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GREEN FUNCTION OF A BOUNDARY VALUE PROBLEM FOR A VECTOR SINGULAR QUASIDIFFERENTIAL EQUATION

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A Green function of a boundary value problem of a vector quasidifferential equation with distributions in coefficients is constructed. With the aid of the method of the introduction of quasiderivatives and obtained expressions for adjoint boundary conditions the properties of Green functions of adjoint boundary value problems are investigated.

Key words: Green function, quasidifferential equation, distributions, quasiderivatives

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Introduction

Linear differential operators generated by differential expressions with smooth coefficients were studied quite comprehensively in the literature (e.g., see [1]). However, applied problems often contain differential equations with discontinuous or even generalized functions in the coefficients. Such problems much worse are investigated.

As early as in the 1950s, boundary value problems were studied for ordinary differential equations of order 2 and 4 describing the free vibrations of a string and a beam that, in addition to a continuously distributed mass, bear lumped masses (beads) [2]. In [3] the Schrodinger operator is explored on an unlimited interval in the case when singular potential is, for example, a finite or infinite sum of Dirac δ-functions.

Real problems often lead to differential expressions that contain terms of the form $(P(x)Y^{(m)})^{(n)}$ and cannot be reduced to conventional differential expressions by n-fold differentiation if the coefficient $P(x)$ is not sufficiently smooth. Such expressions are said to be quasidifferential. The introduction of quasiderivatives is one of the oldest and most efficient methods for their analysis [4, 5]. (The quasiderivatives are the components of the vector used in the reduction of a quasidifferential equation to a system of first-order differential equations.) This method allows to give up the requirements of the smoothness of the coefficients in quasidifferential expressions.

At first mainly quasidifferential expressions with continuous or summable by Lebesgue coefficients were investigated in the works of D. Shin [4, 5] and his following. However, attempts of application of this method to investigation of quasidifferential expressions with distributions were appeared later. In particular, the Green matrix of a boundary value problem for a quasidifferential equation with a self-adjoint quasidifferential expression was constructed in the work [6]. In the case of a boundary value problem for a scalar singular differential and quasidifferential equation the Green function was constructed and the asymptotic behaviour of eigenvalues and eigenfunctions was investigated and also the development by them was fulfilled in the works [7–9].

This work is devoted to the construction of the Green function of a boundary value problem for a vector quasidifferential equation with distributions in the coefficients and homogeneous boundary conditions. With the help of the method of the introduction of quasiderivatives the properties of the Green functions of the adjoint boundary problems are investigated.

I. Formulation of the problem

The real or complex-valued scalar function $f(x)$ is called the function of the boundary variation on the interval $[a,b]$, if the expression

$$v=\sum_{k=0}^{n-1}|f(x_{k+1})-f(x_k)|$$

permits of the fixed supremum for all natural n and all decompositions of the interval $[a,b]$: $a=x_0<x_1<\ldots<x_n=b$. The least general supremum of all such expressions is termed the total variation of the function $f(x)$ on $[a,b]$. The real or complex-valued matrix function $F(x)$ has the boundary variation on the interval $[a,b]$, if every entry of this matrix has the boundary variation on $[a,b]$.

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Consider the differential expression

$$L_{mn}(\mathbf{y}) \equiv \sum_{i=0}^{n}\sum_{j=0}^{m}(-1)^{m-j}(A_{ij}(x)\mathbf{y}^{(n-i)})^{(m-j)},$$

where m, n are natural numbers; $A_{i0}(x)$, $A_{0j}(x)$ are square matrix functions of order l with square summable on the interval $[a,b]$ entries; $A_{ij}(x) = B'_{ij}(x)$, $B_{ij}(x)$, $i=\overline{1,n}$, $j=\overline{1,m}$, are square matrix functions of order l whose entries have bounded variation on the interval $[a,b]$ and are right continuous on it, and $\mathbf{y}(x)$ is a column vector. The prime stands for the generalized differentiation, and therefore, the entries of the matrices $A_{ij}(x)$ are measures [?]. The matrix functions $A_{ij}(x)$ and $B_{ij}(x)$ are assumed to be complex-valued. We assume also, that $A_{00}(x)$ such matrix for which the measurable and limited matrix function $A_{00}^{-1}(x)$ exists.

The quasiderivatives of $\mathbf{y}(x)$ corresponding to the expression $L_{mn}(\mathbf{y})$ are defined as the functions given by the formulas

$$\begin{cases}\mathbf{y}^{[k]} = \mathbf{y}^{(k)},\ k=0,\ldots,n-1;\ \ \mathbf{y}^{[n]} = \sum\limits_{i=0}^{n} A_{i0}\mathbf{y}^{(n-i)};\\ \mathbf{y}^{[n+k]} = -\left(\mathbf{y}^{[n+k-1]}\right)' + \sum\limits_{i=0}^{n} A_{ik}\mathbf{y}^{(n-i)},\ k=1,\ldots,m.\end{cases}$$

We consider also corresponding to the quasidifferential expression $L_{mn}(\mathbf{y})$ the equation

$$L_{mn}(\mathbf{y}) = \lambda\mathbf{y}, \tag{1}$$

where λ is a complex parameter, and the boundary conditions

$$U_\nu(\mathbf{y}) \equiv \sum_{j=0}^{r-1}\Gamma_{\nu j}\mathbf{y}^{[j]}(a) + \sum_{j=0}^{r-1}\Delta_{\nu j}\mathbf{y}^{[j]}(b) = 0, \quad \nu = 1,\ldots,r, \tag{2}$$

which are set with the help of r the linearly independent forms $U_\nu(\mathbf{y})$, $r = n+m$.

Together with the boundary value problem (1), (2) for the vector quasidifferential equation we consider also the associated to it boundary value problem for the matrix quasidifferential equation

$$L_{mn}(Y) = \lambda Y, \tag{3}$$

$$U_\nu(Y) \equiv \sum_{j=0}^{r-1}\Gamma_{\nu j}Y^{[j]}(a) + \sum_{j=0}^{r-1}\Delta_{\nu j}Y^{[j]}(b) = 0, \tag{4}$$

where $Y(x)$ is a square matrix of order l.

By using the rectangular matrix $Y = \left(Y,\, Y^{[1]},\, \ldots,\, Y^{[r-1]}\right)^T$ (where T stands for transposition), one can reduce the equation (3) to the system of first-order differential equations

$$Y' = B'(x)Y, \tag{5}$$

where the matrix-measure

$$B'(x) = \begin{pmatrix} 0 & E_l & \cdots & 0 & 0 & 0 & \cdots & 0\\ \ldots & \ldots & \ddots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \cdots & E_l & 0 & 0 & \cdots & 0\\ \tilde{A}_{n0} & \tilde{A}_{n-1,0} & \cdots & \tilde{A}_{10} & A_{00}^{-1} & 0 & \cdots & 0\\ \tilde{A}_{n1} & \tilde{A}_{n-1,1} & \cdots & \tilde{A}_{11} & A_{01}A_{00}^{-1} & -E_l & \cdots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \ldots\\ \tilde{A}_{n,m-1} & \tilde{A}_{n-1,m-1} & \cdots & \tilde{A}_{1,m-1} & A_{0,m-1}A_{00}^{-1} & 0 & \cdots & -E_l\\ \tilde{A}_{nm}-\lambda E_l & \tilde{A}_{n-1,m} & \cdots & \tilde{A}_{1m} & A_{0m}A_{00}^{-1} & 0 & \cdots & 0 \end{pmatrix},$$

$\tilde{A}_{i0} = -A_{00}^{-1}A_{i0}$, $\tilde{A}_{ij} = A_{ij} - A_{0j}A_{00}^{-1}A_{i0}$, $i=1,\ldots,n$, $j=1,\ldots,m$ (here 0 is the zero matrix of order l, E_l is the identity matrix of order l). Obviously, that the jump of the matrix $B(x)$ looks like

$$\Delta B(x) = B(x) - B(x-0) = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & \cdots & 0 & 0 & \cdots & 0\\ \Delta B_{n1} & \cdots & \Delta B_{11} & 0 & \cdots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \Delta B_{nm} & \cdots & \Delta B_{1m} & 0 & \cdots & 0 \end{pmatrix}.$$

Then, owing to the relation $[\Delta B(x)]^2 \equiv 0$, the system (5) is well posed [11].

The boundary conditions (4) also can be rewrited in the matrix form

$$W_a Y(a) + W_b Y(b) = 0 \tag{6}$$

with the block matrices $W_a = (\Gamma_{\nu,j-1})_{\nu,j=1}^r$, $W_b = (\Delta_{\nu,j-1})_{\nu,j=1}^r$.

A solution of the matrix quasidifferential equation is understood as the first block component $Y(x)$ of the rectangular matrix $Y(x)$ of the system (5), which satisfies it in the sense of distributions. It was proved in [12, 13, p. 134]that there exists a unique solution of the initial problem for the equation (3) in the class of absolutely continuous matrix functions on $[a, b]$, its quasiderivatives of order less than $(n-1)$ are absolutely continuous on $[a, b]$, and all entries of the rest quasiderivatives of order less than $r-1$ inclusive have bounded variation on the interval $[a, b]$ and are right continuous on it.

The system adjoint to the system (5) defines by the matrix relation (see [12])

$$Z' = -(B^*(x))' Z, \tag{7}$$

where $Z = \left(Z^{\{r-1\}}, \ldots, Z^{\{1\}}, Z\right)^T$, the asterisk stands for Hermitian conjugation, and the curly braces are used to denote quasiderivatives in the sense of the adjoint equation. From the relation (7) one can notice [12, 13, p. 135], that they are defined by the formulas

$$\begin{cases} Z^{\{k\}} = Z^{(k)},\ k = 0, \ldots, m-1; \\ Z^{\{m\}} = -\sum\limits_{j=0}^{m} A_{0j}^* Z^{(m-j)}; \\ Z^{\{m+k\}} = -\left(Z^{\{m+k-1\}}\right)' - \sum\limits_{j=0}^{m} A_{kj}^* Z^{(m-j)}, \\ k = 1, \ldots, n. \end{cases}$$

From (7) one can see ([12, 13, p. 135]), that the adjoint quasidifferential equation to the equation (3) has the form

$$L_{mn}^*(Z) \equiv \sum_{j=0}^{m} \sum_{i=0}^{n} (-1)^{n-i} (A_{ij}^*(x) Z^{(m-j)})^{(n-i)} = \bar{\lambda} Z, \tag{8}$$

where the bar stands above λ for complex conjugation.

II. Adjoint boundary conditions

We consider the expression Z^*Y and differentiate it with the help of formulas (5), (7):

$$(Z^*Y)' = (Z^*)' Y + Z^* Y' = -\left((B^*)' Z\right)^* Y + Z^* B' Y = \\ = -Z^* B' Y + Z^* B' Y = 0.$$

Such differentiation is possible, as products $(Z^*)' Y$ and $Z^* Y'$ are correct on the basis of that fact, that Y, $Y^{[1]}, \ldots, Y^{[n-1]}$, Z, $Z^{\{1\}}, \ldots, Z^{\{m-1\}}$ are matrices consisting of absolutely continuous on $[a, b]$ functions, and $Y^{[n]}$, $Y^{[n+1]}$, ..., $Y^{[r-1]}$, $Z^{\{m\}}$, $Z^{\{m+1\}}$, ..., $Z^{\{r-1\}}$ are the matrices whose entries have bounded variation on the interval $[a, b]$ (see [12, 13, p. 136]). Consequently, Z^*Y is the constant and therefore

$$(Z^*Y)\big|_a^b = 0. \tag{9}$$

By the last relation it is possible to determine the adjoint boundary conditions. For this purpose we supplement the linear forms $U_1(Y)$, $U_2(Y)$, ..., $U_r(Y)$ by forms $U_{r+1}(Y)$, $U_{r+2}(Y)$, ..., $U_{2r}(Y)$ to the linearly independent system of $2r$ linear forms. Then the system

$$U_\nu(Y) = \sum_{j=0}^{r-1} \Gamma_{\nu j} Y^{[j]}(a) + \sum_{j=0}^{r-1} \Delta_{\nu j} Y^{[j]}(b), \quad \nu = 1, \ldots, 2r,$$

can be solved uniquely relatively unknowns $Y^{[q]}(a)$, $Y^{[q]}(b)$, which are possible to determine by $U_1(Y)$, ..., $U_{2r}(Y)$. By substituting got $Y^{[q]}(a)$, $Y^{[q]}(b)$ $(q = 0, \ldots, r-1)$ into the bilinear form in the left-hand side of the relation (9), we obtain

$$(Z^*Y)\big|_a^b = \sum_{\nu=1}^{2r} \mathcal{A}_\nu(\xi) \mathcal{B}_\nu(\eta),$$

where $\eta = \left(Y^{[q]}(a), Y^{[q]}(b)\right)$, $\xi = \left(Z^{*\{q\}}(a), Z^{*\{q\}}(b)\right)$, $q = 0, \ldots, n-1$, and $\mathcal{B}_\nu(\eta) = U_\nu(Y)$. We denote $\mathcal{A}_{2r}(\xi) = V_1(Z)$, ..., $\mathcal{A}_1(\xi) = V_{2r}(Z)$. Obviously, in order that the equality (9) holds true, the relations

$$V_\nu(Z) = 0, \quad \nu = 1, \ldots, r \tag{10}$$

must be valid. We say that they are adjoint boundary conditions to the conditions (4) (as the boundary conditions equal zero it is possible to carry the complex conjugation from Z to the matrix of constant coefficients at it). The adjoint boundary conditions can be represented in the vector form $V_\nu(\mathbf{z}) = 0$, $\nu = 1, \ldots, r$.

If $|W_a| \neq 0$ and $|W_b| \neq 0$ simultaneously in the equation (6), then it is readily to make sure in that the boundary conditions for the adjoint equation are in the form $Z^*(a) W_a^{-1} + Z^*(b) W_b^{-1} = 0$.

III. Green function of the boundary value problem

Now consider the nonhomogeneous vector quasidifferential equation

$$L_{mn}(\mathbf{y}) = \lambda \mathbf{y} + \mathbf{f}', \tag{11}$$

where $\mathbf{f}$ is a column vector, whose all entries have bounded variation on $[a, b]$. Also consider the corresponding matrix equation

$$L_{mn}(Y) = \lambda Y + F', \tag{12}$$

where a matrix $F(x)$ consists of l columns $\mathbf{f}(x)$. By using the rectangular matrix Y (see item 1), one can reduce the nonhomogeneous equation (12) to the system of first-order differential equations

$$Y' = B' Y + F', \tag{13}$$

where $F(x) = (0, \ldots, 0, -F(x))^T$. This system is well posed by virtue of the conditions $[\Delta B(x)]^2 \equiv 0$ and $\Delta B(x)\Delta F(x) \equiv 0$ are valid (see [**?**]).

Cauchy matrix function of the equation (3) is understood as the $l \times l$ matrix function $K(x,t,\lambda)$, which satisfies the equation (3) with respect to the first variable, and, besides, $K^{[i]}(t,t,\lambda) = 0$, $i = 0,\ldots,r-2$, $K^{[r-1]}(t,t,\lambda) = E$.

We construct the Green matrix function of the boundary value problem (11), (2). Let $K(x,t,\lambda)$ be the Cauchy matrix function of the homogeneous equation (3). It is known [12, 13, p. 146], that $K(x,a,\lambda)$, $K^{*\{1\}*}(x,a,\lambda)$, $\ldots$, $K^{*\{r-1\}*}(x,a,\lambda)$ form the fundamental solution system and the solution of the equation (12) can be represented in the form

$$Y(x,\lambda) = \sum_{k=1}^{r} K^{*\{k-1\}*}(x,a,\lambda)C_k + \int_a^x K(x,t,\lambda)dF(t). \tag{14}$$

As, in according to [**?**, **?**, p. 156],

$$Y^{[j]}(x,\lambda) = \sum_{k=1}^{r} K^{*\{k-1\}*[j]}(x,a,\lambda)C_k + \\ + \int_a^x K^{[j]}(x,t,\lambda)dF(t), \quad j = 1,\ldots,r,$$

the substitution of the formula (14) to the boundary conditions (4) gives the relations

$$U_\nu(Y) = \sum_{k=1}^{r} U_\nu\left(K^{*\{k-1\}*}(x,a,\lambda)\right)C_k + \\ + \sum_{j=0}^{r-1} \Delta_{\nu j} \int_a^b K^{[j]}(b,t,\lambda)dF(t), \quad \nu = 1,\ldots,r. \tag{15}$$

This can be represented in the form $\mathcal{W}\tilde{C} + \tilde{B} = 0$, where $\tilde{C} = (C_1,\ldots,C_r)^T$, $\tilde{B}$ is a $rl \times l$ rectangular matrix, $\mathcal{W} = \left(U_\nu(K^{*\{k-1\}*}(x,a,\lambda))\right)_{\nu,k=1}^{r}$. In assumption that λ is not an eigenvalue of the boundary value problem (3), (4), the determinant of the system (15) is nonzero $\Delta(\lambda) \equiv \det \mathcal{W} \neq 0$. Then the constant matrices C_k can be obtain from the system (15) uniquely. By substituting this values C_k into the formula (14), we obtain

$$Y(x,\lambda) = -\sum_{\nu=1}^{r}\sum_{k=1}^{r}\sum_{j=0}^{r-1} \int_a^b K^{*\{k-1\}*}(x,a,\lambda) \times \\ \times \frac{W_{\nu k}\Delta_{\nu j}}{\Delta(\lambda)} K^{[j]}(b,t,\lambda)dF(t) + \int_a^x K(x,t,\lambda)dF(t),$$

where $W_{\nu k}$ $(\nu, k = 1,\ldots,r)$ is the matrix of order l, which is transposed to the matrix consisting of the cofactors of the entries of the matrix $U_\nu\left(K^{*\{k-1\}*}(x,a,\lambda)\right)$ in the determinant $\Delta(\lambda)$.

The matrix expression

$$G(x,t,\lambda) = -\sum_{\nu=1}^{r}\sum_{k=1}^{r}\sum_{j=0}^{r-1} K^{*\{k-1\}*}(x,a,\lambda) \times \\ \times \frac{W_{\nu k}}{\Delta(\lambda)}\Delta_{\nu j}K^{[j]}(b,t,\lambda) + P(x,t,\lambda), \tag{16}$$

where

$$P(x,t,\lambda) = \begin{cases} K(x,t,\lambda) & \text{for} \quad x > t \\ 0 & \text{for} \quad x < t, \end{cases}$$

is termed the *Green function* of the boundary value problem (11), (2).

It follows from the uniqueness of the choice of the constants the uniqueness of the Green function. One can see from the next theorem that this Green matrix function, which is constructed only with the help of the Cauchy matrix function and its mixed quasiderivatives, is the analogue of the Green function in the classical interpretation (see, for example, [1], pp. 115-116).

Theorem 1. *In assumption that λ is not eigenvalue of the problem (11), (2), the solution of this problem can be given in the form*

$$\mathbf{y}(x) = \int_a^b G(x,t,\lambda)d\mathbf{f}(t), \tag{17}$$

where the Green matrix function $G(x,t,\lambda)$ is represented by the formula (16) and it possesses the next properties:

1) the quasiderivatives with respect to the first variable $G^{[k]}(x,t,\lambda)$ $(k = 0,\ldots,n-1)$ are the continuous matrix functions of two variables x, t and they are the absolutely continuous matrix functions with respect to the each variable under the second one is fixed;

2) the quasiderivatives $G^{[k]}(x,t,\lambda)$ $(k = n,\ldots,r-1)$ have bounded variation on $[a,b]$ with respect to the first variable and they are absolutely continuous with respect to t;

3) $G(x,t,\lambda)$ satisfies the equation (3) with respect to x on the each of intervals $[a,t),(t,b]$;

4) $G(x,t,\lambda)$ satisfies the boundary conditions (4) with respect to x;

5) under $x = t$ the matrix function $G(x,t,\lambda)$ satisfies the conditions of a jump

$$G^{[k]}(t+0,t,\lambda) - G^{[k]}(t-0,t,\lambda) = 0, \quad k = \overline{0,n-1};$$

$$G^{[n+s]}(t+0,t,\lambda)-G^{[n+s]}(t-0,t,\lambda)=$$
$$=-\sum_{\nu=1}^{r}\sum_{k=1}^{r}\sum_{j=0}^{r-1}\sum_{i=0}^{n-1}\Delta B_{n-i,s+1}(t)K^{(i)*\{k-1\}*}(t,a,\lambda)\frac{W_{\nu k}}{\Delta(\lambda)}\Delta_{\nu j}K^{[j]}(b,t,\lambda),\quad s=0,\dots,m-2;$$

$$G^{[r-1]}(t+0,t,\lambda)-G^{[r-1]}(t-0,t,\lambda)=E-\sum_{\nu=1}^{r}\sum_{k=1}^{r}\sum_{j=0}^{r-1}\sum_{i=0}^{n-1}\Delta B_{n-i,m}(t)K^{(i)*\{k-1\}*}(t,a,\lambda)\frac{W_{\nu k}}{\Delta(\lambda)}\Delta_{\nu j}K^{[j]}(b,t,\lambda).$$

□ *Proof.* The formula (17) is proved above. By using the above-mentioned properties of the solutions of the equation (3) and the adjoint one to it, it is easy to prove the properties 1) – 4). For the proof of the property 5) the relations

$$K^{*\{k-1\}*}(x,t,\lambda)=\sum_{j=1}^{r}Y_j(x,\lambda)C_{jk}(t,\lambda),\quad k=1,\dots,r, \tag{18}$$

are used. They follow from the fact that all $K^{*\{k-1\}*}(x,t,\lambda)$ are the solutions of the equation (3); $Y_j(x)$, $j=1,\dots,r$, are a fundamental solution system of the equation (3). Then, in consequence of the relation from [12, 13, p. 134],

$$\Delta Y^{[n+s]}(x)=\sum_{i=0}^{n-1}\Delta B_{n-i,s+1}(x)Y^{[i]}(x),$$
$$s=0,\dots,m-1,$$

by expanding (18), one can obtain the property 5), which completes the proof. ■

Remark 1. Note that in the case of $\Delta B_{ij}(x)=0$, $i=1,\dots,n$, $j=1,\dots,m$, the property 5) takes the "classical" form

$$G^{[k]}(t+0,t,\lambda)-G^{[k]}(t-0,t,\lambda)=0,\quad k=0,\dots,r-2;\quad G^{[r-1]}(t+0,t,\lambda)-G^{[r-1]}(t-0,t,\lambda)=E.$$

Remark 2. The matrix function $G(x,t,\lambda)$ can be also represented in the form

$$G(x,t,\lambda)=(-1)^{rl}\frac{1}{\Delta(\lambda)}\begin{pmatrix}Q_{11}&\cdots&Q_{1l}\\\cdots&\cdots&\cdots\\Q_{l1}&\cdots&Q_{ll}\end{pmatrix}, \tag{19}$$

where

$$Q_{ij}(x,t,\lambda)=\begin{vmatrix}K_{i1}(x,a,\lambda)&\cdots&K_{il}^{*\{r-1\}*}(x,a,\lambda)&P_{ij}(x,t,\lambda)\\U_1(K_{11}(x,a,\lambda))&\cdots&U_1(K_{1l}^{*\{r-1\}*}(x,a,\lambda))&U_1(P_{1j}(x,t,\lambda))\\\cdots&\cdots&\cdots&\cdots\\U_r(K_{l1}(x,a,\lambda))&\cdots&U_r(K_{ll}^{*\{r-1\}*}(x,a,\lambda))&U_r(P_{lj}(x,t,\lambda))\end{vmatrix}. \tag{20}$$

Indeed, by expanding (20) by the entries of the last column and the first row it is not difficult to go from (19) to the relation (16).

IV. Resolving kernel of the problem (13), (6)

If λ is not an eigenvalue, the solution of the problem (13), (6) can be represented in the form of the integral of the resolving kernel (the Green matrix function of the problem (13), (6)) and the vector F. This result is necessary for further investigations of the properties of the Green function of the problem (11), (2).

For the problem (13), (6) the formula (see [12])

$$Y(x)=\Phi(x,a)Y(a)+\int_a^x\Phi(x,t)dF(t) \tag{21}$$

holds true. Here $\Phi(x,t)=\Phi(x,t,\lambda)$ is the fundamental matrix of the system (5); it is represented in the form $\Phi(x,t,\lambda)=R(x,\lambda)R^{-1}(t,\lambda)$, where $R(x,\lambda)$ is the integral matrix of the system (5). We can represent the relation (21) as follows:

$$Y(x)=R(x,\lambda)C+\int_a^x\Phi(x,t,\lambda)dF(t), \tag{22}$$

where $C=R^{-1}(a,\lambda)Y(a)$ is the rectangular matrix. By substituting (22) in the conditions (6), owing to $|W_aR(a,\lambda)+W_bR(b,\lambda)|\neq 0$ (because λ is not an eigenvalue of the boundary problem), it is possible to obtain the expression for the matrix C

$$C=-\left\{W_aR\left(a,\lambda\right)+W_bR\left(b,\lambda\right)\right\}^{-1}\int_a^bW_b\Phi\left(b,t,\lambda\right)dF(t).$$

Therefore the formula (22) becomes

$$Y(x) = \int_a^b M(x,t,\lambda)dF(t), \qquad (23)$$

where the resolving kernel

$$M(x,t,\lambda) = \begin{cases} \Phi(x,t,\lambda) - R(x,\lambda)\{W_a R(a,\lambda) + W_b R(b,\lambda)\}^{-1} W_b \Phi(b,t,\lambda) & \text{for} \quad x \geq t \\ -R(x,\lambda)\{W_a R(a,\lambda) + W_b R(b,\lambda)\}^{-1} W_b \Phi(b,t,\lambda) & \text{for} \quad x < t. \end{cases}$$

V. Relationship between the Green functions of the adjoint boundary value problems

Let $H(x,t,\lambda)$ be the Green matrix function of the adjoint boundary value problem (8), (10). It is constructed with the aid of the Cauchy matrix function $R(x,t,\lambda)$ of the homogeneous equation $L^*_{mn}(Z) = \bar{\lambda}Z$ and its mixed quasiderivatives in the sense of the initial quasidifferential equation and the adjoint one. It is not difficult to satisfy oneself that the formulas, which are analogous to (17) and (23), hold true also for the function $H(x,t,\lambda)$. Besides, it has the properties, which are similar to the presented in the theorem 1. In particular, $H(x,t,\lambda)$ and its quasiderivatives $H^{\{k\}}(x,t,\lambda)$ $(k = 1,\ldots,m-1)$ with respect to the first variable are the absolutely continuous with respect to the each of x, t under the second one is fixed and they are the continuous with respect to the totality of the variables x and t, but the rest quasiderivatives to order $r-1$ have the bounded variation with respect to x and they are the absolutely continuous with respect to t. The next theorem holds true.

Theorem 1. *Under $x \neq t$, if λ is not an eigenvalue of the boundary value problem (11), (2), the Green functions of the adjoint boundary value problem are related through the relationship*

$$G(x,t,\lambda) = H^*(t,x,\lambda).$$

□ *Proof.* We assume lossless the generality that $G(x,t,\lambda)$ and $H(x,t,\lambda)$ are the Green functions of the adjoint boundary value problems

$$L_{mn}(Y) - \lambda Y = -F_1(x), \qquad (24)$$

$$U_\nu(Y) = \sum_{j=0}^{r-1} \Gamma_{\nu j} Y^{[j]}(a) + \sum_{j=0}^{r-1} \Delta_{\nu j} Y^{[j]}(b) = 0, \quad \nu = 1,\ldots,r, \qquad (25)$$

$$L^*_{mn}(Z) - \bar{\lambda}Z = F_2(x), \qquad (26)$$

$$V_\nu(Z) = \sum_{j=0}^{r-1} \tilde{\Gamma}_{\nu j} Z^{\{j\}}(a) + \sum_{j=0}^{r-1} \tilde{\Delta}_{\nu j} Z^{\{j\}}(b) = 0, \quad \nu = 1,\ldots,r, \qquad (27)$$

correspondingly, where $F_1(x)$, $F_2(x)$ are the continuous matrices on $[a,b]$ and they consist of l identical columns of the form $\mathbf{f}_1(x)$ and $\mathbf{f}_2(x)$. Owing to introduction the rectangular matrices $Y = \left(Y, Y^{[1]}, \ldots, Y^{[r-1]}\right)^T$ and $Z = \left(Z^{\{r-1\}}, \ldots, Z^{\{1\}}, Z\right)^T$, this problems reduce to the problems

$$\begin{cases} Y' = B'Y + F_1, \\ W_a Y(a) + W_b Y(b) = 0, \end{cases} \qquad \begin{cases} Z' = -\left(B^*\right)' Z + F_2, \\ \tilde{W}_a Z(a) + \tilde{W}_b Z(b) = 0 \end{cases}$$

correspondingly, where $F_1(x) = (0,\ldots,0,F_1(x))^T$, $F_2(x) = (F_2(x),0,\ldots,0)^T$, and W_a, W_b, $\tilde{W}_a$, $\tilde{W}_b$ are $rl \times rl$ numerical matrices.

Since the products $(Z^*)' Y$ and $Z^* Y'$ are well posed,

$$(Z^*Y)' = (Z^*)' Y + Z^*Y' =$$
$$= -Z^*B'Y + F_2^*Y + Z^*B'Y + Z^*F_1 = Z^*F_1 + F_2^*Y.$$

On the other hand, by taking into account the method of the constructing the boundary conditions of the adjoint boundary value problem (10), one can satisfy oneself in validity of the relation (9) for the nonhomogeneous adjoint boundary value problem (24)–(27). Then

$$\int_a^b \left(Z^*(x)F_1(x) + F_2^*(x)Y(x)\right) dx = 0.$$

According to the formula (23)

$$Y(x) = \int_a^b M(x,t,\lambda)F_1(t)dt,$$

$$Z(t) = \int_a^b N(t,x,\lambda)F_2(x)dx,$$

by taking into consideration (17), it is possible to conclude that the last entry of the first row of block matrix $M(x,t,\lambda)$ equals $-G(x,t,\lambda)$, and the first entry of the last row of the block matrix $N(t,x,\lambda)$ coincide with $H(t,x,\lambda)$. Moreover,

$$\int_a^b \left(\int_a^b N(t,x,\lambda)F_2(x)dx\right)^* F_1(t)dt +$$
$$+ \int_a^b F_2^*(x) \int_a^b M(x,t,\lambda)F_1(t)dtdx =$$
$$= \int_a^b \int_a^b F_2^*(x) \left\{N^*(t,x,\lambda) + M(x,t,\lambda)\right\} F_1(t)dxdt = 0,$$

that is

$$\int_a^b \int_a^b \mathbf{f}_2^*(x) \{H^*(t,x,\lambda) - G(x,t,\lambda)\} \mathbf{f}_1(t) dx dt = 0. \tag{28}$$

Let $G(x,t,\lambda) = (g_{ij}(x,t,\lambda))_{i,j=1}^l$, $H(x,t,\lambda) = (h_{ij}(x,t,\lambda))_{i,j=1}^l$, $\mathbf{f}_1(x) = (f_{11},\ldots,f_{1l})^T$, $\mathbf{f}_2(x) = (f_{21},\ldots,f_{2l})^T$, let (x_0,t_0) be any point of the domain $a \le x,\ t \le b,\ x \ne t$. We choose arbitrarily the small rectangular Δs, surrounding it, with the sides $t = t_0 \pm \Delta t$ and $x = x_0 \pm \Delta x$ and such vector functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(x)$, so as to $f_{1j}(t) \equiv 0$ under $j \ne j_0$, $f_{1,j_0}(t) \ne 0$ in Δs, $f_{1,j_0}(t) \equiv 0$ out of Δs, $f_{2i}(t) \equiv 0$ under $i \ne i_0$, $f_{2,i_0}(t) \ne 0$ in Δs, $f_{2,i_0}(t) \equiv 0$ out of Δs. For this selection the equation (28) is equivalent to

$$\int_{t_0-\Delta t}^{t_0+\Delta t} \int_{x_0-\Delta x}^{x_0+\Delta x} f_{2,i_0}(x) \left[\overline{h_{j_0,i_0}(t,x,\lambda)} - g_{i_0,j_0}(x,t,\lambda)\right] \times$$
$$\times f_{1,j_0}(t) dx dt = 0.$$

Since $f_{2,i_0}(x) f_{1,j_0}(t) \ne 0$ in Δs, obviously, the expression in the square brackets turns into zero somewhere in this domain. Let Δx and Δt tend to zero. Then in the limit we obtain $\overline{h_{j_0,i_0}(t_0,x_0,\lambda)} = g_{i_0,j_0}(x_0,t_0,\lambda)$ and as a result of randomness of the selection of the vectors $\mathbf{f}_1(t)$, $\mathbf{f}_2(x)$ and points x_0, t_0 $(x_0 \ne t_0)$ we obtain statement of the theorem. ■

Conclusions

The obtained results allow to investigate other problems, including the problems of the eigenfunction expansions. The analogous results can be also obtained in the case of the boundary value problem for the vector singular differential equation.

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ФУНКЦИЯ ГРИНА КРАЕВОЙ ЗАДАЧИ ДЛЯ ВЕКТОРНОГО СИНГУЛЯРНОГО КВАЗИДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

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Построено матрицу-функцию Грина краевой задачи для векторного квазидифференциального уравнения с обобщенными функциями в коэффициентах. С помощью метода введения квазипроизводных и полученных выражений для сопряженных краевых условий исследуются свойства функций Грина сопряженных краевых задач.

Ключевые слова: функция Грина, квазидифференциальное уравнение, обобщенные функции, квазипроизводные.

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ФУНКЦІЯ ҐРІНА КРАЙОВОЇ ЗАДАЧІ ДЛЯ ВЕКТОРНОГО СИНГУЛЯРНОГО КВАЗІДИФЕРЕНЦІАЛЬНОГО РІВНЯННЯ

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Побудовано матрицю-функцію Ґріна крайової задачі для векторного квазідиференціального рівняння з узагальненими функціями в коефіцієнтах. За допомогою методу введення квазіпохідних і отриманих виразів для спряжених крайових умов досліджуються властивості функцій Ґріна спряжених крайових задач.

Ключові слова: функція Ґріна, квазідиференціальне рівняння, узагальнені функції, квазіпохідні

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