to be sufficiently small.

In order to estimate $I_1^{'}$ for fixed $\gamma > 0$, one should make use of the uniform continuity of the function g_0 given by (6) with respect to the arguments $(t, x, y) \in [\gamma, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ for any $\gamma > 0$. It only has to be taken into account that the function v_h may be not integrable over the whole $\mathbb{R}^d$ for $d \ge 2$ (it will be the case if the function $(r(y))_{y\in S}$ is not integrable over S). In the case of $d = 1$, we have $v_h(y) = r \cdot g_0(h, y, 0)$ for $y \in \mathbb{R}^1$ (see considerations prior to Lemma 1 and $\int_{\mathbb{R}^1} v_h(y)\, dy = r \ge 0$. The assertion of Lemma 2 for $d = 1$ can be thus strengthened (see Lemma 3 below).

So, we suppose that $d \ge 2$ and put

$$J_Q^{(h)}(\tau, x) = \int_{B_Q^c} g_0(\tau, x, y)v_h(y)\, dy$$

for $h > 0$, $Q > 0$, $\tau \in (0, T]$ and $x \in \mathbb{R}^d$. Using inequality (8), we can write down for $x \in B_R$ and $Q > R$

$$\begin{aligned} J_Q^{(h)}(\tau, x) \le& N \int_{B_Q^c} \frac{\tau}{(\tau^{1/\alpha} + |y-x|)^{d+\alpha}} v_h(y)\, dy \le \\ \le& N \int_{\mathbb{R}^d} \frac{\tau}{(\tau^{1/\alpha} + |y-x|/2 + (Q-R)/2)^{d+\alpha}} v_h(y)\, dy \end{aligned}$$

According to (10), we have

$$v_h(y) \le N\|r\|\frac{h}{(h^{1/\alpha} + |(y, \nu)|)^{\alpha+1}}, \quad y \in \mathbb{R}^d. \tag{36}$$

Therefore,

$$J_Q^{(h)}(\tau,x) \le$$
$$\le N^2\|r\| \int_{-\infty}^{+\infty} \frac{h\,d\rho}{(h^{1/\alpha}+|\rho|)^{\alpha+1}} \int\limits_{\{(y,\nu)=\rho\}} \frac{\tau\,d\sigma_y}{(\tau^{1/\alpha}+|y-x|/2+(Q-R)/2)^{d+\alpha}} \tag{37}$$

Taking into account that for $y \in \{z \in \mathbb{R}^d : (z,\nu)=\rho\}$ and $x \in \mathbb{R}^d$, the inequality $|y-x| \ge |\tilde{y}-\tilde{x}|$ (we remind that $\tilde{z}$ means the orthogonal projection of z on S) is valid, we arrive at the inequalities

$$J_Q^{(h)}(\tau,x) \le$$
$$\le N^2\|r\| \int_{-\infty}^{+\infty} \frac{d\rho}{(1+|\rho|)^{\alpha+1}} \int_{\mathbb{R}^{d-1}} \frac{\tau\,dz}{(\tau^{1/\alpha}+|z|/2+(Q-R)/2)^{d+\alpha}} \le$$
$$\le \frac{2^{d+\alpha+1}}{\alpha} N^2\|r\|\tau(Q-R)^{-\alpha-1} \int_{\mathbb{R}^{d-1}} \frac{dz}{(1+|z|)^{d+\alpha}}.$$

We have thus proved that uniformly with respect to $h>0$ and $(t,x) \in [0,T]\times B_R$, the relation

$$\int_0^t J_Q^{(h)}(\tau,x)\,d\tau \to 0, \quad \text{as } Q \to +\infty,$$

holds true. It remains now to show that for fixed $\gamma>0$ and $Q>0$ the integral

$$1\!\!1_{\{t>\gamma\}} \int_0^{t-\gamma} d\tau \int_{B_Q} [g_0(t'-\tau,x',y)-g_0(t-\tau,x,y)]\psi(\tau,y)v_h(y)\,dy$$

becomes small enough if the points $(t',x') \in [0,T]\times B_R$ and $(t,x) \in [0,T]\times B_R$ are chosen to be sufficiently close each to other one. It was mentioned above that the function g_0 is uniformly continuous on the set $[\gamma,+\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, hence, the assertion desired will be established if we show that

$$\sup_{h>0} \int_{B_Q} v_h(y)\,dy < +\infty$$

for fixed $Q>0$. Using (36), one can write down the inequality

$$\int_{B_Q} v_h(y)\,dy \le N\|r\| \int_{-Q}^{Q} \frac{h\,d\rho}{(h^{1/\alpha}+|\rho|)^{\alpha+1}} \int\limits_{B_Q\cap\{(y,\nu)=\rho\}} d\sigma_y.$$

It is clear that

$$\int\limits_{B_Q\cap\{(y,\nu)=\rho\}} d\sigma_y \le \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)} Q^{d-1}.$$

Therefore,

$$\int_{B_Q} v_h(y)\,dy \le \frac{2\pi^{(d-1)/2}}{\alpha\Gamma((d+1)/2)} N\|r\|Q^{d-1}. \tag{38}$$

This completes the proof of the lemma in the case of $d \geq 2$. As was noticed above, in the case of $d = 1$ the following strengthened version of Lemma 2 has been proved.

Lemma 3. *If $d = 1$, then for given numbers $\varepsilon > 0$, $L > 0$ and $T > 0$ there exists a number $\delta > 0$ such that the inequality*

$$|\psi_h(t', x') - \psi_h(t, x)| < \varepsilon$$

is held for all $h > 0$, $t \in [0, T]$, $t' \in [0, T]$, $x \in \mathbb{R}^1$, $x' \in \mathbb{R}^1$ and all measurable function $(\psi(t, x))_{t \geq 0, x \in \mathbb{R}^1}$ with the property $\sup_{(t,x) \in [0,T] \times \mathbb{R}^1} |\psi(t, x)| \leq L$, if only the inequality $|t' - t| + |x' - x| < \delta$ is fulfilled.

□

Now we can pass to the limit, as $h \to 0+$, in equation (34). First of all, using the diagonal method, we can choose a sequence $h_n \to 0+$ in such a way that $u^{(h_n)}(t, x, \varphi)$ converges to a function $u(t, x, \varphi)$ locally uniformly with respect to $t \geq 0$ and $x \in \mathbb{R}^d$. Then (32) implies the following equation for the limiting function u

$$u(t, x, \varphi) = u_0(t, x, \varphi) - \int_0^t \int_S g_0(t - \tau, x, y) u(\tau, y, \varphi) r(y) \, d\sigma_y. \tag{39}$$

Notice, that the estimates for $u_k^{(h)}$, $k = 0, 1, 2, \ldots$, were uniform with respect to $h > 0$. So, the limit equation, that is equation (39), can be solved in the same way as the equation for $u^{(h)}$ has been solved. Accordingly, we can conclude that the solution to (39) possessing the property $\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |u(t, x, \varphi)| < +\infty$ for any $T < +\infty$ is unique. It means, first, that $\lim_{h \to 0+} u^{(h)}(t, x, \varphi) = u(t, x, \varphi)$ and, second, that the values of the function $(u(t, x, \varphi))_{t \geq 0, x \in \mathbb{R}^d}$ are non-negative if $\varphi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

Now, let us observe that equation (39) can be obtained by multiplying the first one of equations (24) by $\varphi(y)$ ($\varphi \in \mathbb{C}_b(\mathbb{R}^d)$) and integrating both sides of it with respect to $y \in \mathbb{R}^d$. This means that for $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g(t, x, y) \varphi(y) \, dy, \quad t > 0, \ x \in \mathbb{R}^d. \tag{40}$$

As a consequence, we have that the function g takes on only non-negative values.

It follows from equation (39) and Subsections 3.3, 3.4 that for any $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$ the function $(u(t, x, \varphi))_{t > 0, x \in \mathbb{R}^d}$ satisfies the conditions (i) and (ii). Besides, the equalities

$$\mathbf{B}_\nu u(t, \cdot, \varphi)(x\pm) = u_0^{(\nu)}(t, x, \varphi) \pm r(x) u(t, x, \varphi)$$

are valid for $t > 0$ and $x \in S$. Therefore,

$$\frac{1}{2} \mathbf{B}_\nu u(t, \cdot, \varphi)(x+) - \frac{1}{2} \mathbf{B}_\nu u(t, \cdot, \varphi)(x-) = r(x) u(t, x, \varphi), \quad t > 0, \ x \in S.$$

It remains to prove that for a fixed $y \in \mathbb{R}^d$ the function $(g(t,x,y))_{t>0,x\in\mathbb{R}^d}$ satisfies the equation (4) in the region $t > 0$, $x \notin S$. This is now a non-difficult exercise left for a reader.

We have thus proved the following assertion.

Theorem 1. *The function g constructed as the sum* (25) *is the fundamental solution to the symmetric initial-boundary value problem.*

5. The fundamental solution of the asymmetric problem

5.1. The equations of perturbations

Our starting point is now the function $G_0(t,x,y)$, $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. By the analogy to equations (24), we first write down those equations with the function g_0 being replaced by G_0. The function to be found is denoted by G. Taking into account (17) and the fact

$$\lim_{h\to 0+} \int_{\mathbb{R}^d} G_0(t,x,y) v_h(y)\, dy = \int_S g_0(t,x,y) r(y)\, d\sigma_y,$$

we arrive at the following pair of equations

$$\begin{aligned} G(t,x,y) &= G_0(t,x,y) - \int_0^t d\tau \int_S g_0(t-\tau,x,z) G(\tau,z,y) r(z)\, d\sigma_z, \\ G(t,x,y) &= G_0(t,x,y) - \int_0^t d\tau \int_S G(t-\tau,x,z) G_0(\tau,z,y) r(z)\, d\sigma_z. \end{aligned} \tag{41}$$

For $t > 0$, $x \in \mathbb{R}^d$ and $y \in S$, we have $G_0(t,x,y) = g_0(t,x,y)$, and the first equation can be rewritten in this case as follows

$$G(t,x,y) = g_0(t,x,y) - \int_0^t d\tau \int_S g_0(t-\tau,x,z) G(\tau,z,y) r(z)\, d\sigma_z. \tag{42}$$

This implies the equality $G(t,x,y) = g(t,x,y)$ for $t > 0$, $x \in \mathbb{R}^d$ and $y \in S$. From the second equation we obtain the following formula for G

$$G(t,x,y) = G_0(t,x,y) - \int_0^t d\tau \int_S g(t-\tau,x,z) G_0(\tau,z,y) r(z)\, d\sigma_z \tag{43}$$

that is held true for all $t > 0$, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$. It gives the representation for the function G in terms of G_0 and g constructed above. The dual representation can be obtained from (43) and the second equation in (24) by very simple calculations

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S g(t-\tau,x,z) g_0^{(\nu)}(\tau,z,y) q(z)\, d\sigma_z. \tag{44}$$

The equality $G(t,x,y\pm) = (1 \pm q(y)) g(t,x,y)$ valid for $t > 0$, $x \in \mathbb{R}^d$ and $y \in S$ is a consequence of (44).

5.2. The initial-boundary value problem

For $t > 0$, $x \in \mathbb{R}^d$ and a given function $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$, we put

$$U(t,x,\varphi) = \int_{\mathbb{R}^d} G(t,x,y)\varphi(y)\,dy.$$

Theorem 2. *The function U is a solution to the initial-boundary value problem (i) — (iii).*

Proof. Multiplying both sides of (42) by $\varphi(y)$ and integrating with respect to $y \in \mathbb{R}^d$, we get the following equation for the function U

$$U(t,x,\varphi) = U_0(t,x,\varphi) - \int_0^t d\tau \int_S g_0(t-\tau,x,z)U(\tau,z,\varphi)r(z)\,d\sigma_z. \quad (45)$$

As was shown in Subsection 3.5, the function U_0 satisfies the conditions (i), (ii) and the boundary condition (iii) with $r(x) \equiv 0$. We put for $t > 0$ and $x \in \mathbb{R}^d$

$$V(t,x,\varphi) = \int_0^t d\tau \int_S g_0(t-\tau,x,z)U(\tau,z,\varphi)r(z)\,d\sigma_z.$$

This is a simple-layer potential. It satisfies the condition (i) and the initial condition $V(0+,x,\varphi) \equiv 0$. It follows from Subsection 3.4 that V possesses the property

$$\mathbf{B}_\nu V(t,\cdot,\varphi)(x\pm) = \mp r(x)U(t,x,\varphi)$$

for $t > 0$ and $x \in S$. This completes the proof of the theorem. □

Remark. The function G defined by (43) or (44) is the fundamental solution of the problem (i) — (iii).

6. The probabilistic interpretation

6.1. The symmetric α-stable process

The function g_0 is the transition probability density of a standard Markov process in $\mathbb{R}^d$ in the sense of [2], Theorem 3.14. Denote that process by $(x(t), \mathcal{M}_t, \mathbb{P}_x)$ or somewhat shorter $(x(t))_{t\geq 0}$. It is called a symmetric α-stable process. The function $f_t(x)$, $t \geq 0$, $x \in \mathbb{R}^d$, defined by

$$f_t(x) = \int_0^t d\tau \int_S g_0(\tau,x,y)r(y)\,d\sigma_y$$

is a W-function for this process satisfying the condition $\sup_{x\in\mathbb{R}^d} f_t(x) \to 0$, as $t \to 0+$ (see (10)). According to Theorem 6.6 from [2], there exists a W-functional $(\eta_t(r))_{t\geq 0}$ of the process $(x(t))_{t\geq 0}$ such that $f_t(x) = \mathbb{E}_x\eta_t(r)$ for all $t > 0$ and $x \in \mathbb{R}^d$. Let $r_0(x) \equiv 1$ and $\eta_t = \eta_t(r_0)$, $t \geq 0$. The functional $(\eta_t)_{t\geq 0}$ is called the local time on S for the process $(x(t))_{t\geq 0}$. It is clear that

$$\eta_t(r) = \int_0^t r(x(s))\,d\eta_s, \quad t \geq 0.$$

For $h > 0$, we put

$$\eta_t^{(h)}(r) = \int_0^t v_h(x(s))\, ds, \quad t \geq 0,$$

where v_h is defined in Subsection 4.3. A very simple calculation shows that

$$\mathbb{E}_x \eta_t^{(h)}(r) = \int_h^{t+h} d\tau \int_S g_0(\tau, x, y) r(y)\, d\sigma_y, \quad t \geq 0,\ h > 0,\ x \in \mathbb{R}^d.$$

This implies the estimate

$$\left|\mathbb{E}_x \eta_t^{(h)}(r) - \mathbb{E}_x \eta_t(r)\right| \leq \|r\| N \left[(t+h)^{1-1/\alpha} - t^{1-1/\alpha} + h^{1-1/\alpha}\right]$$

valid for $t \geq 0$, $x \in \mathbb{R}^d$, $h > 0$. Applying Theorem 6.4 from [2] leads us to the relation

$$\lim_{h \to 0+} \eta_t^{(h)}(r) = \eta_t(r) \tag{46}$$

that takes place in the sense of mean square convergence. The function u and $u^{(h)}$ introduced in Section 4 have the following probabilistic sense

$$u^{(h)}(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta_t^{(h)}(r)}\right), \quad u(t, x, \varphi) = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta_t(r)}\right).$$

Relation (46) implies the pointwise convergence

$$u(t, x, \varphi) = \lim_{h \to 0+} u^{(h)}(t, x, \varphi). \tag{47}$$

The fact that the function $u^{(h)}$ is a solution to equation (34) is a consequence of the Feynman-Kac formula. Lemma 2 allows one to assert that the convergence in (47) is locally uniform with respect to $t \geq 0$ and $x \in \mathbb{R}^d$ (if $d = 1$ that convergence is uniform with respect to $x \in \mathbb{R}^1$, see Lemma 3). Finally, we have the following probabilistic representation for the function u

$$u(t, x, \varphi) = \int_{\mathbb{R}^d} g(t, x, y) \varphi(y)\, dy = \mathbb{E}_x \left(\varphi(x(t)) e^{-\eta_t(r)}\right) \tag{48}$$

that holds true for $t > 0$, $x \in \mathbb{R}^d$ and $\varphi \in \mathbb{C}_b(\mathbb{R}^d)$.

Further, the function g constructed in Section 4 is the transition probability density of the process $(x(t))_{t \geq 0}$ killed at some stopping time ζ. It is clear that

$$\mathbb{P}_x(\{\zeta > t\}) = \int_{\mathbb{R}^d} g(t, x, y)\, dy, \quad t > 0,\ x \in \mathbb{R}^d.$$

From the second equation in (24), we conclude

$$\mathbb{P}_x(\{\zeta > t\}) = 1 - \int_0^t d\tau \int_S g(\tau, x, y) r(y)\, d\sigma_y.$$

Therefore, the density of the distribution function of ζ is given by

$$-\frac{d}{dt} \mathbb{P}_x(\{\zeta > t\}) = \int_S g(t, x, y) r(y)\, d\sigma_y$$

It is curious to find out the conditions imposed on the function $(r(x))_{x \in S}$ under which $\mathbb{P}_x(\{\zeta < +\infty\}) = 1$ for all $x \in \mathbb{R}^d$.

In the case of $d = 1$, formula (27) gives the resolvent kernel for the process with its transition probability density given by g.

6.2. The pseudo-process

As was mentioned in Section 2, the function G_0 cannot be transition probability density of any Markov process. But it can be considered as that for a pseudo-process $(y(t))_{t\geq 0}$ in the sense of [1] and the function G constructed in Section 5 must be connected with that pseudo-process by an analogy to (48), that is

$$U(t,x,\varphi) = \int_{\mathbb{R}^d} G(t,x,y)\varphi(y)\,dy = \hat{\mathbb{E}}_x\left(\varphi(y(t))e^{-\hat{\eta}_t(r)}\right), \quad t>0,\ x\in\mathbb{R}^d,$$

where $\hat{\mathbb{E}}_x$ denotes "expectation" with respect to the pseudo-process and $(\hat{\eta}_t(r))_{t\geq 0}$ denotes some "additive functional of the pseudo-process $(y(t))_{t\geq 0}$". In particular,

$$\hat{\mathbb{E}}_x e^{-\hat{\eta}_t(r)} = \int_{\mathbb{R}^d} G(t,x,y)\,dy = 1 - \int_0^t d\tau \int_S g(\tau,x,z)r(z)\,d\sigma_z,$$

as follows from (43). In other words, the distribution function of $\hat{\eta}_t(r)$ is the same as that of $\eta_t(r)$.

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M. M. Osypchuk
Department of Statistics and High Mathematics,
Vasyl Stefanyk Precarpathian National University,
57, Shevchenko str.,
76018, Ivano-Frankivsk
Ukraine
e-mail: mykhailo.osypchuk@pu.if.ua

M. I. Portenko
Department of Theory of Random Processes,
Institute of Mathematics of the National Academy of Sciences of Ukraine,
3, Tereschenkivska str.,
01004, Kyiv,
Ukraine
e-mail: portenko@imath.kiev.ua