SOME COMBINATORIAL IDENTITIES FOR NARAYANA'S COWS SEQUENCE

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We investigate some families of Toeplitz-Hessenberg determinants the entries of which are Narayana's cows numbers with successive, even, and odd subscripts.

Keywords: Narayana's cows sequence, Fibonacci-Narayana sequence, Fibonacci sequence, Toeplitz-Hessenberg matrix.

1. Narayana's cows sequence. The Fibonacci sequence $\{F_n\}_{n\geq 0}$ is defined by the initial values $F_0 = 0$, $F_1 = 1$ and the recurrence relation

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2.$$

Among the many generalizations of the Fibonacci sequence, one of the most known is the *Narayana's cows sequence* (or *Fibonacci-Narayana sequence*) $\{b_n\}_{n\geq 0}$, which defined by the following third-order recurrence relation

$$b_n = b_{n-1} + b_{n-3}, \qquad b_0 = 0,\ b_1 = b_2 = 1,$$

for $n \geq 3$ (sequence A000930 in On-Line Encyclopedia of Integer Sequences). The first few Narayana's cows numbers are 0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60,….

Narayana's cows sequence was introduced by the Indian mathematician Narayana in the 14th century, while studying the following problem: A cow produces one calf every year. Beginning in its fourth year, each calf produces one calf at the beginning of each year. How many cows are there altogether after n years?

Many authors studied the Narayana's cows sequence and its generalizations (see, for example, Bilgici, 2016; Didkivska & St'opochkina, 2013; Flaut & Shpakivskyi, 2013; Ramirez & Sirvent, 2015; Zatorsky & Goy, 2016 and the references given therein).

We study some families of Toeplitz-Hessenberg determinants whose entries are Narayana's cows numbers. This leads to discover new identities for these numbers.

Our approach is similar to Goy, 2017a; Goy, 2017b; Goy, 2017c; Goy, 2017d.

2. Toeplitz-Hessenberg matrices and determinants. A *Toeplitz-Hessenberg matrix* is an $n \times n$ matrix of the form

$$M_n(a_0; a_1, \ldots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix},$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k \geq 1$.

The following result gives the multinomial extension for $\det M_n$.

Lemma 1 (Muir, 1960). *Let n be a positive integer. Then*

$$\det M_n = \sum_{(s_1,\ldots,s_n)} (-a_0)^{n-(s_1+\cdots+s_n)} p_n(s) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \qquad (1)$$

where the summation is over integers $s_j \ge 0$ satisfying $s_1 + 2s_2 + \cdots + ns_n = n$, and $p_n(s) = \dfrac{(s_1+\cdots+s_n)!}{s_1!\cdots s_n!}$ is the multinomial coefficient.

For brevity and clarity, we will denote

$$D(a_1, a_2, \ldots, a_n) = \det M_n(1; a_1, a_2, \ldots, a_n).$$

3. Toeplitz-Hessenberg determinants with Narayana's cows numbers entries. Now we evaluate $D(a_1, a_2, \ldots, a_n)$ with special entries a_i.

Theorem 2. *Let $n \ge 1$, except when noted otherwise. Then*

$$D(b_1, b_3, \ldots, b_{2n-1}) = 1 - (-1)^n F_{n-1},$$

$$D(b_0, b_2, \ldots, b_{2n-2}) = (-1)^{n-1} F_n,$$

$$D(b_0, b_1, \ldots, b_{n-1}) = \frac{(-1)^{n-1} + (-1)^{\lfloor (n+1)/2 \rfloor}}{2},$$

$$D(b_1, b_2, \ldots, b_n) = \frac{(-1)^{n-1} + (-1)^{\lfloor n/3 \rfloor}}{2},$$

$$D(b_3, b_4, \ldots, b_{n+2}) = \frac{1 + (-1)^n}{2(-1)^{n/2}}, \quad n \ge 2,$$

$$D(b_3, b_5, \ldots, b_{2n+1}) = 0, \quad n \ge 4,$$

$$D(b_4, b_5, \ldots, b_{n+3}) = \frac{(-1)^{\lfloor (n-1)/3 \rfloor} + (-1)^{\lfloor n/3 \rfloor}}{2},$$

$$D(b_4, b_6, \ldots, b_{2n+2}) = 1, \quad n \ge 3,$$

$$D(b_5, b_7, \ldots, b_{2n+3}) = n + 1, \quad n \ge 2,$$

where $\lfloor \alpha \rfloor$ is the floor function of α, F_n is the n^{th} Fibonacci number.

3. Multinomial extension of Toeplitz-Hessenberg determinants. In this section, we focus on multinomial extension of Theorems 2, using Lemma 1.

Theorem 3. *Let $n \ge 1$, except when noted otherwise. Then*

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_0^{s_1} b_1^{s_2} \cdots b_{n-1}^{s_n} = \frac{(-1)^{\lfloor (3n+1)/2 \rfloor} - 1}{2},$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_0^{s_1} b_2^{s_2} \cdots b_{2n-2}^{s_n} = -F_n,$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_1^{s_1} b_2^{s_2} \cdots b_n^{s_n} = \frac{(-1)^{\lfloor 4n/3 \rfloor} - 1}{2},$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_1^{s_1} b_3^{s_2} \cdots b_{2n-1}^{s_n} = (-1)^n - F_{n-1},$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_2^{s_1} b_3^{s_2} \cdots b_{n+1}^{s_n} = 0, \quad n \ge 4,$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_3^{s_1} b_5^{s_2} \cdots b_{2n+1}^{s_n} = 0, \quad n \ge 4,$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_4^{s_1} b_5^{s_2} \cdots b_{n+3}^{s_n} = \frac{(-1)^{\lfloor (4n-1)/3 \rfloor} + (-1)^{\lfloor 4n/3 \rfloor}}{2},$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_4^{s_1} b_6^{s_2} \cdots b_{2n+2}^{s_n} = (-1)^n, \quad n \ge 3,$$

$$\sum_{(s_1,\ldots,s_n)} (-1)^{\sigma_n} p_n(s) b_5^{s_1} b_7^{s_2} \cdots b_{2n+3}^{s_n} = (-1)^n (n+1), \quad n \ge 2,$$

where the summation is over integers $s_i \ge 0$ *satisfying* $s_1 + 2s_2 + \cdots + ns_n = n$, $\sigma_n = s_1 + \cdots + s_n$*, and* F_n *is the* n^{th} *Fibonacci number.*

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