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ORDINARY DIFFERENTIAL EQUATIONS

Solvability of a Nonhomogeneous Boundary Value Problem for a Differential System with Measures

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INTRODUCTION

We consider linear nonhomogeneous systems of differential equations containing distributions as coefficients. Descriptions of such systems typically contain some form of products of discontinuous functions by generalized derivatives of functions of bounded variation. These products do not always exist in the sense of distribution theory, and accordingly, different approaches to the definition of a solution may give different results. A detailed review of the literature on this topic can be found in [1], where the main approaches to the definition of a solution of the corresponding equations are also described. We also note the papers [2–6] dealing with the analysis of classes of linear and quasilinear differential systems with measures whose solutions are treated in the sense of distribution theory, whereby their definition (under some restrictions on the coefficients) is independent of the interpretation of the product of a measure by a function of locally bounded variation.

Here we study the solvability of a nonhomogeneous boundary value problem for a differential system with measures assuming that the value of a parameter λ occurring linearly in the equation coincides with some eigenvalue of the corresponding homogeneous problem. The simpler case in which λ does not coincide with any eigenvalue was considered in [4].

1. NOTATION

Let C^n be the n-dimensional complex Euclidean space with inner product

$$(x,y)=y^*x=\overline{y_1}x_1+\overline{y_2}x_2+\cdots+\overline{y_n}x_n,\qquad x=(x_1,\ldots,x_n)^\tau,\qquad y=(y_1,\ldots,y_n)^\tau,$$

where $*$ stands for Hermitian conjugation (that is, complex conjugation $\bar{\cdot}$ followed by transposition τ). We denote the space of right continuous functions of locally bounded variation on the compact set $[a,b]$ by $BV^+[a,b]$. The space of $n\times m$ matrices whose entries are complex-valued functions of the real variable $x\in[a,b]$ is denoted by $\mathscr{L}([a,b]^{n\times m})$. If $A\in BV^+[a,b]$ and $A\in\mathscr{L}([a,b]^{n\times m})$ simultaneously, then we write $A\in BV^+([a,b]^{n\times m})$ and assume that each entry of the matrix $A(x)$ belongs to $BV^+[a,b]$. By $\Delta C(x)=C(x)-C(x-0)$ we denote the jump of a function $C\in BV^+([a,b]^{n\times m})$ at a point $x\in[a,b]$.

Let $A=(a_{k\nu})_1^n\in BV^+[a,b]^{n\times n}$ be a function nondecreasing in the matrix sense [7, p. 539 of the Russian translation] on $[a,b]$. By $\bar{L}_A^2[a,b]$ we denote the Hilbert space of classes of A-equivalent[1] vector functions $f=(f_k)_1^n\in BV^+[a,b]^{n\times 1}$ with dA-measurable coordinates $f_k(x)$ such that

$$\|f\|^2_{\bar{L}_A^2[a,b]}=\|f\|_A^2=\int\limits_a^b f^*(t)dA(t)f(t)=\int\limits_a^b\sum_{k,\nu=1}^n\overline{f_k}(t)f_\nu(t)da_{k\nu}(t)<\infty.$$

The inner product in $\bar{L}_A^2[a,b]$ is defined in the standard way:

$$\langle f,g\rangle_{\bar{L}_A^2[a,b]}=\langle f,g\rangle_A=\int\limits_a^b g^*(t)dA(t)f(t)=\int\limits_a^b\sum_{k,\nu=1}^n\overline{g_k}(t)f_\nu(t)da_{k\nu}(t).$$

[1] As usual, functions $f(x)$ and $g(x)$ are said to be *A-equivalent* on $[a,b]$ if they coincide dA-almost everywhere on $[a,b]$.

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The space $\bar{L}_A^2[a,b]$ will be called the space of A-square integrable functions, and convergence in the norm $\|\cdot\|_{\bar{L}_A^2[a,b]}$ in $\bar{L}_A^2[a,b]$ will be referred to as mean(-square) convergence on $[a,b]$.

2. STATEMENT OF THE PROBLEM

Consider the nonhomogeneous system of differential equations

$$JY' = [B'(x) + \lambda A'(x)]\, Y + F'(x), \qquad a \le x \le b, \tag{1}$$

where A, and B are $n\times n$ matrices, J is a constant $n\times n$ matrix, Y and F are n-vectors, and λ is a scalar (complex) parameter. We assume that $A, B \in BV^+[a,b]^{n\times n}$, $F \in BV^+[a,b]^{n\times 1}$, and $A(x)$ is nondecreasing (in the matrix sense) on $[a,b]$; by A', B', and F' we denote their generalized derivatives, which are Stieltjes measures on $[a,b]$ (see [8, p. 160 of the Russian translation]). Further, we assume that

$$J^* = -J, \qquad J^*J = E, \qquad A^*(x) = A(x), \qquad B^*(x) = B(x), \tag{2_{1-4}}$$

where E is the identity matrix.

A *solution* of Eq. (1) is defined as a vector function $Y \in BV^+[a,b]$ satisfying this equation in the generalized sense, i.e., with the differentiation and equality in (1) treated in the sense of distribution theory [9, 10]. To make this definition unambiguous, we must subject the functions $A(x)$, $B(x)$, and $F(x)$ to the well-posedness conditions [2, 3], which in the case under consideration have the form

$$\{J[\Delta B(x_s) + \lambda\Delta A(x_s)]\}^2 = 0, \qquad [\Delta B(x_s) + \lambda\Delta A(x_s)]\, J\Delta F(x_s) = 0 \qquad \forall x_s \in [a,b]. \tag{3}$$

Let M and N be $n\times n$ matrices such that

$$M^*JM = N^*JN \tag{4}$$

and moreover, it follows from the relation $Mv = Nv = 0$ ($v \in \mathrm{C}^n$) that $v \equiv 0$. For Eq. (1), we consider the boundary value problem

$$Y(a) = Mv, \qquad Y(b) = Nv, \tag{5}$$

where $v \ne 0$ is a given vector.

In what follows, we assume that an arbitrary nontrivial solution $Y(x)$ of the corresponding homogeneous system

$$JY' = [B'(x) + \lambda A'(x)]\, Y \tag{6}$$

satisfies the positive definiteness condition

$$\int_a^b Y^*(x)dA(x)Y(x) > 0 \tag{7}$$

with a classical[2] matrix[3] Riemann–Stieltjes integral.

Note that the eigenvalue problem (6), (5) under the additional condition (7) was comprehensively studied in [4] (see also [6]). Problem (1), (5) was also posed and partially solved there. In particular, the following main result was obtained.

Theorem A [6, p. 29]. *Let conditions* (2)–(4) *and* (7) *be satisfied. If* λ *is not an eigenvalue of problem* (6), (5), *then the nonhomogeneous problem* (1), (5) *has a unique solution* $Y \in \bar{L}_A^2[a,b]$, *which can be represented in the form*

$$Y(x) = \int_a^b K(x,t,\lambda)dF(t), \tag{8}$$

[2] For the existence of the integral in the classical sense, it is sufficient that the so-called strengthened well-posedness condition $\Delta A(x_s)\, J\Delta B(x_s) = 0$ for all $x_s \in [a,b]$ be satisfied [6].
[3] That is actually a set of integrals of scalar type [7, p. 492 of the Russian translation].

where the resolving kernel $K(x,t,\lambda)$ *has the form*

$$K(x,t,\lambda)=\begin{cases}\mathscr{Y}(x,a,\lambda)M[\mathscr{Y}(b,a,\lambda)M-N]^{-1}\mathscr{Y}(b,t,\lambda)J & \text{for } x<t\\ \mathscr{Y}(x,a,\lambda)M[\mathscr{Y}(b,a,\lambda)M-N]^{-1}\mathscr{Y}(b,t,\lambda)J-\mathscr{Y}(x,t,\lambda)J & \text{for } x\ge t;\end{cases} \tag{9}$$

here $\mathscr{Y}(x,t,\lambda)$ *is the evolution operator* (*the principal solution matrix*) *of Eq.* (6). *In particular, it has the following properties* (*is harmonic and* J*-unitary*) [6] *for any* $x,t,s\in[a,b]$:

(1) $\mathscr{Y}(x,t,\lambda)\mathscr{Y}(t,x,\lambda)=E$;

(2) $\mathscr{Y}(x,t,\lambda)\mathscr{Y}(t,s,\lambda)=\ cY(x,s,\lambda)$;

(3) $\mathscr{Y}^*\left(x,t,\bar\lambda\right)J\mathscr{Y}(x,t,\lambda)=J$;

(4) $\mathscr{Y}(x,t,\lambda)$ *belongs to* $BV^+[a,b]$ *with respect to* x *as well as with respect to* t *and is an entire function of the parameter* λ.

In the present paper, we study solvability conditions for the nonhomogeneous problem (1), (5) for the case in which λ coincides with some eigenvalue of problem (6), (5). In this case, the reduction of problem (1), (5) to an equivalent loaded integral equation (of Fredholm–Stieltjes type) with matrix kernel is an effective method.

3. AUXILIARY ASSERTIONS

The results of [11] and [6] imply the following assertion.

Lemma 1. *Let* $h\in BV^+[a,b]$ *with respect to each variable, and let* $f,g\in BV^+[a,b]$; *moreover, suppose that* $h(x,y)$ *is a square matrix function of the variables* (x,y) *and* $f(x)$ *and* $g(y)$ *are matrix functions admitting the formal product* $g'(y)h(x,y)f'(x)$. *In addition, let the conditions* $\Delta g\left(y_s\right)\Delta_y h\left(x,y_s\right)=0$ *for all* $x\in[a,b]$ *and* $\Delta_x h\left(x_p,y\right)\Delta f\left(x_p\right)=0$ *for all* $y\in[a,b]$ *be satisfied at each point* $y_s\in[a,b]$ *of discontinuity of the functions* $g(y)$ *and* $h(x,y)$ *and at each point* $x_p\in[a,b]$ *of discontinuity of the functions* $h(x,y)$ *and* $f(x)$. *Then the repeated integrals* $\int_a^b dg(y)\int_a^b h(x,y)df(x)$ *and* $\int_a^b\left(\int_a^b dg(y)h(x,y)\right)df(x)$ *exist as classical matrix Riemann–Stieltjes integrals, and*

$$\int_a^b dg(y)\int_a^b h(x,y)df(x)=\int_a^b\left(\int_a^b dg(y)h(x,y)\right)df(x). \tag{10}$$

Lemma 2. *If* $x\ne t$ *and* λ *is not an eigenvalue of problem* (6), (5), *then the kernel* $K(x,t,\lambda)$ *given by* (9) *is a right continuous matrix function of bounded variation on* $[a,b]$ *with respect to each of the arguments* x *and* t; *moreover,*

$$K(x,t,\lambda)=K^*\left(t,x,\bar\lambda\right). \tag{11}$$

If $x=t$, *then*

$$K(x,x,\lambda)=K^*\left(x,x,\bar\lambda\right)-J. \tag{12}$$

Proof. The first assertion readily follows from (9) and property (4). Let us show that the kernel $K(x,t,\lambda)$ can be represented in the different (equivalent) form

$$K(x,t,\lambda)=\begin{cases}\mathscr{Y}(x,a,\lambda)M[\mathscr{Y}(b,a,\lambda)M-N]^{-1}\mathscr{Y}(b,t,\lambda)J & \text{for } x<t\\ \mathscr{Y}(x,b,\lambda)N[M-\mathscr{Y}(a,b,\lambda)N]^{-1}\mathscr{Y}(a,t,\lambda)J & \text{for } x>t.\end{cases} \tag{13}$$

To verify that the right-hand sides of (13) and (9) coincide for $x>t$, we rewrite the right-hand side of (9) on the basis of properties (1) and (2) in the form

$$\mathscr{Y}(x,a,\lambda)\{M-[M-\mathscr{Y}(a,b,\lambda)N]\}\times[M-\mathscr{Y}(a,b,\lambda)N]^{-1}\mathscr{Y}(a,t,\lambda)J,$$

which can obviously be reduced to (13) for $x>t$ after some simplifications.

Now let us prove relation (11). To be definite, we assume that $a \le x < t \le b$ (for $x > t$ or $x = t$, the proof follows the same scheme). Consider the difference

$$\begin{aligned} K(x,t,\lambda) - K^*\left(t,x,\bar\lambda\right) &= \mathscr{Y}(x,a,\lambda)M[\mathscr{Y}(b,a,\lambda)M - N]^{-1}\mathscr{Y}(b,t,\lambda)J \\ &\quad - \left\{\mathscr{Y}\left(t,b,\bar\lambda\right)N\left[M - \mathscr{Y}\left(a,b,\bar\lambda\right)N\right]^{-1}\mathscr{Y}\left(a,x,\bar\lambda\right)J\right\}^*. \end{aligned} \tag{14}$$

Performing the conjugation and taking account of the first two relations in (2) and properties (1)–(3), we obtain

$$\begin{aligned} K(x,t,\lambda) - K^*\left(t,x,\bar\lambda\right) &= \mathscr{Y}(x,a,\lambda)M[\mathscr{Y}(b,a,\lambda)M - N]^{-1}\mathscr{Y}(b,t,\lambda)J \\ &\quad + \mathscr{Y}(x,a,\lambda)\left[M^*J - N^*J\mathscr{Y}(b,a,\lambda)\right]^{-1}N^*J\mathscr{Y}(b,t,\lambda)J \\ &= \mathscr{Y}(x,a,\lambda)\left[M^*J - N^*J\mathscr{Y}(b,a,\lambda)\right]^{-1}\Big\{\left[M^*J - N^*J\mathscr{Y}(b,a,\lambda)\right]M \\ &\quad + N^*J[\mathscr{Y}(b,a,\lambda)M - N]\Big\}[\mathscr{Y}(b,a,\lambda)M - N]^{-1}\mathscr{Y}(b,t,\lambda)J \\ &= \mathscr{Y}(x,a,\lambda)\left[M^*J - N^*J\mathscr{Y}(b,a,\lambda)\right]^{-1}\left\{M^*JM - N^*JN\right\} \\ &\quad \times [\mathscr{Y}(b,a,\lambda)M - N]^{-1}\mathscr{Y}(b,t,\lambda)J. \end{aligned}$$

This, together with (4), implies that the difference (14) is zero and hence relation (11) is valid. The proof of the lemma is complete.

Let $\{\lambda_k\}_1^\infty$ be the sequence of eigenvalues of problem (6), (5) in which each λ_k is repeated according to its multiplicity;[4] therefore, to each λ_k there corresponds a set of r_k ($1 \le r_k \le n$) successive indices. Let $\{Y_k(x)\}_1^\infty$ be the corresponding system of eigenfunctions, which, by [6], is complete and orthonormal in the sense that $\langle Y_k, Y_\nu\rangle_A = \delta_{k\nu}$, where $\delta_{k\nu}$ is the Kronecker delta.

The following assertion can be derived from [5] in a straightforward manner.

Lemma 3. *If a vector function $\varphi \in \bar L_A^2[a,b]$ can be sourcewise represented in the form*

$$\varphi(x) = \int_a^b K(x,t,\lambda)\,dA(t)\chi(t),$$

where the function $A(x)$ and the kernel $K(x,t,\lambda)$ are defined in Section 2 and $\chi \in \bar L_A^2[a,b]$, then

$$\varphi(x) = \sum_{k=1}^\infty Y_k(x)c_k; \tag{15}$$

moreover, $c_k = \langle\varphi, Y_k\rangle_A$, and the series (15) converges absolutely and uniformly with respect to x[5] to $\varphi(x)$ on the interval $[a,b]$ and converges also in mean, i.e., $\|\varphi(x) - \sum_{k=1}^p Y_k(x)c_k\| \underset{p\to\infty}{\longrightarrow} 0$.

Corollary 1. *Let λ not be an eigenvalue of problem* (6), (5). *Then*

$$\langle\varphi, Y_k\rangle_A = \langle\chi, Y_k\rangle_A / (\lambda_k - \lambda).$$

Lemma 3 and Corollary 1 imply the following assertion.

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[4] That is, the multiplicity of λ_k viewed as a root of the characteristic equation $\det[N - \mathscr{Y}(b,a,\lambda)M] = 0$ of problem (6), (5).

[5] The convergence is understood as the absolute and uniform convergence of the n series formed by n elements of each of the vectors $Y_k(x)c_k$.

Corollary 2. *Let λ not be an eigenvalue of problem* (6), (5), *and let $\chi \in \bar{L}_A^2[a,b]$. Then*

$$\int_a^b K(x,t,\lambda)dA(t)\chi(t) = \sum_{k=1}^{\infty} Y_k(x)\frac{\langle \chi, Y_k\rangle_A}{\lambda_k - \lambda}, \tag{16}$$

and moreover, the series (16) *converges absolutely and uniformly with respect to x on $[a,b]$.*

Corollary 3. *Let λ not be an eigenvalue of problem* (6), (5), *and let $\chi, \psi \in \bar{L}_A^2[a,b]$. Then the analog*

$$\int_a^b\int_a^b \chi^*(t)dA(t)K(t,x,\lambda)dA(x)\psi(x) = \sum_{k=1}^{\infty} \frac{\overline{\langle \chi, Y_k\rangle_A}\,\langle \psi, Y_k\rangle_A}{\lambda_k - \lambda}$$

of the Hilbert formula is valid.

Corollary 4. *Let λ not be an eigenvalue of problem* (6), (5). *Then*

$$\int_a^b K(x,\tau,\lambda)dA(\tau)K^*(t,\tau,\lambda) = \sum_{k=1}^{\infty} \frac{Y_k(x)Y_k^*(t)}{|\lambda_k - \lambda|^2}; \tag{17}$$

moreover, the series (17) *is convergent*[6] *absolutely and uniformly with respect to each argument, the other argument being fixed.*

Indeed, the eigenfunction $Y_k(x)$ corresponding to the eigenvalue λ_k satisfies the differential equation

$$JY_k(x) = [B'(x) + \lambda A'(x)]\,Y + (\lambda_k - \lambda)\,A'(x)Y.$$

Therefore, by Theorem A,

$$Y_k(x) = (\lambda_k - \lambda)\int_a^b K(x,t,\lambda)dA(t)Y_k(t) \tag{18}$$

provided that λ is not an eigenvalue of problem (6), (5). Since [6, p. 28] $\lambda_k \in \mathrm{R}$, it follows from (18) and (2_3) that

$$\langle \chi, Y_k\rangle_A = \int_a^b Y_k^*(\tau)dA(\tau)K^*(t,\tau,\lambda)u = \frac{Y_k^*(t)}{\lambda_k - \bar{\lambda}}u. \tag{19}$$

Therefore, by using formula (16) for $\chi(\tau) = K^*(t,\tau,\lambda)u$, where u is an arbitrary constant vector, and by taking account of (19), we obtain (17).

Lemma 4. *The kernel $K(x,t,\lambda)$ can be expanded in the bilinear series*

$$\sum_{k=1}^{\infty} Y_k(x)Y_k^*(t)/\left(\lambda_k - \lambda\right)$$

in eigenfunctions of problem (6), (5), *which converges to $K(x,t,\lambda)$ in the sense that*

$$\int_a^b \left(K(x,t,\lambda) - \sum_{k=1}^{p}\frac{Y_k(x)Y_k^*(t)}{\lambda_k - \lambda}\right) dA(t) \left(K(x,t,\lambda) - \sum_{k=1}^{p}\frac{Y_k(x)Y_k^*(t)}{\lambda_k - \lambda}\right)^* \underset{p\to\infty}{\longrightarrow} 0 \tag{20}$$

for each given $x \in [a,b]$.

[6] In the sense that the n^2 matrix entries form n^2 absolutely and uniformly convergent function series.

Proof. By (17),

$$\int_a^b K(x,\tau,\lambda)dA(\tau)K^*(x,\tau,\lambda) = \sum_{k=1}^{\infty} \frac{Y_k(x)Y_k^*(x)}{|\lambda_k - \lambda|^2}. \tag{21}$$

Moreover, the series is convergent on the interval $[a, b]$ absolutely with respect to x; however, one cannot claim that it is uniformly convergent. By using (18) and (2_3) again, we obtain

$$\int_a^b \left(K(x,t,\lambda) - \sum_{k=1}^{p} \frac{Y_k(x)Y_k^*(t)}{\lambda_k - \lambda} \right) dA(t) \left(K(x,t,\lambda) - \sum_{k=1}^{p} \frac{Y_k(x)Y_k^*(t)}{\lambda_k - \lambda} \right)^*$$
$$= \int_a^b K(x,t,\lambda)dA(t)K^*(x,t,\lambda) - \sum_{k=1}^{p} \left(\int_a^b K(x,t,\lambda)dA(t)Y_k(t) \right) \frac{Y_k^*(x)}{\lambda_k - \bar\lambda}$$
$$- \sum_{k=1}^{p} \frac{Y_k(x)}{\lambda_k - \lambda} \int_a^b Y_k^*(t)dA(t)K^*(x,t,\lambda) + \sum_{k,\nu=1}^{p} \frac{Y_k(x)}{\lambda_k - \lambda} \left(\int_a^b Y_k^*(t)dA(t)Y_\nu(t) \right) \frac{Y_\nu^*(x)}{\lambda_\nu - \bar\lambda}$$
$$= \int_a^b K(x,t,\lambda)dA(t)K^*(x,t,\lambda) - \sum_{k=1}^{p} \frac{Y_k(x)Y_k^*(x)}{|\lambda_k - \lambda|^2}.$$

This, together with (21), implies (20).

4. STATEMENT OF AN EQUIVALENT PROBLEM

Without loss of generality, we assume that $\lambda = 0$ is not an eigenvalue of problem (6), (5). Let us treat Eq. (6) as a nonhomogeneous equation (with nonhomogeneity $\lambda A'Y$). Then, on the basis of Theorem A, the solution of problem (6), (5) can be represented via the loaded Fredholm–Stieltjes integral equation

$$Y(x) = \lambda \int_a^b K(x,t)dA(t)Y(t) \tag{22}$$

with Hermitian matrix (see Lemma 2) kernel $K(x,t)$ given by formula (9) with $\lambda = 0$. Further, obviously, the solution of problem (1), (5) can be represented in the form

$$Y(x) = \lambda \int_a^b K(x,t)dA(t)Y(t) + f(x), \tag{23}$$

where

$$f(x) = \int_a^b K(x,t)dF(t). \tag{24}$$

If $\lambda = 0$ is an eigenvalue of problem (6), (5), then we take a value λ^0 of the parameter λ that is not an eigenvalue (such a value always exists by [6]). Further, we rewrite Eq. (6) in the form $JY' = \left(B'(x) + \lambda^0 A'(x)\right) Y + \left(\lambda - \lambda^0\right) A'(x)Y$, or

$$JY' = \left({B^+}'(x) + \lambda^- A'(x) \right) Y, \tag{25}$$

where $B^+(x) = B(x) + \lambda^0 A(x)$ and $\lambda^- = \lambda - \lambda^0$. Then, obviously, $\lambda^- = 0$ is not an eigenvalue of problem (25), (5), and therefore, one can use the above-represented considerations. Note also that the eigenfunctions of problem (25), (5) coincide with those of problem (6), (5) with the only difference that $Y_k(x)$ corresponds to the "shifted" eigenvalue $\lambda_k^- = \lambda_k - \lambda^0$.

5. MAIN RESULTS

First, we consider the case in which λ is not an eigenvalue of problem (6), (5) and show that the solution of the nonhomogeneous problem (1), (5) can be represented in an equivalent form (an analog of the Schmidt formula) that has an advantage over the Fredholm formula (8) in that it explicitly shows the meromorphic character of the solution with respect to the parameter λ and indicates the leading part of the solution with respect to each pole $\lambda = \lambda_\nu$.

Theorem 1. *Let assumptions* (2)–(4) *and* (7) *be valid, and let* λ *not be an eigenvalue of problem* (6), (5). *Then the nonhomogeneous problem* (1), (5) *has the unique solution*

$$Y(x) = f(x) + \lambda \sum_{\nu=1}^{\infty} \left(Y_\nu(x) \frac{1}{\lambda_\nu (\lambda_\nu - \lambda)} \int_a^b Y_\nu^*(t) dF(t) \right), \tag{26}$$

where $f(x)$ *is given by* (24); *moreover, the series on the right-hand side in* (26) *is convergent absolutely and uniformly with respect to* x *on the interval* $[a, b]$.

Proof. It was shown in Section 4 that problem (1), (5) is equivalent to the integral equation (23). Consider the vector function

$$\varphi(x) \equiv \frac{Y(x) - f(x)}{\lambda} \equiv \int_a^b K(x,t) dA(t) Y(t). \tag{27}$$

It satisfies all assumptions of Lemma 1; therefore,

$$\varphi(x) = \sum_{\nu=1}^{\infty} \langle \varphi, Y_\nu \rangle_A Y_\nu(x), \tag{28}$$

and the series (28) converges absolutely and uniformly with respect to x on the interval $[a, b]$.

By Corollary 1, we have

$$\langle \varphi, Y_\nu \rangle_A = \langle Y, Y_\nu \rangle_A / \lambda_\nu. \tag{29}$$

On the other hand, $\langle \varphi, Y_\nu \rangle_A = \langle (Y - f)/\lambda, Y_\nu \rangle_A = \left(\langle Y, Y_\nu \rangle_A - \langle f, Y_\nu \rangle_A \right) / \lambda$. This, together with (29), implies that

$$(\lambda_\nu - \lambda) \langle Y, Y_\nu \rangle_A = \lambda_\nu \langle f, Y_\nu \rangle_A . \tag{30}$$

Since λ is not an eigenvalue, i.e., $\lambda_\nu - \lambda \neq 0$, we have

$$\langle Y, Y_\nu \rangle_A = \frac{\lambda_\nu}{\lambda_\nu - \lambda} \langle f, Y_\nu \rangle_A . \tag{31}$$

By taking account of (28) and (29), we obtain

$$\varphi(x) = \sum_{\nu=1}^{\infty} \frac{\langle f, Y_\nu \rangle_A}{\lambda_\nu - \lambda} Y_\nu(x). \tag{32}$$

This, together with (27), implies that $Y(x) = f(x) + \lambda \sum_{\nu=1}^{\infty} (\lambda_\nu - \lambda)^{-1} \langle f, Y_\nu \rangle_A Y_\nu(x)$. Further, since the eigenfunction $Y_\nu(x)$ satisfies Eq. (22) for $\lambda = \lambda_\nu$, it follows from (2_3), (11), and (12) that

$$\int_a^b Y_\nu^*(x) dA(x) K(x,t) = \frac{Y_\nu^*(t)}{\lambda_\nu - \bar{\lambda}} - Y_\nu^*(t) \Delta A(t) J.$$

By using formula (24) and the last relation, we obtain

$$\langle f, Y_\nu \rangle_A = \frac{1}{\lambda_\nu} \int_a^b Y_\nu^*(t) dF(t) - \int_a^b Y_\nu^*(t) \Delta A(t) J dF(t).$$

Since

$$\int_a^b Y_\nu^*(t) \Delta A(t) J dF(t) = \sum_{a \le t_k \le b} Y_\nu^*(t_k) \, \Delta A(t_k) \, J \Delta F(t_k),$$

where the sum is taken over all points of discontinuity of the functions $A(t)$ and $F(t)$ and is zero by (3), we have

$$\langle f, Y_\nu \rangle_A = \frac{1}{\lambda_\nu} \int_a^b Y_\nu^*(t) dF(t). \tag{33}$$

Therefore, for the solution of the integral equation (23) [and hence of problem (1), (5)], we have the Schmidt type formula (26).

Let us show that the series on the right-hand side in (26) is convergent on the interval $[a, b]$ absolutely and uniformly with respect to x. To this end, we rewrite it in the form

$$\sum_{\nu=1}^{\infty} (1 - \lambda/\lambda_\nu)^{-1} \lambda_\nu^{-1} \langle f, Y_\nu \rangle_A Y_\nu(x).$$

By Corollary 2, the series $\sum_{\nu=1}^{\infty} \lambda_\nu^{-1} \langle f, Y_\nu \rangle_A Y_\nu(x)$ is absolutely and uniformly convergent. Moreover, by [6, p. 28], the eigenvalues of problem (6), (5) have no finite limit points, i.e., $\lambda_\nu \to \infty$ as $\nu \to \infty$; therefore, $(1 - \lambda/\lambda_\nu)^{-1} \to 1$. Hence it follows that the series is absolutely and uniformly convergent with respect to x on the interval $[a, b]$. The solution (26) is unique. Indeed, if $Y^1(x)$ and $Y^2(x)$ $[Y^1(x) \neq Y^2(x)]$ are solutions of Eq. (23), then the function $Y^1(x) - Y^2(x)$ $(\neq 0)$ is a solution of the homogeneous equation (22); consequently, λ is an eigenvalue, which contradicts our assumptions. The proof of the theorem is complete.

Remark. If for a given value of the parameter λ the series $\sum_{\nu=1}^{\infty} Y_\nu(x) Y_\nu^*(t)/(\lambda_\nu - \lambda)$ is convergent on the interval $[a, b]$ uniformly with respect to t, then, by setting (see Lemma 5)

$$K(x, t, \lambda) = \sum_{\nu=1}^{\infty} Y_\nu(x) Y_\nu^*(t)/(\lambda_\nu - \lambda),$$

one can rewrite the solution of problem (1), (5) in the form (8).

Now consider the case in which λ coincides with some eigenvalue of problem (6), (5). For example, suppose that $\lambda = \lambda_k$ is an eigenvalue of multiplicity r_k $(1 \le r_k \le n)$, so that

$$\lambda = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+r_k-1}. \tag{34}$$

Theorem 2. *Let assumptions* (2)–(4) *and* (7) *be valid, and let* $\lambda = \lambda_k$ *be an eigenvalue of problem* (6), (5) *of rank* r_k. *The nonhomogeneous problem* (1), (5) *has a solution if and only if the following* r_k *conditions are satisfied*:

$$\int_a^b Y_\nu^*(t) dF(t) = 0, \qquad \nu \in \mathscr{K} \equiv \{k, \ldots, k + r_k - 1\}. \tag{35}$$

In this case, the solution is given by the formula

$$Y(x) = f(x) + \lambda \sum_{\nu=1,\, \nu \notin \mathscr{K}}^{\infty} \left(Y_\nu(x) \frac{1}{\lambda_\nu (\lambda_\nu - \lambda)} \int_a^b Y_\nu^*(t) dF(t) \right) + \sum_{i \in \mathscr{K}} c_i Y_i(x), \tag{36}$$

where the c_i *are arbitrary constants and* $f(x)$ *is given by* (24).

Proof of necessity. Arguing as in the previous case, we obtain (30). Since, by (34), $\lambda = \lambda_\nu$, $\nu \in \mathscr{K}$, it follows from the last relation that

$$\langle f, Y_\nu \rangle_A = 0, \qquad \nu \in \mathscr{K}. \tag{37}$$

Therefore, we obtain r_k necessary conditions for the existence of a solution of the integral equation (23) and hence of problem (1), (5). By (33), these conditions are equivalent to (35).

Proof of sufficiency. If $\nu \notin \mathscr{K}$, then $\lambda_\nu - \lambda \neq 0$; just as above, relation (31) is valid for such values of ν, and $\varphi(x)$ is given by (32), where the summation is performed over $\nu \in N \backslash \mathscr{K}$. However, for $\nu \in \mathscr{K}$, the coefficients $\langle Y, Y_\nu \rangle_A / \lambda_\nu$ become indefinite. Therefore, we write

$$Y(x) = f(x) + \sum_{\nu \in \mathscr{K}} c_\nu Y_\nu(x) + \lambda \sum_{\nu=1,\, \nu \notin \mathscr{K}}^{\infty} \frac{\langle f, Y_\nu \rangle_A}{\lambda_\nu - \lambda} Y_\nu(x), \tag{38}$$

which, by (33), coincides with (36).

The convergence of the series in (36) is obvious from the argument carried out in the previous case. The proof of the fact that the vector function $Y(x)$ is a solution of the nonhomogeneous problem (1), (5) can be carried out by a straightforward substitution of the expression (38) into Eq. (23) with regard to Corollary 2, relations (37), and the fact that the eigenfunctions $Y_\nu(x)$, $\nu \in \mathscr{K}$, of problem (6), (5), and hence their arbitrary linear combination, are solutions of the homogeneous equation (22). The proof of the theorem is complete.

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