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On Automatic Continuity and Three Problems of "The Scottish Book" Concerning the Boundedness of Polynomial Functionals

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In this paper we introduce and study the notions of isotropic mapping and essential kernel. In addition some theorems on the Borel graph and Baire mapping for polynomial operators are proved. It is shown that a polynomial functional from an infinite dimensional complex linear space into the field of complex numbers vanishes on some infinite dimensional affine subspace.

1. INTRODUCTION

Let $X^k := X \times \cdots \times X$ be the kth Cartesian product of a normed space X. Consider an operator $B_k(x_1, \dots, x_k)$ from X^k into a normed space Y, linear on each of its arguments; we shall call this a k-linear operator. Setting $x_j = x \in X$ $(j = 1, \dots, k)$, we obtain the operator $P_k(x) \equiv B_k(x, \dots, x)$ from X into Y; P_k is called a homogeneous polynomial operator of degree k, provided that $P_k(x)$ is not identically zero. A constant operator will be denoted by $P_0(x)$. An operator P: $X \to Y$ of the form $P(x) = P_0(x) + P_1(x) + \cdots + P_m(x)$ will be called a polynomial operator of degree m, provided that P_m is not identically zero. If Y is the field of complex or real numbers then we call the polynomial operator P: $X \to Y$ a polynomial functional. The main results of the theory of polyno-

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mial operators are contained in the books [5, 10]. It is well known that a linear operator on a Banach space becomes automatically continuous (or bounded, which is the same) under some additional conditions. The Closed Graph Theorem, the inverse mapping theorem, the theorem on the continuity of Baire operators [4], and many others (see, for instance, [27]) belong to results of this kind. Less is known about automatic continuity of polynomial operators. The polynomial analogs of the Banach–Steinhaus theorem and the uniform boundedness principle are proved in [16], see also [1]. The Open Mapping Theorem for k-linear operators is not true [12], even in the finite-dimensional case [11]. The Banach inverse mapping theorem is not true for polynomial operators [24, 21]. The Closed Graph Theorem for polynomial functionals (even for G-holomorphic functionals) is proved in [6].

We start our research with a problem of "The Scottish Book" [26].

It is obvious that for each discontinuous linear operator T on the normed space X unboundedness of a sequence $(T(x_i))$ implies that the sequence $(T(x + x_i))$ is unbounded for all $x \in X$. A simple example (see below) shows that this fact is not true for polynomial functionals of degree 2. It is possible that in connection with this fact there appears:

Problem 56 (Mazur, Orlicz). Let f be a discontinuous polynomial functional of degree n on a Banach space X. ("Of degree n" here means that for every $x, y \in X$ there exist numbers $a_0, \dots, a_n$ such that $f(x + ty) = a_0 + a_1 t + \cdots + a_n t^n$ for all rational numbers t.) Does there exist a sequence $x_i \in X$ such that $x_i \to 0$ and $f(x + x_i) \to \infty$ or at least $\overline{\lim}_i |f(x + x_i)| = \infty$ for all $x \in X$?

We will consider for a while the same definition of polynomial functional of degree n that is in Problem 56. But mainly we will investigate usual polynomial operators and prove the following theorems.

THEOREM 1. *Let P be a polynomial operator from a normed space X into a normed space Y. If P is discontinuous then there exists a sequence $z_i \in X$, $z_i \to 0$ as $i \to \infty$ such that* $\sup_i \|P(x_0 + z_i)\| = \infty$ *for each element $x_0 \in X$.*

THEOREM 2. *On every normed space X, that has the linear dimension ω_1, there exists a discontinuous polynomial functional p of degree 2 for which there is no sequence $x_i \to 0$ such that $p(x + x_i) \to \infty$ as $i \to \infty$ for each element $x \in X$.*

We shall introduce the following definition which is motivated by Problem 56 and Theorem 1.

DEFINITION 1. Let X be a metric group (not necessarily commutative) with group operation $+$ and Y be a metric space. We call a mapping F: $X \to Y$ isotropic if either it is everywhere continuous or there exists a sequence $x_i \in X$, $x_i \to 0$, such that for some number $c > 0$

$$\overline{\lim_i} \operatorname{dist}\big(F(x + x_i), F(x)\big) \geq c \tag{1}$$

for each $x \in X$.

We call the maximal number c for which (1) is true the isotropy constant; it can be equal to ∞.

We shall consider the isotropy constant of the everywhere continuous mapping equal to zero. Conversely when we speak about isotropic mappings with zero constant we mean everywhere continuous mappings. A linear operator on a Banach space is a typical example of an isotropic mapping.

COROLLARY 1 (of Theorem 1). *Each polynomial operator between normed spaces is isotropic* (*with isotropy constant equal to* 0 *or* ∞).

The main result of Section 2 is the proving of the Borel graph and Baire mapping theorems for isotropic mappings.

In Section 3 we will consider the two following questions from [26].

Problem 75 (Mazur). Let a polynomial functional p on a Banach space X be bounded on an ϵ-neighborhood of a set $M \subset X$. Does there exist for each number a a δ-neighborhood of the set aM on which p will also be bounded?

Problem 55 (Mazur). Let a polynomial functional p on a Banach space X be bounded on an ϵ-neighborhood of a certain set $M \subset X$. Does there exist a polynomial functional q and a linear operator T on the space X such that $p = qT$ and the set $T(M)$ is bounded?

In Problem 55 it is not required (formally) that the polynomial functional q be continuous. Problem 55 with the requirement of the continuity of q will be called Problem 55′.

The answer to Problem 75 for linear operators is affirmative, of course. It is easy to see that the affirmative answer to Problem 75 is the result of an affirmative answer to Problem 55′. We will write this in the following way: Pr.55′ $\to$ Pr.75. For (as hinted at in [26]) given X, p, and M the set $p(aM) = q(aT(M))$ is bounded since T is linear and q is continuous (bounded). In [2] it is proved that the answer to Problem 55′ and also to Problem 75 is affirmative for finite-dimensional spaces. We will show that the answer to Problem 75 (also to Problem 55′) is negative, that Pr.55 $\not\to$ Pr.55′ and Pr.75 $\not\to$ Pr.55′, i.e.,

$$\text{Pr.55} \overset{\not\to}{\leftarrow} \text{Pr.55}' \overset{\to}{\not\leftarrow} \text{Pr.75}.$$

Problem 75 has also recently been shown to have a solution in the negative by Pestov in [18]. We do not know the answer to Problem 55. The following question is a version of it.

QUESTION 1. *Let p be a polynomial functional on a separable Banach space X, which is bounded on a neighborhood of some (unbounded) set M. Does there exist a (generally speaking, discontinuous) polynomial functional q and a continuous operator T on the space X such that $p = qT$ and the set $T(M)$ is bounded*?

The simple example of the polynomial $p(x, y) = xy$, $x, y \in \mathbf{R}$ shows that the set $\{x : p(x) = p(0)\}$ may not be a linear manifold and the polynomial p will be unbounded on each ϵ-neighborhood of this set. Problem 55 may have been an attempt to give an analog to the null-space of linear operators for polynomials. For this reason we shall introduce the notion of essential kernel for the polynomial functionals. At the end of the article we will show that any complex polynomial functional $p(x), p(0) = 0$, defined on an infinite-dimensional linear space, vanishes on an infinite-dimensional linear subspace.

2. AUTOMATIC CONTINUITY AND ISOTROPIC MAPPINGS

First of all we will give the example of a polynomial functional p of degree 2 for which unboundedness of sequence $(p(x + x_i))$ for an arbitrary vector $x \in X$ does not follow from the unboundedness of sequence $(p(x_i))$ (this was mentioned in the Introduction).

EXAMPLE 1. Let X be an infinite-dimensional normed space, f and g be linear functionals on X. Let $f \neq 0$ and g is unbounded on the kernel of the functional f: $\exists (x_i) \subset \ker f, x_i \to 0, g(x_i) \to \infty$ as $i \to \infty$. Put $p(x) = f(x_0 - x)g(x)$ for a fixed point $x_0 \in \ker g, x_0 \notin \ker f$. Then

$$p(x_i) = (f(x_0) - f(x_i))g(x_i) = f(x_0)g(x_i) \to \infty$$

as $i \to \infty$. But $p(x_i + x_0) = -f(x_i)g(x_i) = 0$.

The proof of next lemma is not complicated.

LEMMA 1. *Let $(a_i^k)_{i=1}^{\infty}$, $k = 1, \dots, n$ be a collection of nonnegative scalars such that for some k, $a_i^k \to \infty$ as $i \to \infty$. Then there exist a number $k_0, 1 \le k_0 \le n$ and a subsequence i_s, $s = 1, \dots, \infty$, of positive integers such that $a_{i_s}^k / a_{i_s}^{k_0} \to 0$ as $s \to \infty$ for each $k < k_0$ and $\sup_s a_{i_s}^k / a_{i_s}^{k_0} < \infty$ for $k > k_0$.*

It is well known that the continuity of every term in decomposition $P = \sum_0^n P_k$ follows from the continuity of the polynomial P. The next proposition makes this result more exact.

PROPOSITION 1. *Let P be a polynomial operator from a normed space X into a normed space Y. In addition, let there exist a sequence $x_i \to 0$ and a term P_k in the decomposition $P = \sum_0^n P_k$ such that $\|P_k(x_i)\| \to \infty$ as $i \to \infty$. Then $\sup_i \|P(m^{-1}x_i)\| = \infty$ for some positive integer m.*

Proof. It follows from Lemma 1 that it is possible to select a number k_0 and a subsequence of (x_i) (this subsequence we denote by the same symbol) such that

$$\text{for } k < k_0 \qquad \frac{\|P_k(x_i)\|}{\|P_{k_0}(x_i)\|} \to 0 \text{ as } i \to \infty \tag{2}$$

$$\text{for } k > k_0 \qquad \sup_i \frac{\|P_k(x_i)\|}{\|P_{k_0}(x_i)\|} = c < \infty. \tag{3}$$

It follows from (2) that for $k < k_0$ we have

$$\frac{\|P_k(m^{-1}x_i)\|}{\|P_{k_0}(m^{-1}x_i)\|} = m^{-k}m^{k_0}\frac{\|P_k(x_i)\|}{\|P_{k_0}(x_i)\|} \to 0$$

as $i \to \infty$. Consequently

$$\frac{1}{n}\|P_{k_0}(m^{-1}x_i)\| - \|P_k(m^{-1}x_i)\| \to \infty \tag{2$'$}$$

as $i \to \infty$ for $k < k_0$.

According to (3) for $k > k_0$ we have

$$\frac{\|P_k(m^{-1}x_i)\|}{\|P_{k_0}(m^{-1}x_i)\|} = m^{-k}m^{k_0}\frac{\|P_k(x_i)\|}{\|P_{k_0}(x_i)\|} \le cm^{-k}m^{k_0}.$$

Consequently, if m is large enough, then for $k > k_0$

$$\frac{1}{n}\|P_{k_0}(m^{-1}x_i)\| - \|P_k(m^{-1}x_i)\| \to \infty \tag{3$'$}$$

as $i \to \infty$. Then

$$\begin{aligned}\left\| P\left(m^{-1}x_i\right)\right\| &\geq \left\| P_{k_0}\left(m^{-1}x_i\right)\right\| - \sum_{k \neq k_0} \left\| P\left(m^{-1}x_i\right)\right\| \\ &= \sum_{k<k_0} \left(\frac{1}{n}\left\| P_{k_0}\left(m^{-1}x_i\right)\right\| - \left\| P_k\left(m^{-1}x_i\right)\right\|\right) \\ &\quad + \sum_{k>k_0} \left(\frac{1}{n}\left\| P_{k_0}\left(m^{-1}x_i\right)\right\| - \left\| P_k\left(m^{-1}x_i\right)\right\|\right) \to \infty\end{aligned}$$

as $i \to \infty$, because according to (2′), (3′) each term of this sum tends to ∞ if m is large enough. The proposition is proved.

Proof of Theorem 1. Certainly we can suppose that in the decomposition $P = \sum_0^n P_k$ of the operator P into homogeneous polynomials the term with the biggest degree is discontinuous, that is, there exists a sequence $x_i \to 0$ such that $P_n(x_i) \to \infty$. Fixing a vector $x_0 \in X$ we decompose the operator $P_{x_0}(x) := P(x_0 + x)$ into homogeneous polynomials,

$$P_{x_0}(x) := \sum_{k=0}^{n} B_k(x_0, x),$$

where $B_k(x_0, x)$ is a homogeneous polynomial of degree k of the argument x for fixed x_0 and $B_n(x_0, x) = P_n(x)$. As we are supposing that P_n is discontinuous, more exactly, that $P_n(x_i) \to \infty$ as $i \to \infty$, then it follows from Proposition 1 that there exists a number m (depending on x_0) such that $\sup_i \|P(x_0 + m^{-1}x_i)\| = \infty$. Let $y_{m,i} = m^{-1}x_i$. Let $\phi(m,i)$: $\mathbf{N} \times \mathbf{N} \to \mathbf{N}$ be a one-to-one mapping such that $y_{m,i} \to 0$ as $\phi(m,i) \to \infty$, $\mathbf{N}$ is the set of positive integers. Put $z_{\phi(m,i)} = y_{m,i}$ and Theorem 1 is proved.

Proof of Theorem 2. Let e_α, $1 \leq \alpha < \omega_1$, be a normalized Hamel basis for X. Put $f(e_n) = n$ if $n < \omega_0$ and $f(\alpha) = 0$ if $\alpha \geq \omega_0$ and extend f to the whole space X as a linear functional.

Now we will construct a symmetric bilinear functional $B(x, y)$ in the following way. We index all linear independent sequences $x_n \to 0$: $(x_n)_\beta$, $1 \leq \beta < \omega_1$, and put in correspondence to each sequence $(x_n)_\beta$ an element e_{α_β}, so that:

(1) $\alpha_\beta > \omega_0$,

(2) $\alpha_{\beta_1} > \alpha_{\beta_2}$ if $\beta_1 > \beta_2$,

(3) $\alpha_\beta > \min(\alpha : (x_n)_\beta \in \operatorname{lin}(e_\gamma : \gamma < \alpha))$.

Put $X_\beta = \operatorname{lin}(e_\alpha : \alpha < \alpha_\beta)$, $1 \leq \beta < \omega_1$; obviously $\bigcup_\beta X_\beta = X$. Put $B(x, y) = 0$ if $x, y \in X_1$. If $B(x, y)$ is defined on X_γ for all $\gamma < \beta$ and β is a limit ordinal then $B(x, y)$ is defined also on $X_\beta = \bigcup_{\gamma<\beta} X_\gamma$.

Now let $B(x, y)$ be defined on X_β. We shall define it on $X_{\beta+1}$. For this purpose we define in $X_{\beta+1}$ a complement to $\operatorname{lin}(x_n)_\beta$—the subspace $Y_\beta \supset (e_\gamma : \alpha_\beta \le \gamma < \alpha_{\beta+1})$, and put $B(x, e_\gamma) = 0$ if $x \in Y_\beta$, $\alpha_\beta \le \gamma < \alpha_{\beta+1}$. Put

$$B(x_n, e_\gamma) = \begin{cases} -(1/2)\big(f(x_n) + B(x_n, x_n)\big) & \text{if } \gamma = \alpha_\beta \\ 0 & \text{if } \alpha_\beta < \gamma < \alpha_{\beta+1} \end{cases}$$

for $x_n \in (x_n)_\beta$. Finally we extend B to the whole of $X_{\beta+1}$ as a symmetric bilinear functional. Therefore, the polynomial functional $p(x) = f(x) + B(x, x)$ is unbounded because $B(e_\alpha, e_\alpha) = 0$ if $\alpha < \omega_1$.

Suppose a linearly independent sequence $(x_n)_\alpha$ converges to zero. Then for the corresponding (e_{α_β}) we have

$$\begin{aligned} p(x_n + e_{\alpha_\beta}) &= f(x_n) + f(e_{\alpha_\beta}) + B(x_n, x_n) + 2B(x_n, e_{\alpha_\beta}) + B(e_{\alpha_\beta}, e_{\alpha_\beta}) \\ &= f(e_{\alpha_\beta}) + B(e_{\alpha_\beta}, e_{\alpha_\beta}). \end{aligned}$$

So for $x_0 = e_{\alpha_\beta}$ we have that $p(x_0 + x_n)$ does not converge to infinity.

Now let (x_n) be an arbitrary sequence in X which converges to zero. If (x_n) does not contain a linearly independent subsequence then $\operatorname{lin}(x_i)$ is a finite-dimensional subspace. But arbitrary polynomial functionals on a finite-dimensional space are continuous so $\lim_{n\to\infty} p(x + x_n) = p(x) < \infty$. If (x_n) contains a linearly independent subsequence (x_{n_i}) then from the condition $\lim_{n\to\infty} p(x + x_n) = \infty\ \forall x \in X$ it follows that $\lim_{i\to\infty} p(x + x_{n_i}) = \infty\ \forall x \in X$, but we showed that for some point x_0 this is not true. Theorem 2 is proved.

Remark 1. If we suppose the continuum hypothesis and consider the field of real numbers as a normed space over the field of rational numbers then the construction of Theorem 2 will be an example of a polynomial functional of degree 2, in the sense of Problem 56, which gives the negative answer to the first part of this problem.

The following proposition shows that the result similar to Theorem 1 is true for symmetric n-linear operators.

THEOREM 3. *Let X, Y be linear normed spaces and B_n: $X^n \to Y$ be a symmetric n-linear operator. If this operator is discontinuous then there exists a sequence $\bar z_i = (z_i^1, \dots, z_i^n) \in X^n$, $\bar z_i \to 0$ as $i \to \infty$ such that* $\sup_i \|B_n(\bar x_0 + \bar z_i)\| = \infty$ *for each $\bar x_0 = (x_0^1, \dots, x_0^n) \in X^n$.*

Proof. If the mapping B_n is discontinuous then it follows from the polarization formula [5, p. 4] that the homogeneous polynomial $P_n(x) =$

$B_n(x,\dots,x)$ is discontinuous. So there exists a sequence $x_i \in X$, $x_i \to \infty$ such that $\|P_n(x_i)\| \to \infty$ as $i \to \infty$. If we fix an element $\bar{x}_0 \in X^n$ then $B_n(\bar{x}_0 + (x,\dots,x))$ will be a polynomial of degree n in the variable $x \in X$,

$$B_n\big(\bar{x}_0 + (x,\dots,x)\big) = P_0(x) + P_1(x) + \cdots + P_{n-1}(x) + P_n(x)$$

moreover $P_0(x) = B_n(\bar{x}_0)$ and $P_n(x) = B_n(x,\dots,x)$. Since $P_n(x_i) \to \infty$ as $i \to \infty$, according to Proposition 1 there exists a number m (dependent on x_0) such that $\sup_i \|B_n(\bar{x}_0 + m^{-1}(x_i,\dots,x_i)\| = \infty$. Now define (z_i) as in the proof of Theorem 1.

COROLLARY 2. *Every n-linear symmetric mapping between normed spaces is isotropic* (*with the isotropy constant equal to* 0 *or* ∞).

We will give some properties of isotropic mappings.

PROPOSITION 2. *Let F_n, $n = 1,\dots,\infty$, be a sequence of isotropic mappings from a metric group X into a metric space Y which converges uniformly on each bounded set to a mapping $F\colon X \to Y$. Assume its isotropy constants c_n satisfy* $\inf_n c_n =: c > 0$. *Then the mapping F is isotropic and its isotropy constant is not smaller than c.*

Proof. For each n there exists a sequence $x_i^n \to 0$ as $i \to \infty$ such that $\overline{\lim}_i \operatorname{dist}(F_n(x + x_i^n), F_n(x)) \geq c_n \geq c$ for any element $x \in X$. Of course, we can assume that $\sup_i \operatorname{dist}(x_i^n, 0) \to 0$ as $n \to \infty$. Let us index the elements of sequence $(x_i^n)_{i=1,\infty}^{n=1,\infty}$ with one index j: $(z_j)_{j=1}^{\infty} = (x_i^n)_{i=1,\infty}^{n=1,\infty}$, so that $z_j \to 0$ as $j \to \infty$. Since (F_n) converges to F uniformly on each bounded set then $\overline{\lim}_i \operatorname{dist}(F(x + z_i), F(x)) \geq c$ for every element $x \in X$.

COROLLARY 3. *The limit in the topology of uniform convergence on bounded sets of Banach space polynomial mappings is an isotropic mapping with the isotropy constant equal to zero or infinity.*

The following proposition enables us to construct other examples of isotropic mappings.

PROPOSITION 3. *Let X, Y, Z be Banach spaces, $F\colon X \to Y$ be an isotropic mapping with the isotropy constant $c > 0$, and $G\colon Y \to Z$ be a continuous mapping for which there exists a constant $a > 0$ such that $\|G(y)\| > a\|y\|$ for each $y \in Y$. Then the composition $G \circ F$ is an isotropic mapping with the isotropy constant not smaller than ac.*

The proof is clear.

We will now give an example of an isotropic mapping with isotropy constant not equal to 0 or ∞. Let T be a linear unbounded operator on a Banach space $(X, \|\ \|)$. We introduce on X the new metric $\rho(x, y) =$

$\min(\|x-y\|, c)$ where $c > 0$. Then (X, ρ) is a metric group, moreover ρ generates the same topology as generated by norm $\| \ \|$. If $x_i \to 0$ and $\|T(x_i)\| \to \infty$, then $\rho(T(x+x_i), T(x)) = c$ for sufficiently large i, i.e., the isotropy constant of mapping T on (X, ρ) is equal to c exactly.

Let us recall some definitions from the theory of metric spaces. A subset M of a metric space X is called residual if its complement, $X \setminus M$, is of the first category, i.e., a countable union of nowhere dense sets. A set $M \subset X$ is called perfect if it is closed and does not contain isolated points. A mapping F from a metric space X into a metric space Y satisfies the condition of Baire if in each nonempty perfect set $M \subset X$ there exists a residual in M subset $N \subset M$ such that the restriction $F|_N$ is continuous. Finally, the mapping $F\colon X \to Y$ is called a Baire mapping (measurable in the terminology of [4]), if it belongs to the smallest class of mappings, which includes continuous mappings and is closed under pointwise limits.

PROPOSITION 4. *Let F be an isotropic mapping from a complete metric group X into a complete metric space Y. If F is discontinuous at zero then it is discontinuous on each residual set $M \subset X$.*

Proof. Let (x_i) be the sequence from Definition 1 and $M \subset X$ be a residual set in X. Then the set $M - x_i$ is residual for each i so $N = \bigcap_i (M \cap (M - x_i))$ is also residual and $N \subset M$.

Let $x_0 \in N$; then $x_0 + x_i \in M$. If we suppose that the mapping F is continuous on M then $F(x_0 + x_i) \to F(x_0)$ as $i \to \infty$. But this contradicts Definition 1.

COROLLARY 4. *Let an isotropic mapping F from a complete metric group X into a metric space Y satisfy the condition of Baire (or at least be continuous on a residual set). Then F is continuous.*

Since a Baire mapping satisfies the condition of Baire we have:

COROLLARY 5 (Generalization of Theorem 4 [4, Chap. 1, Sect. 3]. *A Baire isotropic mapping from a complete metric group X into a metric space Y is continuous.*

COROLLARY 6 (a similar result is in [9]). *A polynomial operator from a Banach space X to Banach space Y, which satisfies the condition of Baire (or at least is continuous on a residual set) is continuous.*

COROLLARY 7. *Let F be a continuous bijective mapping from a Polish space (i.e., separable complete metric space) X onto a complete metric group Y such that F^{-1} is isotropic. Then F^{-1} is continuous.*

Since, F^{-1} is a Baire mapping under the conditions of this corollary (this follows from [14, Chap. 3, Sect. 39.4, and Chap. 2, Sect. 31.9]) it remains to use Corollary 5.

THEOREM 4. *An isotropic mapping F from an arbitrary complete metric group X into a Polish group Y which has Borel graph is continuous.*

Proof. First we suppose that the group X is separable. Let $\Gamma = \{(x, F(x)) : x \in X\}$ be the graph of F. We let $F_*(x) := (x, F(x)) \in \Gamma$, $x \in X$. It is clear that F_* is a bijective mapping from X onto Γ and F_*^{-1} is continuous. As F_* is a Borel mapping [14, Chap. 3, Sect. 39.5] it is a Baire mapping [14, Chap. 2, Sect. 31.9]. If F is discontinuous then according to Definition 1 there exists a sequence $x_i \to 0$ as $i \to 0$ such that for each $x \in X$

$$\overline{\lim_i} \operatorname{dist}\big(F(x + x_i), F(x)\big) \geq c > 0.$$

So $\overline{\lim}_i \operatorname{dist}((x + x_i, F(x + x_i)), (x, F(x))) \geq c$. Therefore F_* is isotropic. But according to Corollary 4 this mapping is necessarily continuous. So the mapping F is continuous too.

Now let X be an arbitrary complete metric group. If F is discontinuous then it is discontinuous on a separable closed subgroup $X_0 \subset X$. The graph of the mapping $F: X_0 \to Y$ is the intersection of the Borel set Γ and the closed subset $X_0 \times Y \subset X \times Y$. So from the first part of this proof it follows that $F: X_0 \to Y$ is a continuous mapping. This contradiction proves the theorem.

COROLLARY 8. *A polynomial operator from a Banach space X into a separable Banach space Y which has Borel graph is continuous.*

Remark 2. It is shown in [22] that the Borel graph theorem is not true for linear operators if it is not supposed that Y is separable. We do not know if the Closed Graph Theorem is true for polynomial operators on arbitrary Banach spaces. The next proposition shows that it is true for n-linear operators.

PROPOSITION 5 (a similar result is in [8]). *Let $B: X^n \to Y$ be an n-linear operator, where X, Y are Banach spaces and the graph of B is closed. Then B is continuous.*

Proof. If the graph of B is closed then the graph of linear operator

$$B_{x_1 \cdots x_{k-1} x_{k+1} \cdots x_n}(x) := B(x_1, \ldots, x_{k-1}, x, x_{k+1}, \ldots, x_n)$$

is closed for every fixed $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in X$. According to classical Closed Graph Theorem for linear operators the mapping $B_{x_1 \cdots x_{k-1} x_{k+1} \cdots x_n}$ is continuous. But it follows from the separate continuity of an n-linear operator that this operator is continuous on Banach space [16].

We want to point out one more question from [26].

Problem 45 (Banach). Let G be a complete and non-Abelian metric group, $F_1(x), F_2(x), \dots, F_n(x)$ be multiplicative mappings from G to G. Prove that if the mapping $F(x) = F_1(x)F_2(x) \cdots F_n(x)$ satisfies the condition of Baire then it must be continuous.

We do not know the answer to this question. In connection with Problem 45 we raise the following:

QUESTION 2. *Is the mapping F in Problem* 45 *isotropic*?

3. BOUNDEDNESS OF POLYNOMIAL FUNCTIONALS ON UNBOUNDED SETS

Let us recall that a sequence of elements in a normed space is called minimal if each term does not belong to the closed linear span of the other terms of the sequence. With the help of a standard biorthogonalization procedure we can construct a minimal sequence in arbitrary normed space. The following proposition gives the negative answer to Problem 75 of [26].

PROPOSITION 6. *Let (x_m) be a minimal sequence in a normed space X. Then there exist numbers $a_m > 0$ and a continuous polynomial functional of degree two which is bounded on the neighborhood of radius 1 of the set $M = \{a_m x_m\}$ but which is unbounded on the set aM if $a \neq a^2$.*

Proof. Take numbers $b_m > 0$ and linear continuous functionals f_n on X such that for $y_m = b_m x_m$ we have $f_n(y_m) = \delta_{nm}$ (δ_{nm} is the Kronecker symbol) and $\|f_n\| = 1$. Put $p_1(x) = \sum_1^\infty n^{-2} f_n(x)$, $p_2(x) = \sum_1^\infty n^{-5}[f_n(x)]^2$, $p = p_1 - p_2$, and $a_m = m^3 b_m$. Let $\|x\| < 1$. Then we have for each m,

$$
\begin{aligned}
&\left| p\left(x + m^3 y_m\right)\right| \\
&\quad = \left|(p_1 - p_2)\left(x + m^3 y_m\right)\right| \\
&\quad = \left| p_1(x) + m - \sum_{n=1}^{\infty} n^{-5}\left[f_n\left(x + m^3 y_m\right)\right]^2 \right| \\
&\quad = \left| \sum_1^\infty n^{-2} f_n(x) + m - \sum_{n \neq m} n^{-5}\left[f_n(x)\right]^2 - m^{-5}\left[f_m(x) + m^3\right]^2 \right| \\
&\quad \leq \sum_1^\infty n^{-2} + \sum_{n \neq m} n^{-5} + m^{-5} + 2m^{-2} < c,
\end{aligned}
$$

where the number c is independent on m. But

$$p\left(am^3 y_m\right) = (p_1 - p_2)\left(am^3 y_m\right) = (a - a^2)m \to \infty \qquad \text{if } a \neq a^2.$$

The proposition is proved.

The following example shows that Pr.55 $\nrightarrow$ Pr.55$'$.

EXAMPLE 2. Let X be a separable infinite-dimensional Banach space and (x_m) a bounded Markushevich basis, i.e., a minimal sequence such that the closed linear span $[x_m]_1^\infty$ is equal to X and for biorthogonal functionals (f_m), $\sup_m \|x_m\| \|f_m\| < \infty$ and for each $x \in X \setminus \{0\}$ there exists a number m such that $f_m(x) \neq 0$. It is well known that a bounded Markushevich basis exists in arbitrary separable Banach space [15, p. 44]. Let p and M be as in Proposition 6. Since Pr.55$'$ $\to$ Pr.75 we must only prove the existence of a linear (and bounded) operator T on X and a (unbounded) polynomial functional q such that $p = qT$ and the set $T(M)$ is bounded.

We define the linear bounded injective operator T: $X \to X$ by formula $Tx = \sum_1^\infty m^{-4} f_m(x) x_m$. It is clear that the set $T(M)$ is bounded. We define the polynomial q on image TX by the formula $q(y) = pT^{-1}(y)$. Let Z be an algebraic complement to the subspace TX in X. We extend q to the whole space X by the formula $q(y + z) = q(y)$, $y \in TX$, $z \in Z$. It is clear that q is a polynomial and $p = qT$.

The next example shows that Pr.75 $\nrightarrow$ Pr.55$'$.

EXAMPLE 3. Let $X = c_0$ be the space of all null sequences with the maximum norm. There exists a homogeneous polynomial functional p on X of degree 2 and a set $M \subset X$ such that p is bounded on a neighborhood of radius 1 of the set aM for arbitrary number a but there does not exist a linear operator T in X and a continuous polynomial functional q such that $p = qT$ and the set $T(M)$ is bounded.

Put $p(x) = \sum_1^\infty n^{-2} a_n^2$ for $x = (a_1, a_2, \dots) \in c_0$ and $M = \{me_m\}_1^\infty$ where (e_m) is the standard basis for c_0. Then for arbitrary $a > 0$ and $x = (a_1, a_2, \dots)$, $\|x\| < 1$ we have

$$\begin{aligned} p(x + ame_m) &= \sum_{n \neq m} n^{-2} a_n^2 + m^{-2}(a_m + am)^2 \\ &\leq 2 + m^{-2} a_m^2 + m^{-2} 2 a_m am + a^2 \leq 3 + 2a + a^2. \end{aligned}$$

So p is bounded on the neighborhood of radius 1 of the set aM for arbitrary a. Now we will show that there does not exist a linear operator on

c_0 and a continuous polynomial q such that $p = qT$ and the set $T(M)$ is bounded. Since the sequence $y_m = T(me_m)$ is bounded and c_0 does not contain any subspaces isomorphic to l_1 by the Rosenthal alternative [15, p. 99] we can choose a weak Cauchy subsequence (y_{m_k}) from (y_n). Then $(y_{m_{2k+1}} - y_{m_{2k}})_{k=1}^{\infty}$ converges weakly to zero and we can choose from it a subsequence $(y_{\phi'(l)} - y_{\phi(l)})_{l=1}^{\infty}$ where $\phi(l) = m_{2k_l}$ and $\phi'(l) = m_{2k_l+1}$, which either converges to zero or is equivalent to the standard basis (e_n) of c_0 [15, p. 5].

In the first case we have $T(\phi'(l)e_{\phi'(l)} - \phi(l)e_{\phi(l)}) \to 0$ but $p(\phi'(l)e_{\phi'(l)} - \phi(l)e_{\phi(l)}) = 2$, therefore q must be unbounded. In the second case the sequence $T(\sum_{l=1}^{n}(\phi'(l)e_{\phi'(l)} - \phi(l)e_{\phi(l)}))$, $n = 1, \dots, \infty$, must be bounded but $p(\sum_{l=1}^{n}(\phi'(l)e_{\phi'(l)} - \phi(l)e_{\phi(l)})) = 2n \to \infty$, therefore p is unbounded.

We do not know how the set M must look in order that p be bounded on an ϵ-neighborhood of M. The consideration of this question leads to the notion of "the essential kernel." We shall begin with a lemma which is a simple statement about polynomials of two scalar variables.

LEMMA 2. *Let p be a polynomial functional on a normed space X and $x, y \in X$. If the scalar polynomial $p_{xy}(b) := p(x + by)$ is bounded on a unbounded set B then for each pair of numbers a and b we have*

$$p(ax + by) = p(ax), \tag{4}$$

in particular for each number b

$$p(by) = p(0). \tag{5}$$

Proof. It is obvious that

$$p(ax + by) = \sum_{0}^{n} p_k(ax, y) b^k, \tag{6}$$

where $p_k(ax, y)$ for fixed x, y, is a scalar homogeneous polynomial of some degree in the variable a (i.e., usual scalar homogeneous polynomial; see, for example, [5, pp. 8–9]). The boundedness of $p(ax + by)$ (as a polynomial in b for $a = 0$) on the unbounded set B means that $p_k(x, y) = 0$ for $k > 0$. So in (6) actually $p(ax + by) = p_0(ax, y) = p(ax)$. The equalities (4) and hence (5) are proved.

COROLLARY 9. *If the polynomial functional p is bounded by a number c on an ϵ-neighborhood on an unbounded subset of some line $L \subset X$, then it is bounded on every neighborhood of the whole line L. Moreover, p is bounded on an ϵ-neighborhood of the whole line L by the same number c.*

COROLLARY 10. *If the polynomial functional p is bounded on an ϵ-neighborhood of a linear subspace $Y \subset X$ then for every $x \in X$ and $y \in Y$ we have*

$$p(x+y) = p(x), \tag{7}$$

so $p(y) = p(0)$ for each $y \in Y$.

In particular, p is bounded on arbitrary δ-neighborhood of Y and the supremum of its values on this δ-neighborhood is independent of the subspace Y.

COROLLARY 11. *Let Y and Z be linear subspaces of a normed space X. If the polynomial functional p is bounded on ϵ-neighborhoods of Y and Z by the number c then it is bounded by the same number c on the ϵ-neighborhood of their sum $X + Y$.*

Proof. Let $u = y + z, y \in Y, z \in Z$. By (7) we have $p(x+u) = p(x + y + z) = p(x+y) = p(x)$ for arbitrary element x. This implies the corollary

PROPOSITION 7. *Let p be a continuous polynomial functional on a Banach space X. Then there exists a "maximal" subspace $X_0 \subset X$ such that p is bounded on some (and so on arbitrary) ϵ-neighborhood of X_0. The "maximal" subspace means here that p is unbounded on an arbitrary ϵ-neighborhood of an arbitrary subspace X_1, $X_0 \subset X_1 \subset X$, $X_1 \neq X_0$. Moreover p is unbounded on an arbitrary ϵ-neighborhood of an arbitrary line which intersects X_0 only at the zero. This subspace is closed and unique. In addition, for each $x \in X$ and $y \in X_0$ we have $p(x+y) = p(x)$ so for each $y \in X_0$, $p(y) = p(0)$.*

The proof of Proposition 7 is easily derived from Corollary 11.

From now on, we will call X_0 the "essential kernel" of the polynomial p.

The next statement is certainly well known.

LEMMA 3. *Let X and Y be separable infinite-dimensional Banach spaces and $X_0 \subset X$ be a closed subspace. Then there exists a linear bounded operator $T: X \to Y$ such that* $\ker T = X_0$.

Proof. It is well known (see, for example, [20, p. 190]) that for two separable Banach spaces Z and Y ($\dim Y = \infty$) there exists a bounded linear operator $S: Z \to Y$ with the trivial kernel. In particular, it exists for $Z = X/X_0$ and Y. It is enough to put $T = S\phi$ where $\phi: X \to X/X_0$ is the quotient mapping.

COROLLARY 12. *Let X_0 be the essential kernel of a continuous polynomial functional p on a separable Banach space X. Then there exists a linear continuous operator $T\colon X \to X$ and a polynomial functional q (not necessarily continuous) such that $p = qT$ and $X_0 = \ker T$ (i.e., the answer to Problem 55 is affirmative for $M = X_0$).*

Proof. Let $T\colon X \to X$ be the operator from Lemma 3 and Z be an algebraic complement to X_0 and U be an algebraic complement to TX. Let T^{-1} be the inverse to $T\colon Z \to TX$. Put $q(y+u) = pT^{-1}(y)$, $y \in TX$, $u \in U$. Then $qTx = p(T^{-1}Tx)$. Let $x = x_0 + z$, $x_0 \in X_0$, $z \in Z$. We have $T^{-1}Tx = T^{-1}Tz = z$, hence $T^{-1}Tx - x \in X_0$. As X_0 is the essential kernel of p, it follows that $p(T^{-1}Tx) = p(x)$ (see (7)). As q is the composition of a polynomial functional and two linear operators, it is a polynomial functional.

Remark 3. If the essential kernel X_0 from Corollary 12 is a closed complemented subspace of X then the polynomial q can be chosen to be continuous, i.e., the answer to Problem 55′ is affirmative in this case. It is not necessary to assume the separability of X.

Clearly then it is possible to define the projection of X onto Z as the operator T.

Remark 4. For a nonseparable Banach space X and its closed subspace X_0 the existence of a bounded linear operator $T\colon X \to X$ with $\ker T = X_0$ is not guaranteed, even when there is a square functional p on X with $X_0 = \ker p$. As an example we will consider the space l_∞ of bounded sequences $x = (x_1, \dots, x_i, \dots)$ with norm $\|x\| = \sup_i |x_i|$. By [25], l_∞ contains a closed subspace X_0 such that l_∞/X_0 is isomorphic to a nonseparable Hilbert space. We show that a linear bounded operator $T\colon l_\infty \to l_\infty$ with $\ker T = X_0$ does not exist. Let us suppose that such an operator exists. Then it induces the linear continuous injective operator $\tilde{T}\colon l_\infty/X_0 \to l_\infty$. Since a denumerable total set of continuous linear functionals exists on l_∞ the same set exists on l_∞/X_0. But there is no such set on a nonseparable Hilbert space.

As l_∞/X_0 is isomorphic to a Hilbert space, on it there exists an inner product $(\tilde{x}, \tilde{y})$, which generates the norm $\|\tilde{x}\|_0 = (\tilde{x}, \tilde{x})^{1/2}$ equivalent to the norm of the space l_∞/X_0. Now put $p(x) = (\tilde{x}, \tilde{x})$, where $x \in \tilde{x}$. It is obvious that $\ker p = X_0$.

Now we will show that contrary to the real case the kernel of a complex polynomial functional contains infinite-dimensional subspaces.

THEOREM 5. *Let X be an infinite-dimensional complex linear space and $p\colon X \to \mathbf{C}$ be a homogeneous polynomial of degree $n > 0$. Then there exists an infinite-dimensional subspace $X_0 \subset \ker p$.*

LEMMA 4. *Let Theorem* 5 *be proved for every homogeneous polynomial functional of degree* $\leq n$. *Then for arbitrary homogeneous polynomial functionals* $p_1, \dots, p_m$ *of degree* $\leq n$ *there exists a subspace* $X_0 \subset \ker p_1 \cap \cdots \cap \ker p_m$, $\dim X_0 = \infty$.

Proof. Let $X_1 \subset \ker p_1$, $\dim X_1 = \infty$. Then there exists a subspace $X_2 \subset X_1 \cap \ker p_2$, $\dim X_2 = \infty$. Continue this process to get the subspace $X_0 = X_m \subset X_{m-1} \subset \cdots \subset X_1$, $X_0 \subset \ker p_1 \cap \cdots \cap \ker p_m$, $\dim X_0 = \infty$.

Proof of Theorem 5. We will construct X_0 using the method of mathematical induction. The theorem is, of course, true for linear functionals. Suppose that it is true for homogeneous polynomials of degree $< n$.

Let $x_1 \in X$, $p(x_1) \neq 0$ (if such an x_1 does not exist then the proposition is true automatically). By the induction hypothesis and by Lemma 4 there exists a subspace $X_1 \subset X$, $\dim X_1 = \infty$, on which all of the homogeneous polynomials

$$p_{x_1}(x) := \bar{p}(x_1, x, \dots, x),$$

$$p_{x_1 x_1}(x) := \bar{p}(x_1, x_1, x, \dots, x), \dots, p_{\underbrace{x_1 \cdots x_1}_{n-1}}(x) := \bar{p}(x_1, \dots, x_1, x)$$

vanish, where $\bar{p}$ is the symmetric n-linear functional corresponding to the homogeneous polynomial p.

We choose an element $x_2 \in X_1$ such that $p(x_2) \neq 0$ (if x_2 does not exist then $X_1 \subset \ker p$ and the theorem is proved at once). By the induction hypothesis and by Lemma 4 there exists a subspace $X_2 \subset X_1$, $\dim X_2 = \infty$ on which all homogeneous polynomials

$$p_{\underbrace{x_1 \cdots x_1}_{k}\underbrace{x_2 \cdots x_2}_{l}}(x) := \bar{p}\Big(\underbrace{x_1, \dots, x_1}_{k}, \underbrace{x_2, \dots, x_2}_{l}, x, \dots, x\Big), \qquad 0 < k + l < n.$$

vanish.

We choose an element $x_3 \in X_2$ such that $p(x_3) \neq 0$ (if x_3 does not exist then $X_2 \subset \ker p$ and the theorem is proved). As before there exists a subspace $X_3 \subset X_2$, $\dim X_3 = \infty$ on which all polynomials

$$\bar{p}\Big(\underbrace{x_1, \dots, x_1}_{k}, \underbrace{x_2, \dots, x_2}_{l}, \underbrace{x_3, \dots, x_3}_{m}, x, \dots, x\Big), \qquad 0 < k + l + m < n$$

vanish.

We continue this process in the way written above. If it finishes on the ith step (i.e., $P(X_i) \equiv 0$), then the theorem is proved. Conversely, if it does not finish then we will get an infinite sequence (x_i) consisting of linearly independent terms, $p(x_i) \neq 0$ for every i, such that

$$\bar{p}\Big(\underbrace{x_1, \ldots, x_1}_{k_1}, \underbrace{x_2, \ldots, x_2}_{k_2}, \ldots, \underbrace{x_i, \ldots, x_i}_{k_i}\Big) = 0$$

if at least one $0 < k_i < n$.

Hence it follows that for any finite set of scalars (a_i),

$$p\Big(\sum a_i x_i\Big) = \sum a_i^n p(x_i).$$

Put $y_i = x_i / p(x_i)$, for all i. Then p vanishes on the linear span of elements

$$y_1 + \sqrt[n]{-1}\, y_2,\ y_3 + \sqrt[n]{-1}\, y_4,\ y_5 + \sqrt[n]{-1}\, y_6, \ldots.$$

COROLLARY 13. *If p is a polynomial functional on a complex infinite-dimensional linear space and $p(0) = 0$ then there exists an infinite dimensional linear subspace $X_0 \subset \ker p$.*

The corollary is proved in the same way as Lemma 4.

COROLLARY 14. *If p is a polynomial functional on a complex infinite-dimensional linear space and $p(x_0) = 0$, then there exists an infinite-dimensional affine subspace $X_0 \subset \ker p$ with $x_0 \in X_0$.*

Clearly, it is enough to apply Corollary 13 to the polynomial $p_{x_0}(x) = p(x_0 + x)$.

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