Gibonacci identities using permanents of Toeplitz-Hessenberg matrices

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Gibonacci numbers $\{G_n\}_{n\geq 0}$ are defined by the recurrence relation $G_n = G_{n-1} + G_{n-2}$, where G_0 and G_1 are arbitrary integers, and $n \geq 2$ (see, for example, [4] and the bibliography given there). When $G_0 = 0$ and $G_1 = 1$, $G_n = F_n$, the nth Fibonacci number; and when $G_0 = 2$ and $G_1 = 1$, $G_n = L_n$, the nth Lucas number.

The nth Pell number P_n is defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$, where $P_0 = 0$, $P_1 = 1$, and $n \geq 2$. The nth Jacobsthal number J_n is defined by $J_n = J_{n-1} + 2J_{n-2}$, where $J_0 = 0$, $J_1 = 1$, and $n \geq 2$.

There are large number of sequences indexed in OEIS [7], being in this case

$$\begin{aligned}
\{F_n\}_{n\geq 0} &= \{0,1,1,2,3,5,8,13,21,34,55,89,144,233,377,610,987,1597,\ldots\}: && A000045\\
\{L_n\}_{n\geq 0} &= \{2,1,3,4,7,11,18,29,47,76,123,199,322,521,843,1364,2207,\ldots\}: && A000032\\
\{P_n\}_{n\geq 0} &= \{0,1,2,5,12,29,70,169,408,985,2378,5741,13860,33461,\ldots\}: && A000129\\
\{J_n\}_{n\geq 0} &= \{0,1,1,3,5,11,21,43,85,171,341,683,1365,2731,5461,10923,\ldots\}: && A001045
\end{aligned}$$

A *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0\\ a_2 & a_1 & a_0 & \cdots & 0 & 0\\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

The purpose of this paper is to study Fibonacci and Lucas numbers. We investigate some families of Toeplitz-Hessenberg matrices the entries of which are Fibonacci and Lucas numbers with successive, odd or even subscripts. These permanent formulas may also be rewritten as identities involving sums of products of Fibonacci and Lucas numbers and multinomial coefficients. Similar results we obtained in [1–3].

Recall that permanent of a square matrix is defined in a similar manner to the determinant but all the sign used in the Laplace expansion of minors are positive. Given a $n \times n$ matrix $A = (m_{ij})$, the permanent of A, denoted $\mathrm{Per}(A)$, is defined by

$$\mathrm{Per}(A) = \sum_{\pi\in S_n} \prod_{j=1}^{n} m_{1\pi(1)} m_{2\pi(2)} \cdots m_{n\pi(n)},$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$ [5].

For simplicity of notation, we will write $\mathrm{Per}(a_1, a_2, \ldots, a_n)$ instead of $\mathrm{Per}\big(M_n(1; a_1, a_2, \ldots, a_n)\big)$.

Lemma 1. (Trudi's formula, [6]) *Let n be a positive integer. Then*

$$\mathrm{Per}(a_1, a_2, \ldots, a_n) = a_0^n \cdot \sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) \left(\frac{a_1}{a_0}\right)^{t_1} \left(\frac{a_2}{a_0}\right)^{t_2} \cdots \left(\frac{a_n}{a_0}\right)^{t_n}, \quad (1)$$

where the summation is over nonnegative integers satisfying equation $t_1 + 2t_2 + \cdots + nt_n = n$, and $p_n(t) = \frac{(t_1+t_2+\cdots+t_n)!}{t_1!t_2!\cdots t_n!}$ is the multinomial coefficients.

Next theorem gives connection between Fibonacci numbers and Pell, Jacobsthal numbers via the Toeplitz-Hessenberg permanents.

Theorem 2. *For all* $n \geq 1$,

$$\operatorname{Per}(F_1, F_2, \ldots, F_n) = P_n,$$
$$\operatorname{Per}(F_0, F_1, \ldots, F_{n-1}) = J_n.$$

The following theorem gives the value of $\operatorname{Per}(a_1, a_2, \ldots, a_n)$ for several Fibonacci entries a_i.

Theorem 3. *Let* $n \geq 1$, *except when noted otherwise. Then*

$$\operatorname{Per}(F_0, F_2, \ldots, F_{2n-2}) = 3^{n-2}, \qquad n \geq 2,$$
$$\operatorname{Per}(F_1, F_3, \ldots, F_{2n-1}) = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{4},$$
$$\operatorname{Per}(F_2, F_3, \ldots, F_{n+1}) = \frac{(1+\sqrt{3})^{n+1} - (1-\sqrt{3})^{n+1}}{4\sqrt{3}},$$
$$\operatorname{Per}(F_2, F_4, \ldots, F_{2n}) = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}},$$
$$\operatorname{Per}(F_3, F_4, \ldots, F_{n+2}) = \frac{17+4\sqrt{17}}{17}\left(\frac{3+\sqrt{17}}{2}\right)^{n-1} + \frac{17-4\sqrt{17}}{17}\left(\frac{3-\sqrt{17}}{2}\right)^{n-1},$$
$$\operatorname{Per}(F_3, F_5, \ldots, F_{2n+1}) = \frac{17+4\sqrt{17}}{17}\left(\frac{5+\sqrt{17}}{2}\right)^{n-1} + \frac{17-4\sqrt{17}}{17}\left(\frac{5-\sqrt{17}}{2}\right)^{n-1},$$
$$\operatorname{Per}(F_4, F_6, \ldots, F_{2n+2}) = \frac{8+3\sqrt{7}}{2\sqrt{7}}(3+\sqrt{7})^{n-1} - \frac{8-3\sqrt{7}}{2\sqrt{7}}(3-\sqrt{7})^{n-1}.$$

Next we investigate the Lucas counterparts of some of the results from Theorem 3.

Theorem 4. *Let* $n \geq 1$, *except when noted otherwise. Then*

$$\operatorname{Per}(L_0, L_1, \ldots, L_{n-1}) = 5 \cdot 3^{n-2}, \qquad n \geq 2,$$
$$\operatorname{Per}(L_0, L_2 \ldots, L_{2n-2}) = \frac{5 \cdot 4^{n-1} + 1}{3},$$
$$\operatorname{Per}(L_1, L_2, \ldots, L_n) = \frac{5 \cdot 3^{n-1} + (-1)^n}{4},$$
$$\operatorname{Per}(L_1, L_3, \ldots, L_{2n-1}) = 5 \cdot 4^{n-2}, \qquad n \geq 2.$$

Now from Theorems 2-4, using Trudi's formula (1), we obtain new Fibonacci and Lucas identities with multinomial coefficients.

Theorem 5. *Let* $n \geq 1$, *except when noted otherwise. The following formulas hold*:

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_0^{t_1} F_1^{t_2} \cdots F_{n-1}^{t_n} = J_n,$$
$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_0^{t_1} F_2^{t_2} \cdots F_{2n-2}^{t_n} = 3^{n-2}, \qquad n \geq 2,$$
$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_1^{t_1} F_2^{t_2} \cdots F_n^{t_n} = P_n,$$
$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_1^{t_1} F_3^{t_2} \cdots F_{2n-1}^{t_n} = \frac{(2+\sqrt{2})^n + (2-\sqrt{2})^n}{4},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_2^{t_1} F_3^{t_2} \cdots F_{n+1}^{t_n} = \frac{(1+\sqrt{3})^{n+1}-(1-\sqrt{3})^{n+1}}{4\sqrt{3}},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_2^{t_1} F_4^{t_2} \cdots F_{2n}^{t_n} = \frac{(2+\sqrt{3})^{n}-(2-\sqrt{3})^{n}}{2\sqrt{3}},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_3^{t_1} F_4^{t_2} \cdots F_{n+2}^{t_n} = \frac{17+4\sqrt{17}}{17}\left(\frac{3+\sqrt{17}}{2}\right)^{n-1} + \frac{17-4\sqrt{17}}{17}\left(\frac{3-\sqrt{17}}{2}\right)^{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_3^{t_1} F_5^{t_2} \cdots F_{2n+1}^{t_n} = \frac{17+4\sqrt{17}}{17}\left(\frac{5+\sqrt{17}}{2}\right)^{n-1} + \frac{17-4\sqrt{17}}{17}\left(\frac{5-\sqrt{17}}{2}\right)^{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) F_4^{t_1} F_6^{t_2} \cdots F_{2n+2}^{t_n} = \frac{8+3\sqrt{7}}{2\sqrt{7}}(3+\sqrt{7})^{n-1} - \frac{8-3\sqrt{7}}{2\sqrt{7}}(3-\sqrt{7})^{n-1}.$$

where $p_n(t) = \frac{(t_1+t_2+\cdots+t_n)!}{t_1!t_2!\cdots t_n!}$ *is the multinomial coefficient,* P_n *and* J_n *are the nth Pell and Jacobsthal numbers, respectively.*

The next theorem gives analogous results for the Lucas family.

Theorem 6. *The following formulas hold*:

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) L_0^{t_1} L_1^{t_2} \cdots L_{n-1}^{t_n} = 5 \cdot 3^{n-2}, \qquad n \geq 2,$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) L_0^{t_1} L_2^{t_2} \cdots L_{2n-2}^{t_n} = \frac{5 \cdot 4^{n-1}+1}{3}, \qquad n \geq 1,$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) L_1^{t_1} L_2^{t_2} \cdots L_{n}^{t_n} = \frac{5 \cdot 3^{n-1}+(-1)^n}{4}, \qquad n \geq 1,$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) L_1^{t_1} L_3^{t_2} \cdots L_{2n-1}^{t_n} = 5 \cdot 4^{n-2}, \qquad n \geq 2.$$

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