On new families of the Jacobsthal identities

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Abstract. *Formulas relating determinants to Jacobsthal numbers have been an object of recent interest. In some cases, these sequences arise as the determinant for certain families of matrices having integer entries, while in other cases these sequences are the actual entries of the matrix whose determinant is being evaluated. In this paper, we study some families of Toeplitz-Hessenberg determinants the entries of which are Jacobsthal numbers. The determinant formulas we have obtained may be rewritten as combinatorial identities involving sum of products of Jacobsthal numbers and multinomial coefficients.*

Keywords: *Jacobsthal sequence; Jacobsthal numbers; Toeplitz-Hessenberg determinant; Trudi's formula, multinomial coefficient*.

1. Introduction

The *Jacobsthal sequence* $\{J_n\}_{n\geq 0}$ is defined by the recurrence [10]

$$J_{n+2} = J_{n+1} + 2J_n, \qquad J_0 = 0, \quad J_1 = 1. \tag{1}$$

The list of the first 15 terms of the sequence is given in Table 1.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
J_n	0	1	1	3	5	11	21	43	85	171	341	683	1365	2731	5461

Table 1. Terms of J_n.

The numbers J_n appear as the integer sequence A001045 from The On-Line Encyclopedia of Integer Sequences [15]. The Jacobsthal number at a specific point in the sequence may be calculated directly using the closed-form equation

$$J_n = \frac{2^n - (-1)^n}{3}, \quad n \geq 0,$$

and it also can be expressed in the floor function notation as follows

$$J_n = \frac{1-(-1)^n}{2} + \left\lfloor \frac{2^n}{3} \right\rfloor, \qquad n \geq 0.$$

The Jacobsthal sequence is considered as one of the major sequences among the well-known integer sequences. The Jacobsthal numbers have many interesting properties and applications in many fields of mathematics, as geometry, number theory, combinatorics, and probability theory (see [1–6, 12, 14] and the bibliography giver therein). For instance, Akbulak and Öteleş [1] defined two n-square upper Hessenberg matrices one of which corresponds to the adjacency matrix a directed pseudo graph and investigated relations between determinants and permanents of these Hessenberg matrices and sum formulas of the Jacobsthal sequences. Köken and Bozkurt [12] defined the n-square Jacobsthal matrix and using this matrix derived some properties of Jacobsthal numbers. In [14], Öteleş et al. investigated the relationships between the Hessenberg matrices and the Jacobsthal numbers. Cılasun [5] introduced recurrence relation for multiple-counting Jacobsthal sequences and showed their application with Fermat's little theorem. In [6], Daşdemir extended the Jacobsthal numbers to the terms with negative subscripts and presented many identities for new forms of these numbers. Čerin [4] considered sums of squares of odd and even terms of the Jacobsthal sequence and sums of their products; these sums are related to products of appropriate Jacobsthal numbers and several integer sequences. In [3], Catarino et al. presented new families of sequences that generalize the Jacobsthal numbers and established some identities. The main properties of the Jacobsthal numbers are summarized in [11].

The purpose of the present paper is to study the Jacobsthal numbers. We investigate some families of Toeplitz-Hessenberg determinants the entries of which are Jacobsthal numbers with successive, odd and even subscripts. As a consequence, we obtain for these numbers new combinatorial identities involving multinomial coefficients.

2. Toeplitz-Hessenberg determinants and related formulas

A *Toeplitz-Hessenberg determinant* takes the form

$$T_n(a_0; a_1, \ldots, a_n) = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{vmatrix},$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

The sequence $\{T_n\}_{n\geq 0}$ satisfies the following recurrence: for $n \geq 1$,

$$T_n = \sum_{k=1}^{n} (-a_0)^{k-1} a_k T_{n-k}, \tag{2}$$

where, by definition, $T_0 = 1$.

The following result is known as *Trudi's formula* [12]

$$T_n = \sum_{t_1+2t_2+\cdots+nt_n=n} (-a_0)^{n-(t_1+t_2+\cdots+t_n)} p_n(t) a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}$$

or

$$T_n = (-a_0)^n \cdot \sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+t_2+\cdots+t_n} p_n(t) \left(\frac{a_1}{a_0}\right)^{t_1} \left(\frac{a_2}{a_0}\right)^{t_2} \cdots \left(\frac{a_n}{a_0}\right)^{t_1}, \tag{3}$$

where the summation is over all nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$, and

$$p_n(t) = \frac{(t_1 + t_2 + ... + t_n)!}{t_1! t_2! ... t_n!}$$

denotes the multinomial coefficient.

Note that $n = t_1 + 2t_2 + \cdots + nt_n$ is partitions of the positive integer *n*, where each positive integer *i* appears t_i times. Many combinatorial identities involving sums over integer partitions can be generated in this way. Some of these identities presented in [8, 9] and in the next section of this paper.

3. Jacobsthal determinant formulas

In this section, we find relations involving the Jacobsthal sequence, which arise as certain families of Toeplitz-Hessenberg determinants.

We investigate a particular case of Toeplitz-Hessenberg determinants, in which all subdiagonal elements are 2 or –2. Note that similar results for Toeplitz-Hessenberg determinants with Jacobsthal numbers and polynomials entries and with $a_0 = \pm 1$ we obtained in [7, 8].

Theorem 1. *For all* $n \geq 1$, *the following formulas hold*:

$$T_n(2; J_0, J_1, ..., J_{n-1}) = \frac{\sqrt{7}}{7}\left((-1-\sqrt{7})^{n-1} - (-1+\sqrt{7})^{n-1}\right),$$

$$T_n(-2; J_0, J_1, ..., J_{n-1}) = \frac{\sqrt{11}}{11}\left((1+\sqrt{11})^{n-1} - (1-\sqrt{11})^{n-1}\right),$$

$$T_n(2; J_1, J_2, ..., J_n) = \frac{\sqrt{33}}{33}\left(\left(\frac{-1+\sqrt{33}}{2}\right)^n - \left(\frac{-1-\sqrt{33}}{2}\right)^n\right),$$

$$T_n(-2; J_1, J_2, ..., J_n) = \frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^n - \left(\frac{3-\sqrt{41}}{2}\right)^n\right), \tag{4}$$

$$T_n(2;J_2,J_3,\dots,J_{n+1})=\frac{\sqrt{17}}{17}\left(\left(\frac{-1-\sqrt{17}}{2}\right)^{n+1}-\left(\frac{-1+\sqrt{17}}{2}\right)^{n+1}\right),$$

$$T_n(2;J_0,J_2,\dots,J_{2n-2})=\frac{\sqrt{7}}{7}\left((-5-\sqrt{7})^{n-1}-(-5+\sqrt{7})^{n-1}\right),$$

$$T_n(2;J_2,J_4,\dots,J_{2n})=\frac{\sqrt{17}}{17}\left(\left(\frac{-9+\sqrt{17}}{2}\right)^{n}-\left(\frac{-9-\sqrt{17}}{2}\right)^{n}\right),$$

$$T_n(2;J_4,J_6,\dots,J_{2n+2})=\frac{\sqrt{7}i}{7}\left(\left(\frac{-5+i\sqrt{7}}{2}\right)^{n}-\left(\frac{-5-i\sqrt{7}}{2}\right)^{n}\right),$$

where $i=\sqrt{-1}.$

Proof. We will prove formula (4) by induction; the other proofs, which we omit, are similar. To make the notation simpler, we will write T_n instead of $T_n(-2;J_1,J_2,\dots,J_n)$. When $n=1$ and $n=2$ the formula is seen to hold. Suppose it is true for all $k\le n-1,\ n\ge 2$. Using the recurrence (2), we have

$$\begin{aligned}
T_n &=\sum_{i=1}^{n}(-2)^{i-1}J_iT_{n-i}\\
&=J_1T_{n-1}+\sum_{i=2}^{n}(-2)^{i-1}\left(J_{i-1}+2J_{i-2}\right)T_{n-i}\\
&=T_{n-1}+\sum_{i=2}^{n}(-2)^{i-1}J_{i-1}T_{n-i}+2\sum_{i=2}^{n}(-2)^{i-1}J_{i-2}T_{n-i}\\
&=T_{n-1}+\sum_{i=1}^{n-1}(-2)^{i}J_iT_{n-1-i}+2\sum_{i=0}^{n-2}(-2)^{i+1}J_iT_{n-2-i}\\
&=T_{n-1}-2\sum_{i=1}^{n-1}(-2)^{i-1}J_iT_{n-1-i}+2\left(4\sum_{i=1}^{n-2}(-2)^{i-1}J_iT_{n-2-i}-2J_0T_{n-2}\right)\\
&=T_{n-1}-2T_{n-1}+8T_{n-2}=-T_{n-1}+8T_{n-2}.
\end{aligned}$$

Now, using the induction hypothesis and the Jacobsthal recurrence relation (1), we obtain

$$\begin{aligned}
T_n &=-\frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^{n-1}-\left(\frac{3-\sqrt{41}}{2}\right)^{n-1}\right)+\frac{8\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^{n-2}-\left(\frac{3-\sqrt{41}}{2}\right)^{n-2}\right)\\
&=\frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^{n-2}\left(8\cdot\frac{3+\sqrt{41}}{2}-1\right)+\left(\frac{3-\sqrt{41}}{2}\right)^{n-2}\left(-8\cdot\frac{3-\sqrt{41}}{2}+1\right)\right)\\
&=\frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^{n-2}(11+4\sqrt{41})+\left(\frac{3-\sqrt{41}}{2}\right)^{n-2}(-11+4\sqrt{41})\right)
\end{aligned}$$

$$=\frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{2}\right)^{n}+\left(\frac{3-\sqrt{41}}{2}\right)^{n}\right).$$

Consequently, the formula (4) is true in the case *n*. Therefore, by induction, the formula holds for all positive integers *n*. The proof is complete.

4. Multinomial extension of Toeplitz-Hessenberg determinants

In this section, we focus on multinomial extensions of Theorem 1. All formulas from Theorem 1 may be rewritten in terms of Trudi's formula. As a result, we obtained new combinatorial identities involving products of powers of Jacobsthal numbers and multinomial coefficients.

Theorem 2. *For all* $n\geq 1,$ *the following formulas hold*:

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_0}{2}\right)^{t_1}\left(\frac{J_1}{2}\right)^{t_2}\cdots\left(\frac{J_{n-1}}{2}\right)^{t_n}=\frac{\sqrt{7}}{14}\left(\left(\frac{1-\sqrt{7}}{2}\right)^{n-1}-\left(\frac{1+\sqrt{7}}{2}\right)^{n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}p_n(t)\left(\frac{J_0}{2}\right)^{t_1}\left(\frac{J_1}{2}\right)^{t_2}\cdots\left(\frac{J_{n-1}}{2}\right)^{t_n}=\frac{\sqrt{11}}{22}\left(\left(\frac{1+\sqrt{11}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{11}}{2}\right)^{n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_1}{2}\right)^{t_1}\left(\frac{J_2}{2}\right)^{t_2}\cdots\left(\frac{J_n}{2}\right)^{t_n}=\frac{\sqrt{33}}{33}\left(\left(\frac{1-\sqrt{33}}{4}\right)^{n}-\left(\frac{1+\sqrt{33}}{4}\right)^{n}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}p_n(t)\left(\frac{J_1}{2}\right)^{t_1}\left(\frac{J_2}{2}\right)^{t_2}\cdots\left(\frac{J_n}{2}\right)^{t_n}=\frac{\sqrt{41}}{41}\left(\left(\frac{3+\sqrt{41}}{4}\right)^{n}-\left(\frac{3-\sqrt{41}}{4}\right)^{n}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_2}{2}\right)^{t_1}\left(\frac{J_3}{2}\right)^{t_2}\cdots\left(\frac{J_{n+1}}{2}\right)^{t_n}=\frac{2\sqrt{17}}{17}\left(\left(\frac{1-\sqrt{17}}{4}\right)^{n+1}-\left(\frac{1+\sqrt{17}}{4}\right)^{n+1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_0}{2}\right)^{t_1}\left(\frac{J_2}{2}\right)^{t_2}\cdots\left(\frac{J_{2n-2}}{2}\right)^{t_n}=\frac{\sqrt{7}}{14}\left(\left(\frac{5-\sqrt{7}}{2}\right)^{n-1}-\left(\frac{5+\sqrt{7}}{2}\right)^{n-1n-1}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_2}{2}\right)^{t_1}\left(\frac{J_4}{2}\right)^{t_2}\cdots\left(\frac{J_{2n}}{2}\right)^{t_n}=\frac{\sqrt{17}}{17}\left(\left(\frac{9-\sqrt{17}}{4}\right)^{n}-\left(\frac{9+\sqrt{17}}{4}\right)^{n}\right),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n}(-1)^{t_1+t_2+\cdots+t_n}p_n(t)\left(\frac{J_4}{2}\right)^{t_1}\left(\frac{J_6}{2}\right)^{t_2}\cdots\left(\frac{J_{2n+2}}{2}\right)^{t_n}=\frac{\sqrt{7}i}{7}\left(\left(\frac{5-i\sqrt{7}}{4}\right)^{n}-\left(\frac{5+i\sqrt{7}}{4}\right)^{n}\right),$$

where the summation is over integers $t_i\geq 0$ *satisfying* $t_1+2t_2+\cdots+nt_n=n$.

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