Journal of Vasyl Stefanyk Precarpathian National University
http://jpnu.pu.if.ua
Vol. 2, No. 4 (2015), 75-92

UDC 517.98
MSC 2010: Primary 47A16; Secondary 46E10, 46E20
doi: 10.15330/jpnu.2.4.75-92

HYPERCYCLIC COMPOSITION OPERATORS

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Abstract. In this paper we give survey of hypercyclic composition operators. In pacticular, we represent new classes of hypercyclic composition operators on the spaces of analytic functions.
Keywords: hypercyclic operators, functional spaces, polynomial automorphisms, symmetric functions.

1. INTRODUCTION

Hypercyclicity is a young and rapidly evolving branch of functional analysis, which was probably born in 1982 with the Ph.D. thesis of Kitai [24]. It has become rather popular, thanks to the efforts of many mathematicians. In particular, the seminal paper [17] by Godefroy and Shapiro, the survey [20] by Grosse-Erdmann and useful notes [37] by Shapiro have had a considerable influence on both its internal development. Let us recall a definition of hypercyclic operator.

Definition 1.1. Let X be a Fréchet linear space. A continuous linear operator $T : X \to X$ is called *hypercyclic* if there is a vector $x_0 \in X$ for which the orbit under T, $\mathrm{Orb}(T, x_0) = \{x_0, Tx_0, T^2x_0, \ldots\}$ is dense in X. Every such vector x_0 is called a *hypercyclic vector* of T.

The investigation of hypercyclic operators has relation to invariant subspaces problem. It is easy to check that if every nonzero vector of X is hypercyclic for T, then T has no closed invariant subsets, and so no closed invariant subspaces as well. In his paper [32] Read shows that there exists continuous linear operator on ℓ_1 for which every nonzero vector is hypercyclic. It is still open problem does exist a linear continuous operator on a separable Hilbert space without closed invariant subspaces.

The classical Birkhoff's theorem [7] asserts that any operator of composition with translation $x \mapsto x + a$, $T_a \colon f(x) \mapsto f(x + a)$, $(a \neq 0)$ is hypercyclic on the space of entire functions $H(\mathbb{C})$ on the complex plane $\mathbb{C}$, endowed with the topology of uniform convergence on compact subsets. The Birkhoff's translation T_a has also been regarded as a differentiation operator

$$T_a(f) = \sum_{n=0}^{\infty} \frac{a^n}{n!} D^n f.$$

In 1941, Seidel and Walsh [35] obtained an analogue for non-Euclidean translates in the unit disk $\mathbb{D}$. Variants and strengthenings of the theorems of Birkhoff and Seidel and Walsh were found by Heins [22], Luh [25], [26] and Shapiro [36], while Gauthier [15] gave a new proof of Birkhoff's theorem.

In 1952, MacLane [27] showed that there exists an entire function f whose derivatives $f^{(n)}$ $(n \in \mathbb{N}_0)$ form a dense set in the space $H(\mathbb{C})$ of entire functions, in other words, that the differentiation operator D is hypercyclic on $H(\mathbb{C})$. This result was rederived by Blair and Rubel [8]. Duyos-Ruiz [14] showed the residuality of set of entire functions that are hypercyclic for D; see also [16] and [19].

The most remarkable generalization of MacLane's theorem, which at the same time also includes Birkhoff's theorem was proved by Godefroy and Shapiro in [17]. They showed that if $\varphi(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha$ is a non-constant entire function of exponential type on $\mathbb{C}^n$, then the operator

$$f \mapsto \sum_{|\alpha| \geq 0} c_\alpha D^\alpha f, \qquad f \in H(\mathbb{C}^n) \tag{1.1}$$

is hypercyclic.

We fix $n \in \mathbb{N}$ and denote by $T_a : \mathbb{C}^n \to \mathbb{C}^n$ the translation operator $T_a f(z) = f(z+a)$ for $a \in \mathbb{C}^n$ and by $D_k : \mathbb{C}^n \to \mathbb{C}^n$ the differentiation operator $D_k f(z) = \frac{\partial f}{\partial z_k}(z)$ for $1 \leq k \leq n$.

Theorem 1.2. *(Godefroy, Shapiro). Let T be a continuous linear operator on $H(\mathbb{C}^n)$ that commutes with all translation operators T_a, $a \in \mathbb{C}^n$ (or, equivalently, with all differentiation operators D_k, $1 \leq k \leq n$). If T is not a scalar multiple of the identity, then T is hypercyclic.*

Further hypercyclicity for differential and related operators are obtained by Mathew [28], Bernal [4] for spaces $H(O)$, $O \subset \mathbb{C}$ open; by Bonet [9] for weighted inductive limits of spaces of holomorphic functions.

Let us recall that an operator C_φ on $H(\mathbb{C}^n)$ is said to be a *composition operator* if $C_\varphi f(x) = f(\varphi(x))$ for some analytic map $\varphi\colon \mathbb{C}^n \to \mathbb{C}^n$. It is known that only translation operator T_a for some $a \neq 0$ is a hypercyclic composition operator on $H(\mathbb{C})$ [6]. However, if $n > 1$, $H(\mathbb{C}^n)$ supports more hypercyclic composition operators. In [5] Bernal-González established some necessary and sufficient conditions for a composition operator by an affine map to be hypercyclic. In particular, in [5] it is proved that a given affine automorphism $S = A + b$ on $\mathbb{C}^n$, the composition operator $C_S : f(x) \mapsto f(S(x))$ is hypercyclic if and only if the linear operator A is bijective and the vector b is not in the range of $A - I$.

In [11] Chan and Shapiro show that T_a is hypercyclic in various Hilbert spaces of entire functions on $\mathbb{C}$. More detailed, they considered Hilbert spaces of entire functions of one complex variable $f(z) = \sum_{n=1}^{\infty} f_n z^n$ with norms $\|f\|_{2,\gamma}^2 = \sum_{n=1}^{\infty} \gamma_n^{-2} |f_n|^2$ for appropriated sequence of positive numbers and shown that if $n\gamma_n/\gamma_{n-1}$ is monotonically decreasing, then T_a is hypercyclic.

In [34] Rolewicz proved that even though the backward shift operator $B : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ on the space of square summable sequences defined by

$$B(x_1, x_2, x_3, \ldots) = (x_2, x_3, \ldots)$$

is not hypercyclic, the operator λB (weighted backward shift) is hypercyclic for any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. A related result which came later due to Kitai [24] and Gethner and Shapiro [16] is that, in addition, the set of hypercyclic vectors is G_δ and dense in $\ell^2(\mathbb{N})$. Further results on hypercyclic operators are described in [20].

In this paper we represent the new classes of hypercyclic composition operators on spaces of analytic functions. In Section 1 we consider some examples of hypercyclic composition operators on $H(\mathbb{C})$. In Section 2 we find hypercyclic composition operators on $H(\mathbb{C}^n)$ which can not be described by formula (1.1) but can be obtained from the translation operator using polynomial automorphisms of $\mathbb{C}^n$. To do it we developed a method which involves the theory of symmetric analytic functions on Banach spaces. In the first subsection we discuss some relationship between polynomial automorphisms on $\mathbb{C}^n$ and the operation of changing of polynomial bases in an algebra of symmetric analytic functions on the Banach space of summing sequences, ℓ_1. We also consider operators of the form $C_\Theta{}^{-1} T_b C_\Theta$ for a polynomial automorphism Θ and show that if C_S is a hypercyclic operator for some affine automorphism S on $\mathbb{C}^n$, then there exists a representation of the form $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$ that is we can write $C_S = C_\Theta{}^{-1} T_b C_\Theta T_a$. To do it we use the method of symmetric polynomials on ℓ_1 as an important tool for constructing and computations. In the next subsection we prove the hypercyclicity of a special operator on an algebra of symmetric analytic functions on ℓ_1 which plays the role of translation in this algebra. In Section 3 we propose a simple method how to construct analytic hypercyclic operator on Fréchet spaces and Banach spaces. There are some examples. Some hypercyclic operators on spaces of analytic functions on some algebraic manifolds are described in Section 4.

For details of the theory of analytic functions on Banach spaces we refer the reader to Dineen's book [13]. Note that an analogue of the Godefroy-Shapiro Theorem for a special class of entire functions on Banach space with separable dual was proved by Aron and Bés in [2]. Current state of theory of symmetric analytic functions on Banach spaces can be found in [1, 18]. Detailed information about hypercyclic operators is given in [3].

2. Topological Transitive, Chaotic and Mixing Composition Operators

This chapter provides an introduction to the theory of hypercyclicity. Fundamental concepts such as topologically transitive, chaotic and mixing maps are defined. The Birkhoff transitivity theorem is derived as a crucial tool for showing that a map has a dense orbit.

Definition 2.1. Let X be metric space. A continuous map $T : X \to X$ is called *topologically transitive* if, for any pair U, V of nonempty open subsets of X, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

Topological transitivity can be interpreted as saying that T connects all nontrivial parts of X. This is automatically the case whenever there is a point $x \in X$ with dense orbit under T. What is less obvious is that, in separable complete metric spaces, the converse of this case is also true: topologically transitive maps must have a dense orbit. This result was first obtained in 1920 by G. D. Birkhoff in the context of maps on compact subsets of $\mathbb{R}^N$.

Theorem 2.2. (**Birkhoff transitivity theorem**). *Let T be a continuous map on a separable complete metric space X without isolated points. Then the following assertions are equivalent:*
(i) T is topologically transitive;
(ii) there exists some $x \in X$ such that $\mathrm{Orb}(x, T)$ is dense in X. If one of these conditions holds then the set of points in X with dense orbit is a dense G_δ-set.

Definition 2.3. Let T be a continuous map on a metric space X.
(a) A point $x \in X$ is called a *fixed point* of T if $Tx = x$.
(b) A point $x \in X$ is called a *periodic point* of T if there is some $n \geq 1$ such that $T^n x = x$. The least such number n is called the period of x.

Definition 2.4. (**Devaney chaos**). Let X be metric space. A continuous map $T : X \to X$ is said to be *chaotic* (in the sense Devaney) if it satisfies the following conditions:
(i) T is topologically transitive;
(ii) T has a dense set of periodic points.

Definition 2.5. Let X be metric space. A continuous map $T : X \to X$ is called *mixing* if, for any pair U, V of nonempty open subsets of X, there exists some $N \geq 0$ such that

$$T^n(U) \cap V \neq \varnothing \quad \text{for all} \quad n \geq N.$$

Every mathematical theory has its notion of isomorphism. Let X, Y be metric space. When do we want to consider two continuous operators $S : Y \to Y$ and $T : X \to X$ as equal? There should be a homeomorphism $\phi : Y \to X$ such that, when $x \in X$ corresponds to $y \in Y$ via ϕ then Tx should correspond to Sy via ϕ. In other words, if $x = \phi(y)$ then $Tx = \phi(Sy)$. This is equivalent to saying that $T \circ \phi = \phi \circ S$.

We recall that a homeomorphism is a bijective continuous map whose inverse is also continuous. It is already enough to demand that ϕ is continuous with dense range.

Definition 2.6. Let X, Y be metric space and $S : Y \to Y$, $T : X \to X$ be a continuous map.
(a) Then T is called *quasiconjugate* to S if there exists a continuous map $\phi : Y \to X$ with dense range such that $T \circ \phi = \phi \circ S$, that is, the diagramm

$$\begin{tikzcd} Y \arrow[r, "S"] \arrow[d, "\phi"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "T"] & X \end{tikzcd}$$

commutes.
(b) If ϕ can be chosen to be a homeomorphism then S and T are called *conjugate*.

As we seen, operators may often be interpreted in various ways. MacLane's operator is both a differential operator and a weighted shift. Birkhoff's operators are differential operators as well. Here now we have interpretation of Birkhoff's operators T_a: they are special composition operators. Writing

$$\tau_a(z) = z + a$$

we see that τ_a is an entire function such that

$$T_a f = f \circ \tau_a.$$

In fact, τ_a is even an automorphism of $\mathbb{C}$, that is, a bijective entire function. This observations serve as the starting point of another major investigation: the hypercyclicity of general composition operators.

The further results of this section we also can find in [21].

Let Ω be an arbitrary domain in $\mathbb{C}$, that is, a nonempty connected open set. An automorphism of Ω is a bijective analytic function

$$\varphi : \Omega \to \Omega$$

its inverse is then also holomorphic. The set of all automorphisms of Ω is denoted by $Aut(\Omega)$. Now, for $\varphi \in Aut(\Omega)$ the corresponding composition operator is defined as

$$C_\varphi f = f \circ \varphi,$$

that is, $(C_\varphi f)(z) = f(\varphi(z))$, $z \in \Omega$.

Definition 2.7. Let Ω be a domain in $\mathbb{C}$ and $\varphi_n : \Omega \to \Omega$, $n \geq 1$, holomorphic maps. Then the sequence $(\varphi_n)_n$ is called a *run-away sequence* if, for any compact subset $K \subset \Omega$, there is some $n \in \mathbb{N}$ such that $\varphi_n(K) \cap K = \emptyset$.

We will usually apply this definition to the sequence $(\varphi^n)_n$ of iterates of an automorphism φ on Ω. Let us consider two examples.

Example 2.8. (a) Let $\Omega = \mathbb{C}$. Then the automorphisms of $\mathbb{C}$ are the functions

$$\varphi(z) = az + b, \quad a \neq 0, \quad b \in \mathbb{C},$$

and $(\varphi^n)_n$ is run-away if and only if $a = 1$, $b \neq 0$.

Indeed, let φ be an automorphism of $\mathbb{C}$. If φ is not a polynomial then, by the Casorati-Weierstrass theorem, $\varphi(\{z \in \mathbb{C}; |z| > 1\})$ is dense in $\mathbb{C}$ and therefore intersects the set $\varphi(\mathbb{D})$, which is open by the open mapping theorem. Since this contradicts injectivity, φ must be a polynomial. Again by injectivity, its degree must be one, so that φ is of the stated form. Now, if $a = 1$ then $\varphi^n(z) = z + nb$, so that we have the run-away property if and only if $b \neq 0$; while if $a \neq 1$ then $(1-a)^{-1}b$ is a fixed point of φ so that $(\varphi^n)_n$ cannot be run-away.

(b) Let $\Omega = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the punctured plane. An argument as in (a) shows that the automorphisms of $\mathbb{C}^*$ are the functions

$$\varphi(z) = az \quad \text{or} \quad \varphi(z) = \frac{a}{z}, \quad a \neq 0.$$

Then $(\varphi^n)_n$ is run-away if and only if $\varphi(z) = az$ with $|a| \neq 1$.

We first show that the run-away property is a necessary condition for the hypercyclicity of the composition operator.

Proposition 2.9. *Let Ω be a domain in $\mathbb{C}$ and $\varphi \in Aut(\Omega)$. If C_φ is hypercyclic then $(\varphi^n)_n$ is a run-away sequence.*

Corollary 2.10. *There is no automorphism of $\mathbb{C}^*$ whose composition operator is hypercyclic.*

If $\Omega = \mathbb{C}$, the automorphisms are given by

$$\varphi(z) = az + b, \quad a \neq 0, \quad b \in \mathbb{C},$$

and C_φ is hypercyclic if and only if $a = 1$, $b \neq 0$; see Example 2.8(a). Thus the hypercyclic composition operators on $\mathbb{C}$ are precisely Birkhoff's translation operators.

Let us now consider the simply connected domains Ω other than $\mathbb{C}$. By the Riemann mapping theorem, Ω is conformally equivalent to the unit disk, that is, there is a conformal map $\varphi : \mathbb{D} \to \Omega$. It suffices to study the case when $\Omega = \mathbb{D}$.

Proposition 2.11. *The automorphisms of $\mathbb{D}$ are the linear fractional transformations*

$$\varphi(z) = b\frac{a-z}{1-\bar{a}z}, \quad |a| < 1, |b| = 1.$$

Moreover, φ maps $\mathbb{T}$ bijectively onto itself, where $\mathbb{T}$ is the unit circle.

Now, linear fractional transformations are a very well understood class of analytic maps. Using their properties it is not difficult to determine the dynamical behaviour of the corresponding composition operators; via conjugacy these results can then be carried over to arbitrary simply connected domains.

Theorem 2.12. *Let Ω be a simply connected domain and $\varphi \in Aut(\Omega)$. Then the following assertions are equivalent:*
(i) C_φ is hypercyclic;
(ii) C_φ is mixing;
(iii) C_φ is chaotic;
(iv) $(\varphi^n)_n$ is a run-away sequence;
(v) φ has no fixed point in Ω;
(vi) C_φ is quasiconjugate to a Birkhoff's operator.

2.1. Composition Operators on the Hardy Space

In this section we consider an interesting generalization of the backward shift operator. The underlying space will be the Hardy space H^2. Arguably its easiest definition is the following. If $(a_n)_{n\geq 0}$ is a complex sequence such that

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty,$$

then it is, in particular, bounded, and hence

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \mathbb{C}, |z| < 1,$$

defines a analytic function on the complex unit disk $\mathbb{D}$. The Hardy space is then defined as the space of these functions, that is,

$$H^2 = \left\{ f : \mathbb{D} \to \mathbb{C}; f(z) = \sum_{n=0}^{\infty} a_n z^n, z \in \mathbb{D}, \text{with} \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

In other words, the Hardy space is simply the sequence space $\ell^2(\mathbb{N}_0)$, with its elements written as analytic functions. It is then clear that H^2 is a Banach space under the norm

$$\|f\| = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{\frac{1}{2}} \quad \text{when} f(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and it is even a Hilbert space under the inner product

$$\langle f, g \rangle = \sum_{n=0}^{\infty} a_n \overline{b_n} \quad \text{when} f(z) = \sum_{n=0}^{\infty} a_n z^n, g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

The polynomials form a dense subspace of H^2.

Let φ be an automorphism of the unit disk $\mathbb{D}$ and let $C_\varphi f = f \circ \varphi$ be the corresponding composition operator, where we now demand that f belongs to H^2.

Proposition 2.13. *For any $\varphi \in Aut(\mathbb{D})$, C_φ defines an operator on H^2.*

Our aim now is to characterize when C_φ is hypercyclic on H^2. It will be convenient to consider φ as a particular linear fractional transformation.

Indeed, let

$$\varphi(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0,$$

be an arbitrary linear fractional transformation, which we consider as a map on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then φ has either one or two fixed points in $\widehat{\mathbb{C}}$, or it is the identity.

Theorem 2.14. *Let $\varphi \in Aut(\mathbb{D})$ and C_φ be the corresponding composition operator on H^2. Then the following assertions are equivalent:*
(i) C_φ is hypercyclic;
(ii) C_φ is mixing;
(iii) φ has no fixed point in $\mathbb{D}$.

3. Hypercyclic Composition Operator on Space of Symmetric Functions

In this section we consider hypercyclic composition operators on space of symmetric analytic functions, the basic results are given in [31].

3.1. Polynomial Automorphisms and Symmetric Functions

Definition 3.1. A polynomial map $\Phi = (\Phi_1, \ldots, \Phi_n)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ is said to be a *polynomial automorphism* if it is invertible and the inverse map is also a polynomial.

Definition 3.2. Let X be a Banach space with a symmetric basis $(e_i)_{i=1}^\infty$. A function g on X is called *symmetric* if for every $x = \sum_{i=1}^\infty x_i e_i \in X$,

$$g(x) = g\Big(\sum_{i=1}^\infty x_i e_i\Big) = g\Big(\sum_{i=1}^\infty x_i e_{\sigma(i)}\Big)$$

for an arbitrary permutation σ on the set $\{1, ..., m\}$ for any positive integer m.

Definition 3.3. The sequence of homogeneous polynomials $(P_j)_{j=1}^\infty$, $\deg P_k = k$ is called a *homogeneous algebraic basis* in the algebra of symmetric polynomials if for every symmetric polynomial P of degree n on X there exists a polynomial q on $\mathbb{C}^n$ such that

$$P(x) = q(P_1(x), \ldots, P_n(x)).$$

Throughout this paper we consider the case when $X = \ell_1$. We denote by $\mathcal{P}_s(\ell_1)$ the algebra of all symmetric polynomials on ℓ_1. The next two algebraic bases of $\mathcal{P}_s(\ell_1)$ are useful for us: $(F_k)_{k=1}^\infty$ (see [18]) and $(G_k)_{k=1}^\infty$, where

$$F_k(x) = \sum_{i=1}^\infty x_i^k \quad \text{and} \quad G_k(x) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}.$$

By the Newton formula $G_1 = F_1$ and for every $k > 1$,

$$G_{k+1} = \frac{1}{k+1}((-1)^k F_{k+1} - F_k G_1 + \cdots + F_1 G_k).$$

Denote by $H_s^n(\ell_1)$ the algebra of entire symmetric functions on ℓ_1 which is topologically generated by polynomials $F_1, \ldots, F_n$. It means that $H_s^n(\ell_1)$ is the completion of the algebraic span of $F_1, \ldots, F_n$ in the uniform topology on bounded subsets. We say that polynomials $P_1, \ldots, P_n$ (not necessary homogeneous) form an *algebraic basis* in $H_s^n(\ell_1)$ if they topologically generate $H_s^n(\ell_1)$. Evidently, if $(P_j)_{j=1}^{\infty}$ is a homogeneous algebraic basis in $\mathcal{P}_s(\ell_1)$, then $(P_1, \ldots, P_n)$ is an algebraic basis in $H_s^n(\ell_1)$. We will use notations $\mathbf{F} := (F_k)_{k=1}^n$ and $\mathbf{G} := (G_k)_{k=1}^n$.

Proposition 3.4. *Let $\Phi = (\Phi_1, \ldots, \Phi_n)$ be a polynomial automorphism on $\mathbb{C}^n$. Then $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis in $H_s^n(\ell_1)$ for an arbitrary algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$.*

Conversely, if $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis for some algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$ in $H_s^n(\ell_1)$ and a polynomial map Φ on $\mathbb{C}^n$, then Φ is a polynomial automorphism.

Proof. Suppose that Φ is a polynomial automorphism and

$$\Phi^{-1} = ((\Phi^{-1})_1, \ldots, (\Phi^{-1})_n)$$

is its inverse. Then $P_k = (\Phi^{-1})_k(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P})), 1 \le k \le n$. Hence polynomials $\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P})$ topologically generate $H_s^n(\ell_1)$ and so they form an algebraic basis.

Let now $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ be an algebraic basis in $H_s^n(\ell_1)$ for some algebraic basis $\mathbf{P} = (P_1, \ldots, P_n)$. Then for each P_k, $1 \le k \le n$, there exists a polynomial q_k on $\mathbb{C}^n$ such that $P_k = q_k(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$. Put $(\Phi^{-1})_k(t) := q_k(t)$, $t \in \mathbb{C}^n$. Since $(\Phi_1(\mathbf{P}), \ldots, \Phi_n(\mathbf{P}))$ is an algebraic basis, the map

$$(x_1, \ldots, x_n) \mapsto (\Phi_1(\mathbf{P}(x)), \ldots, \Phi_n(\mathbf{P}(x)))$$

is onto by [1, Lemma 1.1]. Thus $\Phi\colon \mathbb{C}^n \to \mathbb{C}^n$ is a bijection and so the mapping $((\Phi^{-1})_1, \ldots, (\Phi^{-1})_n)$ is the inverse polynomial map for Φ. □

3.2. Similar Translations

We start with an evident statement, which actually is a very special case of the Universal Comparison Principle (see e.g. [20, Proposition 4]).

Proposition 3.5. *Let T be a hypercyclic operator on X and A be an isomorphism of X. Then $A^{-1}TA$ is hypercyclic.*

We will say that $A^{-1}TA$ is a *similar* operator to T. If $T = C_R$ is a composition operator on $H(\mathbb{C}^n)$ and $A = C_\Phi$ is a composition by an analytic automorphism Φ of $\mathbb{C}^n$, then $A^{-1}TA = C_{\Phi\circ R\circ\Phi^{-1}}$ is a composition operator too. If A is a composition with a polynomial automorphism, we will say that $A^{-1}TA$ is *polynomially similar* to T. Now we consider operators which are similar to the translation composition $T_a\colon f(x) \mapsto f(x+a)$ on $H(\mathbb{C}^n)$.

Example 3.6. Let $\Phi(t_1, t_2) = (t_1, t_2 - t_1^m)$ for some positive integer m. Clearly, Φ is a polynomial automorphism and $\Phi^{-1}(z_1, z_2) = (z_1, z_2 + z_1^m)$. So

$$\begin{aligned}\Phi(t+a) &= (t_1 + a_1, t_2 + a_2 - (t_1 + a_1)^m) \\ &= \left(t_1 + a_1, t_2 + a_2 - t_1^m - a_1^m - \sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j\right).\end{aligned}$$

Thus we have

$$\Phi \circ (I + a) \circ \Phi^{-1}(t) = \Phi(\Phi^{-1}(t) + a) = \left(t_1 + a_1, t_2 + a_2 - a_1^m - \sum_{j=1}^{m-1} \binom{m-j}{j} t_1^{m-j} a_1^j \right).$$

Hence the composition operator with the $(m-1)$-degree polynomial $\Phi \circ (I+a) \circ \Phi^{-1}$ is similar to the translation operator $T_a = C_{(I+a)}$ and so it must be hypercyclic. Here I is the identity operator.

It is known (see [1]) that the map

$$\mathcal{F}_n^{\mathbf{F}} \colon f(t_1, \ldots, t_n) \mapsto f(F_1(x), \ldots, F_n(x))$$

is a topological isomorphism from the algebra $H(\mathbb{C}^n)$ to the algebra $H_s^n(\ell_1)$. Now we will prove more general statement.

Lemma 3.7. *Let $\mathbf{P} = (P_k)_{k=1}^n$ be an algebraic basis in $H_s^n(\ell_1)$. Then the map*

$$\mathcal{F}_n^{\mathbf{P}} \colon f(t_1, \ldots, t_n) \mapsto f(P_1(x), \ldots, P_n(x))$$

is a topological isomorphism from $H(\mathbb{C}^n)$ onto $H_s^n(\ell_1)$.

Proof. Evidently, $\mathcal{F}_n^{\mathbf{P}}$ is a homomorphism. It is known [1] that for every vector $(t_1, \ldots, t_n) \in \mathbb{C}^n$ there exists an element $x \in \ell_1$ such that $P_1(x) = t_1, \ldots, P_n(x) = t_n$. Therefore the map $\mathcal{F}_n^{\mathbf{P}}$ is injective. Let us show that $\mathcal{F}_n^{\mathbf{P}}$ is surjective. Let $u \in H_s^n(\ell_1)$ and $u = \sum u_k$ be the Taylor series expansion of u at zero. For every homogeneous polynomial u_k there exists a polynomial q_k on $\mathbb{C}^n$ such that $u_k = q_k(P_1, \ldots, P_n)$. Put $f(t_1, \ldots, t_n) = \sum_{k=1}^{\infty} q_k(t_1, \ldots, t_n)$. Since f is a power series which converges for every vector $(t_1, \ldots, t_n)$, f is an entire analytic function on $\mathbb{C}^n$. Evidently, $\mathcal{F}_n^{\mathbf{P}}(f) = u$. From the known theorem about automatic continuity of an isomorphism between commutative finitely generated Fréchet algebras [23, p. 43] it follows that $\mathcal{F}_n^{\mathbf{P}}$ is continuous. $\square$

Let $x, y \in \ell_1$, $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$. We put

$$x \bullet y := (x_1, y_1, x_2, y_2, \ldots)$$

and define

$$\mathcal{T}_y(f)(x) := f(x \bullet y).$$

We will say that $x \mapsto x \bullet y$ is the *symmetric translation* and the operator $\mathcal{T}_y$ is the *symmetric translation operator*. It is clear that if f is a symmetric function, then $f(x \bullet y)$ is a symmetric function for any fixed y.

In [12] is proved that $\mathcal{T}_y$ is a topological isomorphism from the algebra of symmetric analytic functions to itself. Evidently, we have that

$$F_k(x \bullet y) = F_k(x) + F_k(y) \tag{3.1}$$

for every k.

Let $g \in H_s^n(\ell_1)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Set

$$\mathcal{D}^\alpha g := \mathcal{F}_n^{\mathbf{F}} D^\alpha (\mathcal{F}_n^{\mathbf{F}})^{-1} g = \left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}} f \right) (F_1(\cdot), \ldots, F_n(\cdot)),$$

where $f = (\mathcal{F}_n^{\mathbf{F}})^{-1} g$.

Theorem 3.8. *Let $y \in \ell_1$ such that $(F_1(y), \ldots, F_n(y))$ is a nonzero vector in $\mathbb{C}^n$. Then the symmetric translation operator $\mathcal{T}_y$ is hypercyclic on $H_s^n(\ell_1)$. Moreover, every operator $\mathcal{A}$ on $H_s^n(\ell_1)$ which commutes with $\mathcal{T}_y$ and is not a scalar multiple of the identity is hypercyclic and can be represented by*

$$\mathcal{A}(g) = \sum_{|\alpha| \geq 0} c_\alpha \mathcal{D}^\alpha g, \tag{3.2}$$

where c_α are coefficients of a non-constant entire function of exponential type on $\mathbb{C}^n$.

Proof. Let $a = (F_1(y), \ldots, F_n(y)) \in \mathbb{C}^n$. If $g \in H_s^n(\ell_1)$, then

$$g(x) = \mathcal{F}_n^{\mathbf{F}}(f)(x) = f(F_1(x), \ldots, F_n(x))$$

for some $f \in H_s^n(\ell_1)$ and property (3.1) implies that

$$\mathcal{T}_y(g)(x) = \mathcal{F}_n^{\mathbf{F}} T_a (\mathcal{F}_n^{\mathbf{F}})^{-1}(g)(x).$$

So the proof follows from Proposition 3.5 and the Godefroy-Shapiro Theorem. ☐

A given algebraic basis $\mathbf{P}$ on $H_s^n(\ell_1)$ we set

$$T_{\mathbf{P},y} := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}} \quad \text{and} \quad D_{\mathbf{P}}^\alpha := (\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{D}^\alpha \mathcal{F}_n^{\mathbf{P}}.$$

Corollary 3.9. *Let $\mathbf{P}$ be an algebraic basis on $H_s^n(\ell_1)$ and let $y \in \ell_1$ such that $(F_1(y), \ldots, F_n(y)) \neq 0$. Then the operator $T_{\mathbf{P},y}$ is hypercyclic on $H(\mathbb{C}^n)$. Moreover, every operator A on $H(\mathbb{C}^n)$ which commutes with $T_{\mathbf{P},y}$ and is not a scalar multiple of the identity is hypercyclic and can be represented by the form*

$$A(f) = \sum_{|\alpha| \geq 0} c_\alpha D_{\mathbf{P}}^\alpha f, \tag{3.3}$$

where c_α as in (1.1).

Note that due to Proposition 3.4 the transformation $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ is nothing else than a composition with $\Phi \circ (I + a) \circ \Phi^{-1}$, where $\Phi(F_1, \ldots, F_n) = (P_1, \ldots, P_n)$ and $a = (F_1(y), \ldots, F_n(y))$. Conversely, every polynomially similar operator to the translation can be represented by the form $(\mathcal{F}_n^{\mathbf{P}})^{-1} \mathcal{T}_y \mathcal{F}_n^{\mathbf{P}}$ for some algebraic basis of symmetric polynomials $\mathbf{P}$. This observation can be helpful in order to construct some examples of such operators.

Example 3.10. Let us compute how looks the operator $T_{\mathbf{P},y}$ for $\mathbf{P} = \mathbf{G}$. We observe first that $G_k(x \bullet y) = \sum_{i=0}^k G_i(x) G_{k-i}(y)$, where for the sake of convenience we take $G_0 \equiv 1$. Thus

$$\begin{aligned} \mathcal{T}_y \mathcal{F}_n^{\mathbf{G}} f(t_1, \ldots, t_n) &= \mathcal{T}_y f(G_1(x), \ldots, G_n(x)) = f(G_1(x \bullet y), \ldots, G_n(x \bullet y)) \\ &= f\Big(G_1(x) + G_1(y), \ldots, \sum_{i=0}^n G_i(x) G_{n-i}(y)\Big). \end{aligned}$$

Therefore

$$T_{\mathbf{G},y} f(t_1, \ldots, t_n) = f\Big(t_1 + b_1, \ldots, \sum_{i=0}^k t_i b_{k-i}, \ldots, \sum_{i=0}^n t_i b_{n-i}\Big), \tag{3.4}$$

where $t_0 = 1$, $b_0 = 1$ and $b_k = G_k(y)$ for $1 \leq k \leq n$.

According to the Newton formula and Proposition 3.4 the corresponding polynomial automorphism Φ can be given of recurrence form $\Phi_1(t) = t_1$, $\Phi_{k+1}(t) = 1/(k+1)((-1)^k t_{k+1} - t_k \Phi_1(t) + \cdots + t_1 \Phi_k(t))$ which is not so good for computations.

The hypercyclic operator in Example 3.6 is a composition with an $m-1$ degree polynomial and so does not commute with the translation because it can not be generated by formula (1.1). However, the composition with an affine map in Example 3.10 still does not commute with T_a. Indeed, by (3.4),

$$T_a \circ T_{\mathbf{G},y} f(t_1, \dots, t_n) = f\Big(t_1 + b_1 + a_1, \dots, \sum_{i=0}^{n} t_i b_{n-i} + a_n\Big);$$

$$T_{\mathbf{G},y} \circ T_a f(t_1, \dots, t_n) = f\Big(t_1 + b_1 + a_1, \dots, \sum_{i=0}^{n} (t_i + a_i) b_{n-i}\Big),$$

where $a_0 = 1$. Evidently, $T_a \circ T_{\mathbf{G},y} \neq T_{\mathbf{G},y} \circ T_a$ for some $a \neq 0$ whenever $b \neq (0, \dots, 0, b_n)$.

Corollary 3.11. *There exists a nonzero vector $b \in \mathbb{C}^n$ and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $\Theta \circ (I + b)\Theta^{-1}(t) = A(t) + c$ where A is a linear operator with the matrix of the form*

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix} \tag{3.5}$$

and $c \neq 0$.

Proof. We choose $b \in \mathbb{C}^n$ such that all coordinates b_k, $1 \leq k \leq n$ are positive numbers. Let Φ be a polynomial automorphism associated with $T_{\mathbf{G},y}$ in Example 3.10, where $y \in \ell_1$ is such that $G_k(y) = b_k$, $1 \leq k \leq n$. Then, according to (3.4), we can write $\Phi \circ (I + b)\Phi^{-1}(t) = R(t) + b$, where

$$R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{n-2} & b_{n-3} & \cdots & 1 & 0 \\ b_{n-1} & b_{n-2} & \cdots & b_1 & 1 \end{pmatrix}.$$

We recall that the *index* of an eigenvalue λ of a matrix M is the smallest nonnegative integer k such that $\operatorname{rank}((M - \lambda I)^k) = \operatorname{rank}((M - \lambda I)^{k+1})$. The matrix R has a unique eigenvalue 1 and since all coordinates b_k of b are positive, the index of this eigenvalue is equal to n. Indeed, for each $k < n$, $(R - \lambda I)^k$ contains an $(n-k) \times (n-k)$ triangular matrix with only positive numbers in the main diagonal and $(R - \lambda I)^n = 0$. Therefore, from the Linear Algebra we know that the largest Jordan block A associated with the eigenvalue 1 is $n \times n$ and so it can be represented by (3.5). Thus there is a linear isomorphism L on $\mathbb{C}^n$ such that $A = LRL^{-1}$. Hence

$$(L \circ \Phi) \circ (I + b) \circ (L \circ \Phi)^{-1}(t) = L \circ (R + b) \circ L^{-1}(t) = A(t) + L(b).$$

So it is enough to set $\Theta := L \circ \Phi$ and $c := L(b)$. □

Theorem 3.12. *Let S be an affine automorphism on $\mathbb{C}^n$ such that C_S is hypercyclic. Then there are vectors a, b and a polynomial automorphism Θ on $\mathbb{C}^n$ such that $S = \Theta \circ (I + b) \circ \Theta^{-1} + a$.*

Proof. Let $S(t) = A(t) + c$ be an affine map on $\mathbb{C}^n$ such that C_S is hypercyclic. Without loss of the generality we can assume that A is a direct sum of Jordan blocks $A_1, \dots, A_m$ and each block A_j acts on a subspace V_j of $\mathbb{C}^n$. In the proof of Theorem 3.1 of [5] is shown that the spectrum of each block A_j is the singleton $\{1\}$. So each A_j is of the form as in (3.5). Let $\Theta_{(j)}$ be a polynomial automorphism of V_j as in Corollary 3.11, that is,

$$\Theta_{(j)} \circ (I + b_{(j)}) \circ \Theta_{(j)}^{-1} = A_j + b_{(j)},$$

for some $b_{(j)} \in V_j$. Put $\Theta = \Theta_{(1)} + \cdots + \Theta_{(m)}$ and $b = b_{(1)} + \cdots + b_{(m)}$. Then $\Theta \circ (I + b) \circ \Theta^{-1} = A + b$. Let $a = c - b$. Hence

$$S = A + c = A + b + a = \Theta \circ (I + b) \circ \Theta^{-1} + a.$$

□

Of course, the converse of Theorem 3.12 (with $b \neq 0$) also holds.

We do not know whether it is always possible to choose Θ so that $a = 0$. In other words: *Is every hypercyclic operator which is a composition by an affine automorphism polynomially similar to a translation?* Moreover, we do not know any example of a hypercyclic composition operator on $H(\mathbb{C}^n)$ which is not similar to a translation.

3.3. The Infinity-Dimensional Case

Let us recall a well known Kitai-Gethner-Shapiro theorem which is also known as the Hypercyclicity Criterion.

Theorem 3.13. *Let X be a separable Fréchet space and $T\colon X \to X$ be a linear and continuous operator. Suppose there exist X_0, Y_0 dense subsets of X, a sequence (n_k) of positive integers and a sequence of mappings (possibly nonlinear, possibly not continuous) $S_n\colon Y_0 \to X$ so that*

1. *$T^{n_k}(x) \to 0$ for every $x \in X_0$ as $k \to \infty$.*
2. *$S_{n_k}(y) \to 0$ for every $y \in Y_0$ as $k \to \infty$.*
3. *$T^{n_k} \circ S_{n_k}(y) = y$ for every $y \in Y_0$.*

Then T is hypercyclic.

The operator T is said to satisfy the *Hypercyclicity Criterion for full sequence* if we can chose $n_k = k$. Note that T_a satisfies the Hypercyclicity Criterion for full sequence [17] and so the symmetric shift $\mathcal{T}_y$ on $H_s^n(\ell_1)$ satisfies the Hypercyclicity Criterion for full sequence provided $(F_1(y), \ldots, F_n(y)) \neq 0$.

Finally, we establish our result about hypercyclic operators on the space of symmetric entire functions on ℓ_1. But before this, we need the following general auxiliary statement, which might be of some interest by itself.

Lemma 3.14. *Let X be a Fréchet space and $X_1 \subset X_2 \subset \cdots \subset X_m \subset \cdots$ be a sequence of closed subspaces such that $\bigcup_{m=1}^{\infty} X_m$ is dense in X. Let T be an operator on X such that $T(X_m) \subset X_m$ for each m each restriction $T|_{X_m}$ satisfies the Hypercyclicity Criterion for full sequence on X_m. Then T satisfies the Hypercyclicity Criterion for full sequence on X.*

Proof. Let $Y_0^{(m)}$ and $X_0^{(m)}$ be dense subsets in X_m, and $S_k^{(m)}$ corresponding sequence of mappings associated with $T|_{X_m}$ as in Theorem 3.13. Put $X_0 = \bigcup_{m=1}^{\infty} X_0^{(m)}$ and $Y_0 = \bigcup_{m=1}^{\infty} Y_0^{(m)}$. It is clear that both X_0 and Y_0 are dense in X. For a given $y \in Y_0$, we denote by $m(y)$ the minimal number m such that $y \in Y_0^{(m)}$. We set $S_k(y) := S_k^{(m(y))}(y)$. Then

$$T^k \circ S_k(y) = T^k|_{X_{m(y)}} \circ S_k^{(m(y))}(y) = y, \quad \forall y \in Y_0$$

and $S_k(y) = S_k^{(m(y))}(y) \to 0$ as $k \to \infty$ for every $y \in Y_0$. Similarly, if $x \in X_0$, then $x \in X_0^{(m)}$ for some m and $T^k(x) = T^k|_{X_m}(x) \to 0$ as $k \to \infty$. So T satisfies the Hypercyclicity Criterion for full sequence on X. In particular, T is hypercyclic. □

We denote by $H_{bs}(\ell_1)$ the Fréchet algebra of symmetric entire functions on ℓ_1 which are bounded on bounded subsets. This algebra is the completion of the space of symmetric polynomials on ℓ_1 endowed with the uniform topology on bounded subsets.

Theorem 3.15. *The symmetric operator $\mathcal{T}_y$ is hypercyclic on $H_{bs}(\ell_1)$ for every $y \neq 0$.*

Proof. Since $y \neq 0$, $F_{m_0}(y) \neq 0$ for some m_0 [1]. So, $\mathcal{T}_y$ is hypercyclic (and satisfies the Hypercyclicity Criterion for full sequence) on $H_s^m(\ell_1)$ whenever $m \geq m_0$. The set $\bigcup_{m=m_0}^{\infty} H_s^m(\ell_1)$ contains the space of all symmetric polynomials on ℓ_1 and so it is dense in $H_{bs}(\ell_1)$. Also $H_s^m(\ell_1) \subset H_s^n(\ell_1)$ if $n > m$. Hence by Lemma 3.14, $\mathcal{T}_y$ is hypercyclic. □

4. Analytic Hypercyclic Operators

In this section we will show a simple method how to construct polynomial and analytic hypercyclic operators. Basic results of this section we can find in [29].

Let F be an analytic automorphism of X onto X and T be an hypercyclic operator on X. Then $T_F := FTF^{-1}$ (and $T_{F^{-1}} := F^{-1}TF$ as well) must be hypercyclic [20] and, in the general case, they are nonlinear. The following examples show that T_F are nonlinear for some well known hypercyclic operators T and simple analytic automorphisms F.

Example 4.1. Let $A(D)$ be the disk-algebra of all analytic functions on the unit disk D of $\mathbb{C}$ which are continuous on the closure $\overline{D}$. Denote $X_1 = \{\sum_{k=0}^{\infty} a_{2k+1}t^{2k+1} \in A(D)\}$ and $X_2 = \{\sum_{k=0}^{\infty} a_{2k}t^{2k} \in A(D)\}$. Clearly $A(D) = X_1 \oplus X_2$.

For every $f = f_1 + f_2$, $f_1 \in X_1$, $f_2 \in X_2$ we put

$$\begin{cases} F(f_1) & := & f_1, \\ F(f_2) & := & f_2 + f_1^2. \end{cases} \qquad \text{Then we have} \qquad \begin{cases} F^{-1}(f_1) & = & f_1, \\ F^{-1}(f_2) & = & f_2 - f_1^2. \end{cases}$$

So F is a polynomial automorphism of X. Let $T(f(t)) = f(\frac{t+1}{2})$. It is known that T is hypercyclic on $A(D)$ [10, p. 4].

Let us show that $T_F = FTF^{-1}$ is nonlinear. It is enough to check that $T_F(\lambda f) \neq \lambda T_F(f)$ for some $\lambda \in \mathbb{C}$ and $f \in A(D)$. Let $f(t) = t + t^2 \in A(D)$. Then

$$\begin{aligned} T_F(\lambda f) &= F(T(F^{-1}(\lambda t + \lambda t^2))) = F(T(\lambda t + \lambda t^2 - \lambda^2 t^2)) \\ &= F(T(\lambda t + (\lambda - \lambda^2)t^2)) = F\Big(\lambda\Big(\frac{t+1}{2}\Big) + (\lambda - \lambda^2)\Big(\frac{t+1}{2}\Big)^2\Big) \\ &= \frac{(2\lambda - \lambda^2)t}{2} + \frac{(\lambda + 3\lambda^2 - 4\lambda^3 + \lambda^4)t^2}{4} + \frac{(3\lambda - \lambda^2)}{4} \end{aligned}$$

for any $\lambda \neq 0$, $\lambda \neq 1$. Thus $T_F(\lambda f) \neq \lambda T_F(f)$.

By the similar way in the next example we consider the space of entire analytic functions $H(\mathbb{C})$ and $T(f) = f(x + a)$ to show that $T_{F^{-1}}$ is nonlinear, where F is defined as above.

Example 4.2. Let $f(t) = t + t^2 \in H(\mathbb{C})$ then $F(f) = t + 2t^2$, $F(\lambda f) = \lambda(t + t^2) + \lambda^2 t^2$. Thus

$$T(F(\lambda f)) = \lambda(t + a) + 2\lambda(1 + \lambda)at + (\lambda + \lambda^2)(t^2 + a^2)$$

for any $\lambda \neq 0$. Since $F^{-1}(f) = t - t^2$, we have

$$\begin{aligned} F^{-1}TF(\lambda f) &= \lambda(t + t^2) - 4\lambda^2 a^2(t + t^2) + \lambda(a + a^2 + 2t) + 4\lambda^2 at(t + a) \\ &\quad -4\lambda^3 at(t + a) - 4\lambda^3 a^2 t^2(2 + \lambda) \neq \lambda T_{F^{-1}}(f). \end{aligned}$$

So, the operator $T_{F^{-1}} = F^{-1}TF$ is nonlinear.

Example 4.3. Next we consider the Hilbert space ℓ_2. Let $(e_k)_{k=1}^{\infty}$ be an orthonormal basis in ℓ_2 and $x = \sum_{k=1}^{\infty} x_k e_k \in \ell_2$. We define an analytic automorphism $F : \ell_2 \to \ell_2$ by the formula

$$\begin{cases} F(x_{2k-1}e_{2k-1}) &= x_{2k-1}e_{2k-1}, \\ F(x_{2k}e_{2k}) &= x_{2k}e^{-x_{2k-1}}e_{2k}, \qquad k = 1, 2, \ldots. \end{cases}$$

Let T_μ be a weighted shift

$$T_\mu(x) = (\mu x_2, \mu x_3, \ldots).$$

T_μ is a hypercyclic operator if $|\mu| > 1$ (see [34]). Then the operator $T_F = FT_\mu F^{-1}$ is hypercyclic. We will show that T_F is nonlinear.

Let $a \in \ell_2$, $a = (a_1, a_2, \ldots a_n, \ldots)$, $a = \sum_{k=1}^{\infty} a_k e_k$ and $\lambda \in \mathbb{C}$. We will show that $T_F(\lambda a) \neq \lambda T_F(a)$.

$$F^{-1}T_\mu F(\lambda a) = (\mu\lambda a_2 e^{-\lambda a_1}, \mu\lambda a_3 e^{\mu\lambda a_2 e^{-\lambda a_1}}, \mu\lambda a_4 e^{-\lambda a_3}, \mu\lambda a_5 e^{\mu\lambda a_4 e^{-\lambda a_3}}, \ldots).$$

So, $T_F(\lambda a) \neq \lambda T_F(a)$ and moreover, the map $\lambda \rightsquigarrow T_\mu(\lambda a)$ is not polynomial. Thus T_F is an analytic (not polynomial) hypercyclic map.

5. Hypercyclic Operators on Spaces of Functions on Algebraic Manifolds

In this section we represent the basic results which had obtained in [30].

Let $q_1, \ldots, q_m$ be polynomials on $\mathbb{C}^n$. We consider an ideal which is generated by the polynomials

$$\mathcal{I} = (q_1, \ldots, q_n) := \{q_1 p_1 + \cdots + q_n q_n \mid p_k \in \mathcal{P}(\mathbb{C}^n), k = 1, \ldots, n\}.$$

Let $V(\mathcal{I}) = \cap_{k=1}^{n} \ker q_k$ be set of zeros of the ideal $\mathcal{I}$. The set $V(\mathcal{I})$ is called algebraic set and on this set we can define algebra of polynomials

$$\mathcal{P}(V(\mathcal{I})) := \mathcal{P}(\mathbb{C}^n)/I(V(\mathcal{I})),$$

where $I(V(\mathcal{I}))$ is set of polynomials, which are equal to zero on $V(\mathcal{I})$.

Definition 5.1. The ideal $\mathcal{I}$ is called *simple* if from $p \in \mathcal{I}$ and $p = p_1 p_2$ follows that $p_1 \in \mathcal{I}$ and $p_2 \in \mathcal{I}$. In this case the set $V(\mathcal{I})$ is called *algebraic manifold*.

It is known from algebraic geometry (see [33]), that for simple ideal $\mathcal{I}$, $I(V(\mathcal{I})) = \mathcal{I}$, and algebra $\mathcal{P}(V(\mathcal{I})) = \mathcal{P}(\mathbb{C}^n)/I(V(\mathcal{I}))$ is ring integrity, that is ring without zero divisors. Every element of algebra $\mathcal{P}(V(\mathcal{I}))$ is class of equivalence for some $p \in \mathcal{P}(\mathbb{C}^n)$,

$$[p] = \{p + q : q \in \mathcal{I}\}.$$

We define algebra of entire analytic functions $H(V(\mathcal{I}))$ on the algebraic manifold $V(\mathcal{I})$ as set of classes

$$\{[f] : [f] = \{f + q : q \in \mathcal{I}\}, f \in H(\mathbb{C}^n)\}.$$

Let $\mathcal{N} = \{i_1, \dots, i_k\}$ be some proper subset in $\{1, 2, \dots, n\}$ and $\mathcal{M} = \{j_1, \dots, j_m\} =$
$= \{1, 2, \dots, n\} \backslash \mathcal{N}$.
The equation

$$t_{i_1} = t_{i_2} = \cdots = t_{i_k} = 0 \tag{5.1}$$

sets in $\mathbb{C}^n$ linear subspace $L_{\mathcal{M}}$. From another side, if

$$\begin{cases} t_1 = \Phi_1(z_1, \dots, z_n), \\ \vdots \\ t_n = \Phi_n(z_1, \dots, z_n) \end{cases}$$

for polynomials $\Phi_1, \dots, \Phi_n$, then equation (5.1) in coordinates $z_{i_1}, \dots, z_{i_k}$ sets algebraic manifold $V_{\mathcal{M}}$:

$$\Phi_{i_1}(z_1, \dots, z_n) = 0, \ \dots, \Phi_{i_k}(z_1, \dots, z_n) = 0.$$

By striction map $\Phi = (\Phi_1, \dots, \Phi_n)$ on manifold $V_{\mathcal{M}}$ we get polynomial automorphism

$$\begin{cases} \Phi_{j_1}(z_1, \dots, z_n) = t_{j_1}, \\ \vdots \\ \Phi_{j_m}(z_1, \dots, z_n) = t_{j_m} \end{cases}$$

from $V_{\mathcal{M}}$ on $L_{\mathcal{M}}$, which we denote $\widetilde{\Phi}$.

By the another words, manifold $V_{\mathcal{M}}$ is image of subspace $L_{\mathcal{M}}$ at polynomial map $((\widetilde{\Phi}^{-1})_{j_1}, \dots, (\widetilde{\Phi}^{-1})_{j_m})$.

Theorem 5.2. *Let a be non zero vector in $L_{\mathcal{M}}$. Then composition operator with polynomial map $\widetilde{\Phi}^{-1} \circ (I + a) \circ \widetilde{\Phi}$ is hypercyclic operator on space $H(V_{\mathcal{M}})$.*

Proof. The composition operator with translation $I + a$ is hypercyclic map. Since $\widetilde{\Phi}$ is polynomial automorphism from $V_{\mathcal{M}}$ in $L_{\mathcal{M}}$, thus $C_{\widetilde{\Phi}}$ is continuous homomorphism from $H(L_{\mathcal{M}})$ in $H(V_{\mathcal{M}})$. Then, according to Universal Comparison Principle $C_{\widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}} = C_{\widetilde{\Phi}} \circ T_a \circ C_{\widetilde{\Phi}^{-1}}$ is hypercyclic operator on space $H(V_{\mathcal{M}})$. ☐

Example 5.3. Let $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial automorphism:

$$\begin{cases} t_1 = z_1 \\ t_2 = z_2 + P(z_1), \end{cases}$$

where P is some polynomial on $\mathbb{C}$. Put $\mathcal{N} = \{2\}$, $\mathcal{M} = \{1\}$. Then

$$L_{\mathcal{M}} = \{t = (t_1, t_2) \in \mathbb{C}^2 : t_2 = 0\},$$

$$V_{\mathcal{M}} = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_2 + P(z_1) = 0\}.$$

The map $\widetilde{\Phi} : V_{\mathcal{M}} \to L_{\mathcal{M}}$ is defined by formula $\widetilde{\Phi} : (z_1, z_2) \to (t_1, t_2) = (z_1, 0)$. Thus $\widetilde{\Phi}^{-1} : (t_1, 0) \to (z_1, z_2) = (t_1, -P(t_1))$. Hence, for $a = (a_1, a_2) \in L_{\mathcal{M}}$, $a_1 \neq 0$, $a_2 = 0$, automorphism

$\widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}$ we have a representation

$$\begin{aligned} \widetilde{\Phi}^{-1} \circ (I+a) \circ \widetilde{\Phi}(z_1, z_2) &= \widetilde{\Phi}^{-1} \circ (I+a)(z_1, 0) \\ &= \widetilde{\Phi}^{-1}(z_1 + a_1, 0) \\ &= (z_1 + a_1, -P(z_1 + a_1)), \end{aligned}$$

where I is identity operator on $\mathbb{C}^2$. Thus $C_{\widetilde{\Phi}} T_a C_{\widetilde{\Phi}^{-1}}(f)(z) = f((z_1 + a_1), -P(z_1 + a_1))$ is hypercyclic composition operator on $V_{\mathcal{M}}$.

The following questions are natural: Are there polynomials $P \in \mathcal{P}(\mathbb{C}^n)$ for which there is polynomial automorphism from $\mathbb{C}^{n-1}$ in $\ker P$? That is, for which P we can find polynomials $\Phi_1, \ldots, \Phi_n$ on $\mathbb{C}^n$ such that, the map $\Phi = (P, \Phi_2, \ldots, \Phi_n)$ is polynomial automorphism?

It is known that necessary condition for this Jacobi equality

$$\frac{\partial \Phi}{\partial t} = \begin{vmatrix} \frac{\partial P}{\partial t_1} & \cdots & \frac{\partial P}{\partial t_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \Phi_n}{\partial t_1} & \cdots & \frac{\partial \Phi_n}{\partial t_n} \end{vmatrix} \tag{5.2}$$

is equal to some non zero constant M. Denote by $Q_k^{(t)}$ minors, which are complements to 1-th array and k-th column. Evidently, $Q_k^{(t)}$ are polynomials. Expanding the determinant (5.2) along the first column, we get, that

$$\sum_{k=1}^{n} \frac{\partial P(t)}{\partial t_k} Q_k(t) = M.$$

That is, the polynomials $\frac{\partial P}{\partial t_k}$, $(k = 1, \ldots, n)$ generate ideal, which coincides with the whole space of polynomials on $\mathbb{C}^n$. Thus $\frac{\partial P}{\partial t_k}$ do not have common zeros. So we get the next proposition.

Proposition 5.4. *If there is polynomial automorphism from $\mathbb{C}^{n-1}$ in $\ker P$, then polynomials $\frac{\partial P(t)}{\partial t_k}$, $(k = 1, \ldots, n)$ do not have common zeros.*

Is it true vice versa? This question is related to the well-known Jacobi problem which remains open since 1939.

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Received: 28.07.2015; **revised:** 09.10.2015.

Можировська З.Г. Гіперциклічні оператори композиції. *Журнал Прикарпатського університету імені Василя Стефаника,* **2** (4) (2015), 75–92.

В цій статті міститься огляд теорії гіперциклічних операторів композиції, зокрема представлено нові класи гіперциклічних операторів композиції на просторах аналітичних функцій.

Ключові слова: гіперциклічні оператори, функціональні простори, поліноміальні автоморфізми, симетричні функції.