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SOME ALGEBRAS OF SYMMETRIC ANALYTIC FUNCTIONS AND THEIR SPECTRA

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Abstract In the spectrum of the algebra of symmetric analytic functions of bounded type on ℓ_p, $1 \leqslant p < +\infty$, and along the same lines as the general non-symmetric case, we define and study a convolution operation and give a formula for the 'radius' function. It is also proved that the algebra of analytic functions of bounded type on ℓ_1 is isometrically isomorphic to an algebra of symmetric analytic functions on a polydisc of ℓ_1. We also consider the existence of algebraic projections between algebras of symmetric polynomials and the corresponding subspace of subsymmetric polynomials.

Keywords: Banach spaces; polynomials; symmetric polynomials; symmetric and subsymmetric analytic functions of bounded type; spectra of algebras

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1. Introduction and Preliminaries

Let X be a complex Banach space and let G be a semigroup of isometric operators on X. A function f on X is called *symmetric with respect to* G (or G-*symmetric* for short) if $f(\sigma(x)) = f(x)$ for every $\sigma \in G$. The basic example is $X = \ell_p$, $1 \leqslant p < \infty$, and $G = \mathcal{G}$, the group of permutations on the set of positive integers $\mathbb{N}$. Here we mean that $\sigma \in \mathcal{G}$ acts on ℓ_p by

$$\sigma\biggl(\sum_{i=1}^{\infty} x_i e_i\biggr) = \sum_{i=1}^{\infty} x_i e_{\sigma(i)},$$

where $\{e_1, e_2, \dots\}$ is the standard basis in ℓ_p. Another important example is $X = \ell_p$ and $G = \mathfrak{G}$, the semigroup generated by the isometric operators β_i,

$$\beta_i \colon (x_1, x_2, \dots) \mapsto (x_1, \dots, x_{i-1}, 0, x_i, \dots).$$

In the literature, $\mathcal{G}$-symmetric functions on ℓ_p are called *symmetric* and $\mathfrak{S}$-symmetric functions are called *subsymmetric*.

We use the notation $\mathcal{P}(X)$ for the algebra of all polynomials on X and $\mathcal{P}_{\mathrm{s}}(\ell_p)$ (respectively, $\mathcal{P}_{\mathrm{s}_{\mathrm{b_s}}}(\ell_p)$) for the algebra of all symmetric (respectively, subsymmetric) polynomials on ℓ_p. Also, $\mathcal{P}({}^nX)$ (respectively, $\mathcal{P}_{\mathrm{s}}({}^n\ell_p)$ and $\mathcal{P}_{\mathrm{s}_{\mathrm{b_s}}}({}^n\ell_p)$) denotes the Banach space of n-homogeneous polynomials on X (respectively, n-homogeneous symmetric and n-homogeneous subsymmetric polynomials on ℓ_p). The completion of $\mathcal{P}(X)$ in the metric of uniform convergence on bounded sets coincides with the algebra of entire analytic functions of bounded type $\mathcal{H}_{\mathrm{b}}(X)$ on X. We denote by $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ (respectively, $\mathcal{H}_{\mathrm{bs}_{\mathrm{b_s}}}(\ell_p)$) the subalgebra of all symmetric (respectively, subsymmetric) functions in $\mathcal{H}_{\mathrm{b}}(\ell_p)$. We also use the notation $M_{\mathrm{b}}(\ell_p)$, $M_{\mathrm{bs}}(\ell_p)$ and $M_{\mathrm{bs}_{\mathrm{b_s}}}(\ell_p)$ for spectra of the algebras $\mathcal{H}_{\mathrm{b}}(\ell_p)$, $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ and $\mathcal{H}_{\mathrm{bs}_{\mathrm{b_s}}}(\ell_p)$, respectively, that is, the set of all non-null continuous complex-valued homomorphisms.

We continue the study of the spectra of several algebras of symmetric and subsymmetric analytic functions [**1**, **7**–**9**, **12**] and discuss some connections among them.

We begin § 2 by dealing with the radius function of elements in $M_{\mathrm{bs}}(\ell_p)$ and introduce a convolution product there in the same spirit as in the non-symmetric case. This requires a kind of 'symmetric translation' that we obtain by using a new tool: the *intertwining* operation (see Definition 2.2). Then we construct an algebra of symmetric analytic functions on a kind of polydisc in ℓ_1 that is isometrically algebraically isomorphic to the algebra $\mathcal{H}_{\mathrm{b}}(\ell_1)$. We also construct operators of symmetrization on spaces of bounded functions. In particular, $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ is a complemented subspace of $\mathcal{H}_{\mathrm{b}}(\ell_p)$. Unfortunately, the corresponding projection is not multiplicative. In § 3 we find an algebraic basis for the algebra generated by subsymmetric polynomials of degree less than or equal to 3. It enables us to prove that this algebra is algebraically complemented in the algebra generated by all polynomials of degree less than or equal to 3.

We assume that all polynomials appearing in the paper are continuous. Symmetric polynomials on ℓ_p (with respect to $\mathcal{G}$) and $L_p[0,1]$ (with respect to the group of measure-preserving permutations on $[0,1]$) for $1 \leqslant p < \infty$ were first studied by Nemirovski and Semenov in [**12**]. In [**7**] González *et al.* investigated algebraic bases of various algebras of symmetric polynomials on rearrangement-invariant spaces. In particular, it is shown in [**7**] that a polynomial P on ℓ_p is symmetric with respect to $\mathcal{G}$ if and only if it is symmetric with respect to the subgroup $\mathcal{G}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$, where $\mathcal{G}_n$ is the group of permutations on $\{1,\dots,n\}$. Also, in [**7**] it is proved that, similarly to the classical finite-dimensional case, the polynomials

$$F_k(x) = \sum_{i=1}^{\infty} x_i^k, \quad k = \lceil p \rceil, \lceil p \rceil + 1, \dots, \tag{1.1}$$

form an *algebraic basis* in the algebra of all symmetric polynomials on ℓ_p (here $\lceil p \rceil$ is the smallest integer that is greater than or equal to p). This means that, for every symmetric polynomial P of degree $\lceil p \rceil + n - 1$, $n \geqslant 1$, there is a polynomial q on $\mathbb{C}^n$ such that $P(x) = q(F_{\lceil p \rceil}(x), \dots, F_{\lceil p \rceil + n-1}(x))$. Actually, q is unique, as pointed out in [**1**]. Using these facts, Alencar *et al.* [**1**] investigated the spectrum of the algebra of symmetric uniformly continuous analytic functions on the unit ball B_{ℓ_p}, $A_{\mathrm{us}}(B_{\ell_p})$. Note that if $\deg P < \lceil p \rceil$, then P is not symmetric.

Subsymmetric polynomials were investigated in [**8**, **9**, **12**]. Gonzalo shows in [**8**, Theorem 2.1] that the so-called *standard* polynomials

$$F_{k_1,\dots,k_n}(x) = \sum_{i_1<\dots<i_n} x_{i_1}^{k_1}\dots x_{i_n}^{k_n}, \quad k_1+\dots+k_n = n, \tag{1.2}$$

form a linear basis in the finite-dimensional space of n-homogeneous subsymmetric polynomials for $n \geqslant \lceil p \rceil$.

Let A_P denote the symmetric (here, symmetric has the usual (different) meaning of invariant regarding permutations of the variables) n-linear form associated with the n-homogeneous polynomial P on ℓ_p. We can write

$$P(x) = \sum_{i_1<\dots<i_n} \sum_{k_1+\dots+k_n=n} \frac{n!}{k_1!\cdots k_n!} A_p(e_{i_1}^{k_1},\dots,e_{i_n}^{k_n}) x_{i_1}^{k_1}\cdots x_{i_n}^{k_n}.$$

If P is subsymmetric, then $A_P(e_{i_1}^{k_1},\dots,e_{i_n}^{k_n}) = A_P(e_1^{k_1},\dots,e_n^{k_n})$ for every $i_1<\dots<i_n$ and $k_1+\dots+k_n = n$.

For further details on analytic functions on infinite-dimensional spaces, we refer the interested reader to [**6**, **11**]. For details on spectra of algebras of analytic functions on Banach spaces, we refer the interested reader to [**2**, **3**].

2. The algebra of symmetric analytic functions on ℓ_p

2.1. The radius function on $M_{\mathrm{bs}}(\ell_p)$

Following [**2**] we define the *radius function* R on $M_{\mathrm{bs}}(\ell_p)$ (respectively, $M_{\mathrm{bs}_{\mathrm{bs}}}(\ell_p)$) by assigning to any complex homomorphism $\phi \in M_{\mathrm{bs}}(\ell_p)$ (respectively, $\phi \in M_{\mathrm{bs}_{\mathrm{bs}}}(\ell_p)$) the infimum $R(\phi)$ of all r such that ϕ is continuous with respect to the norm of uniform convergence on the ball rB_{ℓ_p}, that is, $|\phi(f)| \leqslant C_r\|f\|_r$. Furthermore, we have $|\phi(f)| \leqslant \|f\|_{R(\phi)}$.

As in the non-symmetric case, we obtain the following formula for the radius function.

Proposition 2.1. *Let $\phi \in M_{\mathrm{bs}}(\ell_p)$ (respectively, $M_{\mathrm{bs}_{\mathrm{bs}}}(\ell_p)$). Then*

$$R(\phi) = \limsup_{n\to\infty} \|\phi_n\|^{1/n}, \tag{2.1}$$

where ϕ_n is the restriction of ϕ to $\mathcal{P}_{\mathrm{s}}({}^n\ell_p)$ (respectively, $\mathcal{P}_{\mathrm{s}_{\mathrm{bs}}}({}^n\ell_p)$) and $\|\phi_n\|$ is its corresponding norm.

Proof. To prove (2.1) we use arguments from [**2**, Theorem 2.3]. Recall that

$$\|\phi_n\| = \sup\{|\phi_n(P)|\colon P \in \mathcal{P}_{\mathrm{s}}({}^n\ell_p) \text{ with } \|P\| \leqslant 1\}.$$

Suppose that

$$0 < t < \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Then there is a sequence of homogeneous symmetric polynomials P_j of degree $n_j \to \infty$ such that $\|P_j\| = 1$ and $|\phi(P_j)| > t^{n_j}$. If $0 < r < t$, then, by homogeneity,

$$\|P_j\|_r = \sup_{x \in rB_{\ell_p}} |P_j(x)| = r^{n_j},$$

so that

$$|\phi(P_j)| > (t/r)^{n_j} \|P_j\|_r,$$

and ϕ is not continuous for the $\|\cdot\|_r$ norm. It follows that $R(\phi) \geqslant r$, and, on account of the arbitrary choice of r, we obtain

$$R(\phi) \geqslant \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

Now let

$$s > \limsup_{n\to\infty} \|\phi_n\|^{1/n}$$

so that $s^m \geqslant \|\phi_m\|$ for large m. Then there exists $c \geqslant 1$ such that $\|\phi_m\| \leqslant cs^m$ for every m. If $r > s$ is arbitrary and $f \in \mathcal{H}_{\mathrm{bs}}(\ell_p)$ has Taylor series expansion

$$f = \sum_{n=1}^{\infty} f_n,$$

then

$$r^m \|f_m\| = \|f_m\|_r \leqslant \|f\|_r, \quad m \geqslant 0.$$

Hence,

$$|\phi(f_m)| \leqslant \|\phi_m\| \, \|f_m\| \leqslant \frac{cs^m}{r^m} \|f\|_r$$

and so

$$\|\phi(f)\| \leqslant c\biggl(\sum \frac{s^m}{r^m}\biggr)\|f\|_r.$$

Thus, ϕ is continuous with respect to the uniform norm on rB, and $R(\phi) \leqslant r$. Since r and s are arbitrary,

$$R(\phi) \leqslant \limsup_{n\to\infty} \|\phi_n\|^{1/n}.$$

The same arguments work for subsymmetric bounded-type entire functions. □

2.2. The intertwining operator

Recall that in [**2**] the convolution operation '$*$' for elements φ, θ in the spectrum $M_{\mathrm{b}}(X)$ of $\mathcal{H}_{\mathrm{b}}(X)$ is defined by

$$(\varphi * \theta)(f) = \varphi(\theta(f(\cdot + x))), \quad \text{where } f \in \mathcal{H}_{\mathrm{b}}(X). \tag{2.2}$$

Here we introduce the analogous convolution in our symmetric setting.

It is easy to see that if f is a symmetric function on ℓ_p, then, in general, $f(\cdot + y)$ is not symmetric for a fixed y. However, it is possible to introduce an analogue of the translation operator that preserves the space of symmetric functions on ℓ_p.

Definition 2.2. Let $x, y \in \ell_p$, $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$. We define the *intertwining* of x and y, $x \bullet y \in \ell_p$, according to

$$x \bullet y = (x_1, y_1, x_2, y_2, \dots).$$

Let us indicate some elementary properties of the intertwining.

Proposition 2.3. *Given $x, y \in \ell_p$, the following assertions hold.*

(i) *If $x = \sigma_1(u)$ and $y = \sigma_2(v)$, $\sigma_1, \sigma_2 \in \mathcal{G}$, then $x \bullet y = \sigma(u \bullet v)$ for some $\sigma \in \mathcal{G}$.*

(ii) $\|x \bullet y\|^p = \|x\|^p + \|y\|^p$.

(iii) *$F_n(x \bullet y) = F_n(x) + F_n(y)$ for every $n \geqslant p$, where the F_n are given by (1.1).*

Proposition 2.4. *If $f(x) \in \mathcal{H}_{\mathrm{bs}}(\ell_p)$, then $f(x \bullet y) \in \mathcal{H}_{\mathrm{bs}}(\ell_p)$ for every fixed $y \in \ell_p$.*

Proof. Note that $x \bullet y = x \bullet 0 + 0 \bullet y$ and that the map $x \mapsto x \bullet 0$ is linear. Thus, the map $x \mapsto x \bullet y$ is analytic and maps bounded sets into bounded sets, and so does its composition with f. Moreover, $f(x \bullet y)$ is obviously symmetric. Hence, it belongs to $\mathcal{H}_{\mathrm{bs}}(\ell_p)$. □

The mapping $f \mapsto T_y^{\mathrm{s}}(f)$, where $T_y^{\mathrm{s}}(f)(x) = f(x \bullet y)$ will be referred as to the intertwining operator.

Proposition 2.5. *For every $y \in \ell_p$, the intertwining operator T_y^{s} is a continuous endomorphism of $\mathcal{H}_{\mathrm{bs}}(\ell_p)$.*

Proof. Evidently, T_y^{s} is linear and multiplicative. Let x belong to ℓ_p and $\|x\| \leqslant r$. Then $\|x \bullet y\| \leqslant \sqrt[p]{r^p + \|y\|^p}$ and

$$|T_y^{\mathrm{s}} f(x)| \leqslant \sup_{\|z\| \leqslant \sqrt[p]{r^p + \|y\|^p}} |f(z)| = \|f\|_{\sqrt[p]{r^p + \|y\|^p}}, \tag{2.3}$$

so T_y^{s} is continuous. □

Using the intertwining operator we can introduce a symmetric convolution on $\mathcal{H}_{\mathrm{bs}}(\ell_p)'$. For any θ in $\mathcal{H}_{\mathrm{bs}}(\ell_p)'$, according to (2.3), the radius function $R(\theta \circ T_y^{\mathrm{s}}) \leqslant \sqrt[p]{R(\theta)^p + \|y\|^p}$. Then, arguing as in [**2**, Theorem 6.1], it turns out that, for fixed $f \in \mathcal{H}_{\mathrm{bs}}(\ell_p)$, the function $y \mapsto \theta \circ T_y^{\mathrm{s}}(f)$ also belongs to $\mathcal{H}_{\mathrm{bs}}(\ell_p)$.

Definition 2.6. For any ϕ and θ in $\mathcal{H}_{\mathrm{bs}}(\ell_p)'$, their *symmetric convolution* is defined according to

$$(\phi \star \theta)(f) = \phi(y \mapsto \theta(T_y^{\mathrm{s}} f)).$$

Corollary 2.7. *If $\phi, \theta \in M_{\mathrm{bs}}(\ell_p)$, then $\phi \star \theta \in M_{\mathrm{bs}}(\ell_p)$.*

Proof. The multiplicativity of T_y^{s} yields that $\phi \star \theta$ is a character. Using inequality (2.3), we obtain that

$$R(\phi \star \theta) \leqslant \sqrt[p]{R(\phi)^p + R(\theta)^p}.$$

Hence, $\phi \star \theta \in M_{\mathrm{bs}}(\ell_p)$. □

Next, we look at the relationship between the spectra of $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ and $\mathcal{H}_{\mathrm{b}}(\ell_p)$. If $\varphi \in M_{\mathrm{b}}(\ell_p)$, then the restriction φ^{s} of φ to $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ is a complex homomorphism of $\mathcal{H}_{\mathrm{bs}}(\ell_p)$. According to [**13**] or [**14**], there exists a sequence of Banach spaces $(E_n)_{n=1}^{\infty}$ and a sequence of maps $\delta^{(n)}\colon E_n \to M_{\mathrm{b}}(\ell_p)$, where $E_1 = \ell_p$, the space E_n coincides with the subspace of all functionals on $\mathcal{P}(^n\ell_p)$ that vanish on finite sums of products of polynomials of degree less than n, and $\delta^{(1)}(z)(f) = f(z)$ such that, for every $\varphi \in M_{\mathrm{b}}(\ell_p)$,

$$\varphi(f) = \mathop{*}_{n=1}^{\infty} \delta^{(n)}(u_n)(f) \tag{2.4}$$

for some $u_n \in E_n$, $n = 1, 2, \dots$.

Hence, for every $\varphi \in M_{\mathrm{b}}(\ell_p)$, φ^{s} has the representation

$$\varphi^{\mathrm{s}} = \Big(\mathop{*}_{n=1}^{\infty} \delta^{(n)}(u_n)\Big)^{\mathrm{s}}.$$

Can we extend this formula for an arbitrary complex homomorphism of $\mathcal{H}_{\mathrm{bs}}(\ell_p)$? Clearly, it can be done if we can extend each character in $M_{\mathrm{bs}}(\ell_p)$ to a character in $M_{\mathrm{b}}(\ell_p)$.

Proposition 2.8. *If there exists a continuous homomorphism $\Phi\colon \mathcal{H}_{\mathrm{b}}(\ell_p) \to \mathcal{H}_{\mathrm{bs}}(\ell_p)$, then every character $\theta \in M_{\mathrm{bs}}(\ell_p)$ can be extended to a character $\theta^0 \in M_{\mathrm{b}}(\ell_p)$ by the formula $\theta^0 = \theta \circ \Phi$. If, moreover, Φ is a projection, then $(\theta^0)^{\mathrm{s}} = \theta$ and, furthermore, for every φ and $\theta \in M_{\mathrm{bs}}(\ell_p)$, we have ψ and $\xi \in M_{\mathrm{b}}(\ell_p)$ such that*

$$(\varphi \star \theta) = (\psi * \xi)^{\mathrm{s}}.$$

Proof. Let φ and θ be arbitrary elements in $M_{\mathrm{bs}}(\ell_p)$. It is clear that $\theta^0 = \theta \circ \Phi$ and $\phi^0 = \phi \circ \Phi$ belong to $M_{\mathrm{b}}(\ell_p)$. According to [**2**], there exist nets (x_α) and (y_β) in ℓ_p such that $\varphi(\Phi(P)) = \lim_\alpha P(x_\alpha)$ and $\theta(\Phi(P)) = \lim_\beta P(y_\beta)$ for all polynomials $P \in \mathcal{P}(\ell_p)$. Let us suppose that Φ is a projection. Then, $\varphi(P) = \lim_\alpha P(x_\alpha)$ and $\theta(P) = \lim_\beta P(y_\beta)$ for all polynomials $P \in \mathcal{P}_{\mathrm{s}}(\ell_p)$. It is a simple calculation to check that

$$(\varphi \star \theta)(F_k) = \lim_\alpha \lim_\beta F_k(x_\alpha \bullet y_\beta) \quad \text{for all } k.$$

Put $\mathfrak{L}(x) := (x \bullet 0)$ and $\mathfrak{R}(x) := (0 \bullet x)$. Both $\mathfrak{L}$ and $\mathfrak{R}$ are analytic self-maps of ℓ_p. Therefore, we have the composition operators $C_{\mathfrak{L}}(f) = f \circ \mathfrak{L}$ and $C_{\mathfrak{R}}(f) = f \circ \mathfrak{R}$. Both act on $\mathcal{H}_{\mathrm{b}}(\ell_p)$ and leave $\mathcal{H}_{\mathrm{bs}}(\ell_p)$ invariant. Define $\psi := \varphi^0 \circ C_{\mathfrak{L}}$ and $\xi := \theta^0 \circ C_{\mathfrak{R}}$. If we set $x'_\alpha = x_\alpha \bullet 0$ and $y'_\beta = 0 \bullet y_\beta$, it turns out that

$$\psi(P) = \lim_\alpha P(x'_\alpha) \quad \text{and} \quad \xi(P) = \lim_\beta P(x'_\beta).$$

Now, since

$$(\psi * \xi)(F_k) = \lim_\alpha \lim_\beta F_k(x'_\alpha + y'_\beta) = \lim_\alpha \lim_\beta F_k(x_\alpha \bullet y_\beta) = (\varphi \star \theta)(F_k) \quad \text{for all } k,$$

these identities are also true for every symmetric polynomial. Thus, the above equation holds for every function $f \in \mathcal{H}_{\mathrm{bs}}(\ell_p)$. □

Note that, for the finite-dimensional case, there is an algebraic isomorphism Φ_n from the algebra of entire functions $\mathcal{H}(\mathbb{C}^n)$ to the algebra of symmetric entire functions $\mathcal{H}_{\mathrm{s}}(\mathbb{C}^n)$ given by

$$\Phi_n \colon f(t_1, \dots, t_n) \mapsto f\biggl(\sum_{k=1}^{n} t_k, \sum_{k=1}^{n} t_k^2, \dots, \sum_{k=1}^{n} t_k^n\biggr).$$

This holds because the map

$$\mathcal{F}_n \colon (t_1, \dots, t_n) \mapsto \biggl(\sum_{k=1}^{n} t_k, \sum_{k=1}^{n} t_k^2, \dots, \sum_{k=1}^{n} t_k^n\biggr)$$

acts from $\mathbb{C}^n$ onto $\mathbb{C}^n$ (see, for example, [**1**]).

As the following proposition will show, this type of construction fails in the infinite-dimensional case, since it implies that the range $\mathcal{F}(\ell_p)$ of

$$\mathcal{F} \colon x \mapsto (F_{\lceil p \rceil}(x), \dots, F_n(x), \dots)$$

is never a linear space for any $1 \leqslant p \leqslant \infty$.

Proposition 2.9. *If* $(\xi_1, \dots, \xi_n, \dots)$ *is a non-zero sequence in* $\mathcal{F}(\ell_p)$*, then* $(-\xi_1, \dots, -\xi_n, \dots)$ *does not belong to* $\mathcal{F}(\ell_p)$*.*

Proof. Let $x \in \ell_p$, $x \neq 0$, such that $F_n(x) = \xi_n$, $n \geqslant \lceil p \rceil$. Suppose that there exists $y \in \ell_p$ such that $F_n(y) = -\xi_n$, $n \geqslant \lceil p \rceil$. Then $F_n(x \bullet y) = \xi_n - \xi_n = 0$ for all $n \geqslant \lceil p \rceil$. According to [**1**], $x \bullet y = 0$, but this is impossible because $x \neq 0$. $\square$

Let (ξ_n) be a sequence of complex numbers. Consider a map $\Xi \colon F_n \mapsto \xi_n$ for all n, and extend it to a map on $\mathcal{P}_{\mathrm{s}}(\ell_p)$ by linearity and multiplicativity, that is, for

$$P = q(F_{\lceil p \rceil}, \dots, F_{\lceil p \rceil + m - 1}),$$

define

$$\Xi(P) = \Xi(q(F_{\lceil p \rceil}, \dots, F_{\lceil p \rceil + m - 1})) = q(\xi_1, \dots, \xi_m).$$

It is evident that Ξ is a homomorphism.

Question 2.10. Under which conditions on (ξ_n) is the map Ξ continuous on $\mathcal{P}_{\mathrm{s}}(\ell_p)$?

If there exists x such that $F_k(x) = \xi_k$ for all k, then it is clear that Ξ is continuous.

Next we show that there is a continuous homomorphism Ξ which is not an evaluation at some point of ℓ_p. In [**1**, Example 3.1] a continuous homomorphism φ on the uniform algebra $A_{\mathrm{us}}(B_{\ell_p})$ was constructed such that $\varphi(F_p) = 1$ and $\varphi(F_i) = 0$ for all $i > p$. Note that $A_{\mathrm{us}}(B_{\ell_p}) \supset \mathcal{H}_{\mathrm{bs}}(\ell_p)$ and $R(\varphi) = 1$; hence

$$\Xi := \varphi|_{\mathcal{H}_{\mathrm{bs}}(\ell_p)} \in M_{\mathrm{bs}}(\ell_p).$$

However, there is no point $x \in \ell_p$ for which the linear multiplicative functional of evaluation at x is equal to Ξ [**1**, Corollary 1.4].

2.3. An algebra of symmetric functions on the polydisc of ℓ_1

Let us denote
$$\mathbb{D} = \left\{ x = \sum_{i=1}^{\infty} x_i e_i \in \ell_1 \colon \sup_i |x_i| < 1 \right\}.$$
It is easy to see that $\mathbb{D}$ is an open unbounded set. We shall call $\mathbb{D}$ the polydisc in ℓ_1.

Lemma 2.11. *For every $x \in \mathbb{D}$, the sequence $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$ belongs to ℓ_1.*

Proof. First, let us consider $x \in \ell_1$ such that
$$\|x\| = \sum_{i=1}^{\infty} |x_i| < 1$$
and let us calculate $\mathcal{F}(x) = (F_k(x))_{k=1}^{\infty}$. We have
$$\begin{aligned}
\|\mathcal{F}(x)\| &= \sum_{k=1}^{\infty} |F_k(x)| \\
&= \sum_{k=1}^{\infty} \left| \sum_{i=1}^{\infty} x_i^k \right| \\
&\leqslant \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} |x_i|^k \\
&\leqslant \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} |x_i| \right)^k \\
&= \sum_{k=1}^{\infty} \|x\|^k \\
&= \frac{\|x\|}{1 - \|x\|} \\
&< \infty.
\end{aligned}$$

In particular,
$$\|\mathcal{F}(\lambda e_k)\| = \frac{|\lambda|}{1 - |\lambda|} \quad \text{for } |\lambda| < 1.$$
If x is an arbitrary element in $\mathbb{D}$, pick $m \in \mathbb{N}$ such that
$$\sum_{i=m+1}^{\infty} |x_i| < 1.$$
Set $u = x - (x_1, \ldots, x_m, 0 \ldots)$ and note that
$$F_k(x) = F_k(x_1 e_1) + \cdots + F_k(x_m e_m) + F_k(u)$$

with

$$\|x_k e_k\| < 1, \quad k = 1, \dots, m, \qquad \|u\| < 1.$$

Also,

$$\|\mathcal{F}(x_k e_k)\| \leqslant \frac{\|x\|_\infty}{1 - \|x\|_\infty}.$$

Hence,

$$\begin{aligned}\|\mathcal{F}(x)\| &= \left\| \sum_{k=1}^{m} \mathcal{F}(x_k e_k) + \mathcal{F}(u) \right\| \\ &\leqslant \sum_{k=1}^{m} \|\mathcal{F}(x_k e_k)\| + \|\mathcal{F}(u)\| \\ &< \infty.\end{aligned}$$

□

Note that $\mathcal{F}$ is an analytic mapping from $\mathbb{D}$ into ℓ_1, since $\mathcal{F}(x)$ can be represented as a convergent series

$$\mathcal{F}(x) = \sum_{k=1}^{\infty} F_k(x) e_k$$

for every $x \in \mathbb{D}$ and $\mathcal{F}$ is bounded in a neighbourhood of zero [**5**, p. 58].

Proposition 2.12. *Let $g_1, g_2 \in \mathcal{H}_\mathrm{b}(\ell_1)$. If $g_1 \neq g_2$, then there exists $x \in \mathbb{D}$ such that $g_1(\mathcal{F}(x)) \neq g_2(\mathcal{F}(x))$.*

Proof. It is sufficient to show that if, for some $g \in \mathcal{H}_\mathrm{b}(\ell_1)$, we have $g(\mathcal{F}(x)) = 0$ for all $x \in \mathbb{D}$, then $g(x) \equiv 0$.

Let

$$g(x) = \sum_{n=1}^{\infty} Q_n(x),$$

where $Q_n \in \mathcal{P}(^n\ell_1)$ and

$$Q_n\left(\sum_{n=1}^{\infty} x_i e_i\right) = \sum_{k_1+\cdots+k_n=n} \sum_{i_1<\cdots<i_n} q_{n,i_1\cdots i_n} x_{i_1}^{k_1} \cdots x_{i_n}^{k_n}.$$

For any fixed $x \in \mathbb{D}$ and $t \in \mathbb{C}$ such that $tx \in \mathbb{D}$, let

$$g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x)$$

be the Taylor series at the origin. Then

$$\sum_{n=1}^{\infty} Q_n(\mathcal{F}(tx)) = g(\mathcal{F}(tx)) = \sum_{j=1}^{\infty} t^j r_j(x).$$

Let us compute $r_m(x)$. We have

$$r_m(x) = \sum_{\substack{k<m \\ k_1 i_1+\cdots+k_n i_n=m}} q_{k,i_1\cdots i_n} F_{i_1}^{k_1}(x)\cdots F_{i_n}^{k_n}(x). \tag{2.5}$$

It is easy to see that the sum on the right-hand side of (2.5) is finite.

Since $g(\mathcal{F}(x)) = 0$ for every $x \in \mathbb{D}$, we have $r_m(x) = 0$ for every m. Furthermore, as $F_1, \ldots, F_n$ are algebraically independent, $q_{k,i_1\cdots i_n} = 0$ in (2.5) for an arbitrary $k < m$, $k_1 i_1 + \cdots + k_n i_n = m$. Since this is true for every m, $Q_n \equiv 0$ for $n \in \mathbb{N}$. So $g(x) \equiv 0$ on ℓ_1. $\square$

Let us denote by $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ the algebra of all symmetric analytic functions that can be represented by $f(x) = g(\mathcal{F}(x))$, where $g \in \mathcal{H}_{\mathrm{b}}(\ell_1)$, $x \in \mathbb{D}$. In other words, $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ is the range of the one-to-one composition operator $C_{\mathcal{F}}(g) = g \circ \mathcal{F}$ acting on $\mathcal{H}_{\mathrm{b}}(\ell_1)$. According to Proposition 2.12, the correspondence $\Psi\colon f \mapsto g$ is a bijection from $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ onto $\mathcal{H}_{\mathrm{b}}(\ell_1)$. Thus, we endow $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ with the topology that turns the bijection Ψ into a homeomorphism. This topology is the weakest topology on $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ in which the following seminorms are continuous:

$$q_r(f) := \|(\Psi(f))\|_r = \|g\|_r = \sup_{\|x\|_{\ell_1} \leqslant r} |g(x)|, \quad r \in \mathbb{Q}.$$

Note that Ψ is a homomorphism of algebras. So we have proved the following proposition.

Proposition 2.13. *There is an onto isometric homomorphism between the algebras $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ and $\mathcal{H}_{\mathrm{b}}(\ell_1)$.*

Corollary 2.14. *The spectrum $M(\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D}))$ of $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ can be identified with $M_{\mathrm{b}}(\ell_1)$. In particular, $\ell_1 \subset M(\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D}))$, that is, for arbitrary $z \in \ell_1$ there is a homomorphism $\psi_z \in M(\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D}))$ such that $\psi_z(f) = \Psi(f)(z)$.*

The following example shows that there exists a character on $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ that is not an evaluation at any point of $\mathbb{D}$.

Example 2.15. Let us consider a sequence of real numbers (a_n), $|a_n| < 1$, such that $(a_n) \in \ell_2 \setminus \ell_1$ and such that the series $\sum_{n=1}^{\infty} a_n$ conditionally converges to some number C. Despite $(a_n) \notin \ell_1$, evaluations on (a_n) are determined for every symmetric polynomial on ℓ_1. In particular,

$$F_1((a_n)) = C, \qquad F_k((a_n)) = \sum a_n^k < \infty \quad \text{and} \quad \{F_k((a_n))\}_{k=1}^{\infty} \in \ell_1.$$

So (a_n) 'generates' a character on $\mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$ by the formula $\varphi(f) = \Psi(f)(\mathcal{F}((a_n)))$.

Since $(a_n) \in \ell_2$, $F_k((a_{\pi(n)})) = F_k((a_n))$, $k > 1$. Note that there exists a permutation on the set of positive integers, π, such that

$$\sum_{n=1}^{\infty} a_{\pi(n)} = C' \neq C.$$

For such a permutation π we may use the same construction as above and obtain a homomorphism φ_π 'generated by evaluation at $(a_{\pi(n)})$', $\varphi_\pi(f) = \Psi(f)(\mathcal{F}((a_{\pi(n)})))$.

Let us suppose that there exist $x, y \in \mathbb{D}$ such that $\varphi(f) = f(x)$ and $\varphi_\pi(f) = f(y)$ for every function $f \in \mathcal{H}_{\mathrm{s}}^{\ell_1}(\mathbb{D})$. Since $\varphi(F_k) = \varphi_\pi(F_k)$, $k \geqslant 2$, it follows from [**1**, Corollary 1.4] that there is a permutation of the indices that transforms the sequence x into the sequence y. But this cannot be true, because $F_1(x) = \varphi(F_1) \neq \varphi_\pi(F_1) = F_1(y)$. Thus, at least one of the homomorphisms φ and φ_π is not an evaluation at some point of $\mathbb{D}$.

Note that the homomorphism 'generated by evaluation at (a_n)' is a character on $\mathcal{P}_{\mathrm{s}}(\ell_1)$ too, but we do not know whether this character is continuous in the topology of uniform convergence on bounded sets.

2.4. Mean symmetrization

For a given topological semigroup G, $\mathcal{B}(G)$ denotes the Banach algebra of all bounded complex functions on G, and $\mathcal{C}(G)$ denotes its subalgebra of continuous functions.

Let U be a subalgebra of $\mathcal{B}(G)$. A *mean* of U is a complex-valued linear functional φ on U that is positive (that is, $\varphi(f) \geqslant 0$ whenever $f \geqslant 0$, $f \in U$) and $\varphi(1) = 1$. A mean φ is called *invariant* (or *bi-invariant*) if it is invariant with respect to both left and right translation by any element of $g \in G$.

A topological semigroup G is called *amenable* if there is an invariant mean on $\mathcal{B}(G)$. It is well known [**4**, p. 89] that $\mathcal{G}_0$ is an amenable group. Let λ be the (discrete) Haar measure on $\mathcal{G}_0$, $\lambda(\sigma) = 1$ for any $\sigma \in \mathcal{G}_0$. It is easy to see that, for every $\sigma \in \mathcal{G}_0$,

$$\lim_{n\to\infty} \frac{\lambda(\sigma\mathcal{G}_n \Delta \mathcal{G}_n)}{\lambda(\mathcal{G}_n)} = 0.$$

According to [**4**, pp. 80, 147], there is an invariant mean on $\mathcal{C}(\mathcal{G}_0)$ defined as

$$\varphi(g) = \lim_{\mathcal{U}} \lambda(\mathcal{G}_n)^{-1} \int_{\mathcal{G}_n} g(\sigma)\,\mathrm{d}\lambda = \lim_{\mathcal{U}} \frac{1}{n!} \sum_{\sigma \in \mathcal{G}_n} g(\sigma), \tag{2.6}$$

where $\mathcal{U}$ is some free ultrafilter on the set of positive integers.

Now let G be a subgroup of isometric operators on a Banach space X. A subset $V \subset X$ is *G-symmetric* if $\sigma(x) \in V$ for every $x \in X$ and $\sigma \in G$. We assume that G is endowed with the topology of pointwise convergence on X. For a given subalgebra A of bounded functions on a G-symmetric subset V, $f \in A$ and $x \in X$, we define a function on G, $(f, x) \in \mathcal{B}(G)$ by $(f, x)(\sigma) = f(\sigma(x))$. If f is continuous, then (f, x) is continuous as well.

Proposition 2.16. *Let φ be a continuous invariant mean of $U \subset \mathcal{B}(G)$ and let A be a uniform algebra of functions on V such that $(f, x) \in U$ for every $f \in A$ and $x \in V$. Then there exists a continuous operator of symmetrization $\mathcal{S}_\varphi$ that maps A into a uniform algebra of bounded G-symmetric functions on X.*

Proof. Set

$$\mathcal{S}_\varphi(f) = \varphi(f, x).$$

Since φ is an invariant mean of U and $(f,x)\in U$,

$$\mathcal{S}_\varphi(f)(\sigma x)=\varphi(f,\sigma(x))=\varphi(f,\sigma_0(x))=\mathcal{S}_\varphi(f)(x),$$

where σ_0 is the identity in G. So $\mathcal{S}_\varphi(f)$ is symmetric. Evidently, if $\|x\|\leqslant 1$, then $\|(f,x)\|\leqslant\|f\|$ and the set

$$\{(f,x)\colon \|f\|\leqslant 1 \text{ and } \|x\|\leqslant 1\}$$

is a subset of $\{(f,x)\colon \|(f,x)\|\leqslant 1\}$. Hence,

$$\|\mathcal{S}_\varphi\|=\sup_{\|f\|\leqslant 1}\|\varphi(f,\cdot)\|=\sup_{\|x\|\leqslant 1,\|f\|\leqslant 1}|\varphi(f,x)|\leqslant\sup_{\|(f,x)\|\leqslant 1}|\varphi(f,x)|=\|\varphi\|.$$

$\square$

Corollary 2.17. *Let V be a $\mathcal{G}_0$-symmetric subset of ℓ_p, $1\leqslant p<\infty$. There exists a continuous linear projection operator $\mathcal{S}$ on the algebra of continuous functions on V that are bounded on bounded subsets, $C_\mathrm{b}(V)$, into the algebra of $\mathcal{G}_0$-symmetric bounded functions on V, $\mathcal{B}_\mathrm{s}(V)$, such that*

$$\mathcal{S}(f)(x)\ =\lim_{\mathcal{U}}\frac{1}{n!}\sum_{\sigma\in\mathcal{G}_n}f(\sigma(x)) \tag{2.7}$$

and

$$\|\mathcal{S}(f)\|_V:=\sup_{x\in V}|\mathcal{S}(f)(x)|\leqslant\|f\|_V. \tag{2.8}$$

Proof. Let φ be the invariant mean on $\mathcal{G}_0$ that is defined by (2.6). Put $\mathcal{S}:=\mathcal{S}_\varphi(f)$. By Proposition 2.16, $\mathcal{S}$ is a continuous linear map from $C_\mathrm{b}(V)$ to $\mathcal{B}_\mathrm{s}(V)$. Since $\mathcal{S}(f)=f$ for any $f\in\mathcal{B}_\mathrm{s}(V)$, $\mathcal{S}$ is a projection. Formula (2.7) follows immediately from (2.6).

Since V is symmetric, $\|f(\sigma(\cdot))\|_V=\|f\|_V$ for every $\sigma\in\mathcal{G}_0$. Then, for each n,

$$\left\|\frac{1}{n!}\sum_{\sigma\in\mathcal{G}_n}f(\sigma(\cdot))\right\|_V\leqslant\frac{1}{n!}\sum_{\sigma\in\mathcal{G}_n}\|f(\sigma(\cdot))\|_V=\|f\|_V.$$

$\square$

Proposition 2.18. *Let V be a $\mathcal{G}_0$-symmetric subset of ℓ_p, $1\leqslant p<\infty$. If f is uniformly continuous on V, then $\mathcal{S}(f)$ is uniformly continuous on V. If V is open and f is analytic on V, then $\mathcal{S}(f)$ is analytic on V.*

Proof. Let $\varepsilon>0$ be given and let $\delta>0$ be chosen such that if $\|x-y\|<\delta$, then $|f(x)-f(y)|<\varepsilon$. Since $\|x-y\|<\delta$ implies $\|\sigma(x)-\sigma(y)\|<\delta$, it follows that

$$\left|\frac{1}{n!}\sum_{\sigma\in\mathcal{G}_n}f(\sigma(x))-\frac{1}{n!}\sum_{\sigma\in\mathcal{G}_n}f(\sigma(y))\right|<\varepsilon.$$

Consequently, $|\mathcal{S}(f)(x)-\mathcal{S}(f)(y)|\leqslant\varepsilon$.

For the last statement, it is sufficient to show that $\mathcal{S}(P)$ is an n-homogeneous polynomial if P is too. This follows from the linearity of the mapping $\sigma\colon x\mapsto\sigma(x)$ for all $\sigma\in\mathcal{G}_0$. $\square$

Corollary 2.19. *$\mathcal{H}_{\mathrm{bs}}(\ell_p)$ is a complemented closed subspace of $\mathcal{H}_{\mathrm{b}}(\ell_p)$.*

The next example shows that $\mathcal{S}$ is not a homomorphism.

Example 2.20. Let P and Q be two functionals on ℓ_1 given by

$$P(x)=\sum_{i=1}^{\infty} x_{2i-1} \quad \text{and} \quad Q(x)=\sum_{i=1}^{\infty} x_{2i}.$$

Observe that

$$\begin{aligned}\mathcal{S}\biggl(\sum_{i=1}^{\infty} x_i\biggr) &= \lim_{\mathcal{U}} \frac{1}{n!}\biggl(\sum_{\sigma\in\mathcal{G}_n}\sum_{i=1}^{\infty} x_{\sigma(i)}\biggr)\\ &= \lim_{n\to\infty} \frac{1}{n!}\biggl(\sum_{\sigma\in\mathcal{G}_n}\sum_{i=1}^{\infty} x_{\sigma(i)}\biggr)\\ &= \sum_{i=1}^{\infty} x_i.\end{aligned}$$

Since

$$(P+Q)(x)=\sum_{i=1}^{\infty} x_i$$

and Q is the composition of P and the shift operator, it follows that

$$\mathcal{S}(P)(x)=\mathcal{S}(Q)(x)=\frac{1}{2}\sum_{i=1}^{\infty} x_i.$$

So

$$\begin{aligned}\mathcal{S}(P)\mathcal{S}(Q)(x) &= \frac{1}{4}\sum_{i,j=1}^{\infty} x_i x_j\\ &\neq \mathcal{S}\biggl(\sum_{i,j=1}^{\infty} x_{2i-1}x_{2j}\biggr)\\ &= \mathcal{S}(PQ)(x)\end{aligned}$$

because $\mathcal{S}(P)\mathcal{S}(Q)(x)$ contains terms $\frac{1}{4}x_i^2$, $i=1,2,\ldots$, and $\mathcal{S}(PQ)(x)$ does not.

3. The algebra of subsymmetric analytic functions on $\boldsymbol{\ell_p}$

3.1. Projection homomorphisms

In [**8**] a spreading model technique was used to construct a homomorphism from $\mathcal{P}(X)$ to $\mathcal{P}_{\mathrm{s_{b_s}}}(X)$. It was proved that, for a given polynomial P on ℓ_p, there exists an infinite set H of positive integers and a polynomial P^* on ℓ_p such that

$$P^*\biggl(\sum_{i=1}^{k} x_i e_i\biggr) = \lim_{\substack{n_1<\cdots<n_k\\ n_j\in \mathrm{H}}} P\biggl(\sum_{i=1}^{k} x_i e_{n_i}\biggr).$$

According to [**6**, pp. 122, 123], P^* can be described in the following way. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then

$$P^*\biggl(\sum_{i=1}^{k} x_i e_i\biggr) = \lim_{\mathcal{U},1} \cdots \lim_{\mathcal{U},k} P\biggl(\sum_{i=1}^{k} x_i e_{n_i}\biggr). \tag{3.1}$$

Formula (3.1) means that at first we take the limit via the ultrafilter $\mathcal{U}$ over the index $k \rightsquigarrow n_k$ at the basis element e_k with the coordinate x_k. We denote this limit by

$$\lim_{\mathcal{U},k} P(x_1 e_1 + \cdots + x_{k-1} e_{k-1} + x_k e_{n_k}).$$

The limit exists because P is bounded. Next we take the limit over the index $k-1 \rightsquigarrow n_{k-1}$ at e_{k-1}, and so on.

Because of the way P^* is defined, it depends only on P and the ultrafilter $\mathcal{U}$. We denote by $\mathfrak{S}_{\mathrm{s_{b_s}}}$ the map $P \mapsto P^*$ for a fixed free ultrafilter $\mathcal{U}$. It is easy to see that P^* is subsymmetric on ℓ_p. From (3.1), it follows that $\mathfrak{S}_{\mathrm{s_{b_s}}}$ is a homomorphism and $\|P^*\| \leqslant \|P\|$.

Note that in the proof of [**8**, Theorem 3.1] the fact that $P \mapsto P^*$ is a homomorphism was used.

Corollary 3.1. *Let V be a $\mathfrak{G}$-symmetric domain of ℓ_p, $1 \leqslant p < \infty$. Then, for any algebra $\mathcal{A}$ of analytic functions on V such that the symmetric polynomials are dense, $\mathfrak{S}_{\mathrm{s_{b_s}}}$ can be extended to a continuous homomorphism onto its subalgebra $\mathcal{A}_{\mathrm{s_{b_s}}}$ of subsymmetric functions. Moreover, if f is continuous on the closure $\bar{V}$, then $\mathfrak{S}_{\mathrm{s_{b_s}}}(f)$ is continuous on $\bar{V}$, and if f is bounded on some subsymmetric subset $V_0 \subset V$, then $\mathfrak{S}_{\mathrm{s_{b_s}}}(f)$ is also.*

We will denote this extension by the same symbol $\mathfrak{S}_{\mathrm{s_{b_s}}}$.

Corollary 3.2. *Every complex homomorphism $\varphi \in M_{\mathrm{bs_{b_s}}}(\ell_p)$ can be extended to some complex homomorphism $\psi \in M_{\mathrm{b}}(\ell_p)$ by*

$$\psi(f) = \varphi(\mathfrak{S}_{\mathrm{s_{b_s}}}(f)), \quad f \in \mathcal{H}_{\mathrm{b}}(\ell_p).$$

3.2. Relation to the symmetric case

Motivated by Proposition 2.8, we seek to find examples of the existence of projection homomorphisms from algebras of analytic functions onto their symmetric counterpart.

Let us denote by $\mathcal{P}_n(\ell_1)$ (respectively, $\mathcal{P}_{\mathrm{s},n}(\ell_1)$, $\mathcal{P}_{\mathrm{s_{b_s}},n}(\ell_1)$) the algebra of (respectively, symmetric, subsymmetric) polynomials on ℓ_1 generated by all (respectively, symmetric, subsymmetric) polynomials of degree less than or equal to n. The notation $\mathcal{H}_{\mathrm{b},n}(\ell_1)$, $\mathcal{H}_{\mathrm{bs},n}(\ell_1)$, $\mathcal{H}_{\mathrm{s_{b_s}},n}(\ell_1)$, $M_{\mathrm{b},n}(\ell_1)$, $M_{\mathrm{bs},n}(\ell_1)$ and $M_{\mathrm{s_{b_s}},n}(\ell_1)$ is obvious.

First we consider the case where $n = 2$. Since $\mathcal{H}_{\mathrm{bs},2}(\ell_1) = \mathcal{H}_{\mathrm{s_{b_s}},2}(\ell_1)$, the restriction $\mathfrak{S}_{\mathrm{s_{b_s}},2}$ of $\mathfrak{S}_{\mathrm{s_{b_s}}}$ to $\mathcal{H}_{\mathrm{b},2}(\ell_1)$ is a projection homomorphism from $\mathcal{H}_{\mathrm{b},2}(\ell_1)$ onto $\mathcal{H}_{\mathrm{bs},2}(\ell_1)$. Let $\Theta\colon \mathcal{H}_{\mathrm{bs},2}(\ell_1) \to \mathcal{H}_{\mathrm{bs},2}(\ell_1)$ be a homomorphism defined on the basis functions F_1, F_2 by

$$\Theta(F_1) = F_2, \qquad \Theta(F_2) = F_1.$$

According to [**1**], there is a topological isomorphism between the algebra $\mathcal{H}_{\mathrm{bs},2}(\ell_1)$ and the algebra of entire functions of two variables $\mathcal{H}(\mathbb{C}^2)$ given by

$$\mathcal{H}_{\mathrm{bs},2}(\ell_1) \ni u(F_1(x), F_2(x)) \leftrightarrow u(t_1, t_2) \in \mathcal{H}(\mathbb{C}^2).$$

Thus, Θ is evidently continuous. Hence $\Theta \circ \mathfrak{S}_{\mathrm{s_{b_s}}}$ is a continuous homomorphism from $\mathcal{H}_{\mathrm{s},2}(\ell_1)$ to itself with the 'pathological' property that

$$\Theta \circ \mathfrak{S}_{\mathrm{s_{b_s}}}(F_1) = F_2, \qquad \Theta \circ \mathfrak{S}_{\mathrm{s_{b_s}}}(F_2) = F_1.$$

Next we will see that the case where $n = 3$ is much more complicated.

Lemma 3.3. *The polynomials F_1, F_2, F_3 and*

$$F_{1,2}(x) = \sum_{i<j} x_i x_j^2$$

algebraically generate $\mathcal{P}_{\mathrm{s_{b_s}},3}(\ell_1)$.

Proof. Since F_1, F_2, F_3, $F_{1,2}$ and $F_{2,1}$ form a linear basis in $\mathcal{P}_{\mathrm{s_{b_s}},3}(\ell_1)$ and (the non-symmetric polynomial) $F_{1,2}$ cannot be represented by an algebraic combination of (the symmetric polynomials) F_1, F_2 and F_3, it is enough to show that $F_{2,1}$ belongs to the algebraic span of F_1, F_2, F_3 and $F_{1,2}$. But it is easy to see that

$$F_{2,1}(x) = F_1(x)F_2(x) - F_3(x) - F_{1,2}(x)$$

for every $x \in \ell_1$. □

Proposition 3.4. *F_1, F_2, F_3 and $F_{1,2}$ form an algebraic basis in $\mathcal{P}_{\mathrm{s_{b_s}},3}(\ell_1)$.*

Proof. It is enough to check that F_1, F_2, F_3 and $F_{1,2}$ are algebraically independent. Let us suppose that

$$P(F_1(x), F_2(x), F_3(x), F_{1,2}(x)) \equiv 0$$

for some non-trivial polynomial P on $\mathbb{C}^4$.

For any fixed $n \geqslant 3$ and any $x = (x_1, \dots, x_n, 0, \dots, 0, \dots)$, a direct calculation shows that

$$F_{1,2}(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots) = 2(F_1(x)F_2(x) - \tfrac{1}{2}F_3(x)).$$

Then we have

$$\begin{aligned}
&P(F_1(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots),\\
&\quad F_2(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots),\\
&\quad F_3(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots),\\
&\quad F_{1,2}(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots))\\
&\qquad\qquad = P(2F_1(x), 2F_2(x), 2F_3(x), F_{1,2}(x_1, \dots, x_n, x_n, \dots, x_1, 0, \dots, 0, \dots))\\
&\qquad\qquad = P(2F_1(x), 2F_2(x), 2F_3(x), 2(F_1(x)F_2(x) - \tfrac{1}{2}F_3(x)))\\
&\qquad\qquad = 0.
\end{aligned}$$

Since $n \geqslant 3$ for an arbitrary vector $(t_1, t_2, t_3) \in \mathbb{C}^3$, we can find [**1**] an element $x = (x_1, x_2, x_3, 0, \dots, 0, \dots)$ such that

$$F_k(x) = t_k, \quad 1 \leqslant k \leqslant 3.$$

So

$$P(2t_1, 2t_2, 2t_3, 2(t_1t_2 - \tfrac{1}{2}t_3)) \equiv 0.$$

Now we want to show that P can be assumed to be irreducible. Let $P = P_1P_2$. Then

$$P_1(2t_1, 2t_2, 2t_3, 2(t_1t_2 - \tfrac{1}{2}t_3))P_2(2t_1, 2t_2, 2t_3, 2(t_1t_2 - \tfrac{1}{2}t_3)) \equiv 0.$$

Hence, either

$$P_1(2t_1, 2t_2, 2t_3, 2(t_1t_2 - \tfrac{1}{2}t_3)) \equiv 0$$

or

$$P_2(2t_1, 2t_2, 2t_3, 2(t_1t_2 - \tfrac{1}{2}t_3)) \equiv 0.$$

So, without loss of generality, we can assume that P is irreducible.

Consider the polynomial $S(u_1, u_2, u_3, u_4) = \frac{1}{2}(u_1u_2 - u_3) - u_4$. Then $\ker S \subset \ker P$. By the Hilbert Nullstellensatz [**10**, Proposition 1.2, Theorem 1.3A], P belongs to the radical of the ideal (S) generated by S. Since S is irreducible, (S) coincides with its radical. So $P = SQ$ for some polynomial Q. Because P is irreducible too, we have $S = cP$ for some constant c. Thus, $\frac{1}{2}(F_1F_2 - F_3) - F_{1,2} \equiv 0$ or

$$F_{1,2} = \tfrac{1}{2}(F_1F_2 - F_3).$$

However, this is impossible. □

Proposition 3.5. *There exists a continuous projection homomorphism $\mathfrak{S}_{\mathrm{s},3}$ of $\mathcal{P}_3(\ell_1)$ onto $\mathcal{P}_{\mathrm{s},3}(\ell_1)$.*

Proof. First we define a homomorphism J from $\mathcal{P}_{\mathrm{s}_{\mathrm{b_s}},3}(\ell_1)$ onto $\mathcal{P}_{\mathrm{s},3}(\ell_1)$ determining it on the algebraic basis of $\mathcal{P}_{\mathrm{s}_{\mathrm{b_s}},3}(\ell_1)$ by

$$J(F_k) = F_k, \quad k = 1, 2, 3,$$

and

$$J(F_{1,2}) = 0.$$

Evidently, J is a well-defined homomorphism.

Set

$$\mathfrak{S}_{\mathrm{s},3} = J \circ \mathfrak{S}_{\mathrm{s}_{\mathrm{b_s}},3}, \tag{3.2}$$

where $\mathfrak{S}_{\mathrm{s}_{\mathrm{b_s}},3}$ is the restriction of $\mathfrak{S}_{\mathrm{s}_{\mathrm{b_s}}}$ to the space $\mathcal{P}_{\mathrm{s}_{\mathrm{b_s}},3}(\ell_1)$. Since the algebraic basis of subsymmetric polynomials contains the algebraic basis of symmetric polynomials, $J(P) = P$ if P is symmetric.

Arguing as in [**1**, Theorem 2.1], it is not difficult to show that $\mathcal{H}_{\mathrm{s}_{\mathrm{b_s}},3}(\ell_1)$ is isomorphic to $\mathcal{H}(\mathbb{C}^4)$ by means of the map

$$\mathcal{H}_{\mathrm{s}_{\mathrm{b_s}},3}(\ell_1) \ni u(F_1(x), F_2(x), F_3(x), F_{1,2}(x)) \leftrightarrow u(t_1, t_2, t_3, t_4) \in \mathcal{H}(\mathbb{C}^4).$$

So J is continuous. □

We denote by the same symbol $\mathfrak{S}_{\mathrm{s},3}$ the continuous extension of this homomorphism to $\mathcal{H}_{b,3}(\ell_1)$ onto $\mathcal{H}_{\mathrm{bs},3}(\ell_1) \subset \mathcal{H}_{\mathrm{bs}}(\ell_1)$, the closed algebra generated by all symmetric polynomials of degree less than or equal to 3.

Corollary 3.6. *There exists a continuous embedding of the set $M_{\mathrm{bs},3}(\ell_1)$ into $M_{\mathrm{b},3}(\ell_1)$:*

$$M_{\mathrm{bs},3}(\ell_1) \ni \varphi \mapsto \varphi \circ \mathfrak{S}_{\mathrm{s},3} \in M_{\mathrm{b},3}(\ell_1),$$

where $M_{\mathrm{bs},3}(\ell_1)$ is the spectrum of $\mathcal{H}_{\mathrm{bs},3}(\ell_1)$ and $M_{\mathrm{b},3}(\ell_1)$ is the spectrum of $\mathcal{H}_{\mathrm{b},3}(\ell_1)$.

Unfortunately, we do not know how to describe an algebraic basis for the algebra of all subsymmetric polynomials.

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