.

ON COMBINATORIAL IDENTITIES FOR JACOBSTHAL POLYNOMIALS

TARAS GOY

ABSTRACT. In this paper, we consider determinants for some families of Toeplitz–Hessenberg matrices whose entries are the Jacobsthal polynomials. These formulas may also be rewritten as identities involving sums of products of Jacobsthal polynomials (with sequential, odd or even subscripts) and multinomial coefficients.

INTRODUCTION

Many numbers and polynomial sequences can be defined by second-order recurrence relations, such as Fibonacci numbers, Lucas numbers, Pell numbers, Jacobsthal numbers, Chebyshev polynomials, among others. These numbers and polynomials play a fundamental role in mathematics and have numerous important applications in combinatorics, number theory, numerical analysis (see, for example, [3, 10, 11] and the references given there).

In particular, for $n \geq 2$, the *Jacobsthal sequence* $\{J_n\}_{n\geq 0}$ is defined by the recurrence $J_n = J_{n-1} + 2J_{n-2}$, with initial conditions $J_0 = 0$, $J_1 = 1$ (sequence A001045 in On-Line Encyclopedia of Integer Sequences [15]).

A natural extension of the Jacobsthal numbers is given by the *Jacobsthal polynomials* $\{j_n(x)\}_{n\geq 0}$, which are introduced by Horadam in [9] and defined as follows

$$j_n(x) = j_{n-1}(x) + 2j_{n-2}(x), \quad n \geq 2, \tag{1}$$

with $j_0(x) = 0$, $j_1(x) = 1$. Explicit closed form expression for $j_n(x)$ is

$$j_n(x) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-1-i}{i} (2x)^i, \quad n \geq 0,$$

where $\lfloor \alpha \rfloor$ is the floor function.

Note that $j_n(1) = J_n$ and $j_n(1/2) = F_n$ is the n^{th} Fibonacci numbers (sequence A000045 in [15]).

Date: July 15, 2018.

2010 *Mathematics Subject Classification.* Primary 11B37; Secondary 15A15.

Key words and phrases. Jacobsthal polynomial, Jacobsthal sequence, Fibonacci sequence, Toeplitz-Hessenberg matrix, multinomial coefficient.

The first few Jacobsthal polynomials are $j_0(x) = 0$, $j_1(x) = j_2(x) = 1$, $j_3(x) = 2x+1$, $j_4(x) = 4x+1$, $j_5(x) = 4x^2+6x+1$, $j_6(x) = 12x^2+8x+1$, $j_7(x) = 8x^3 + 24x^2 + 10x + 1$, $j_8(x) = 32x^3 + 40x^2 + 12x + 1$.

The Jacobsthal polynomials have many applications (see, for example, [1, 4, 14]). Many interesting properties of Jacobsthal polynomials and their generalizations are studied in [2, 3, 9, 16].

The purpose of this paper is to study the Jacobsthal polynomials. We investigate some families of Toeplitz-Hessenberg matrices the entries of which are Jacobsthal polynomials with successive, odd or even subscripts. As result, we obtain some new identities with multinomial coefficients for these polynomials.

1. Toeplitz–Hessenberg matrices and determinants

A lower *Toeplitz–Hessenberg matrix* is a square matrix of the form

$$M_n(a_0; a_1, \ldots, a_n) = \begin{bmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots & \ldots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix},$$

where $a_0 \neq 0$ and $a_i \neq 0$ for at least one $i > 0$.

Expanding the determinant $\det(M_n)$ repeatedly along the last row, we obtain the recurrence

$$\det(M_n) = \sum_{i=1}^{n} (-a_0)^{i-1} a_i \det(M_{n-i}), \tag{2}$$

where, by definition, $\det(M_0) = 1$.

It is known that the determinant $\det(M_n)$ can be evaluated using the *Trudi formula* as follows

$$\det(M_n) = \sum_{s_1+2s_2+\cdots+ns_n=n} p_n(s)(-a_0)^{n-|s|} a_1^{s_1} \cdots a_n^{s_n}, \tag{3}$$

where $|s| = s_1 + \cdots + s$, $p_n(s) = \binom{|s|}{s_1,\ldots,s_n}$ is the multinomial coefficient, and $n = s_1 + 2s_2 + \cdots + ns_n$ is partitions of the positive integer n, where each positive integer i appear s_i times [12, 13].

Many combinatorial identities involving sums over integers partitions can be generated in this way. Some of these identities presented in [5, 6, 7] and Section 3 of this paper.

2. Toeplitz–Hessenberg determinants whose entries are Jacobsthal polynomials

In this section, we provide determinant formulas for Toeplitz–Hessenberg matrices whose triangular entries are Jacobsthal polynomials with successive, odd or even subscripts.

We investigate particular cases of Toeplitz–Hessenberg matrices, where $a_0 = \pm 1$. For simplicity of notation, we write $\det(\pm 1; a_1, \ldots, a_n)$ in place of $\det\big(M_n(\pm 1; a_1, \ldots, a_n)\big)$.

Theorem 1. *Let $n \geq 1$, except when noted otherwise. Then*

$$\det(1; j_0(x), \ldots, j_{n-1}(x)) = \frac{1}{\sqrt{8x-3}}\left(\left(\frac{-1-\sqrt{8x-3}}{2}\right)^{n-1} - \left(\frac{-1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\det(-1; j_0(x), \ldots, j_{n-1}(x)) = \frac{1}{\sqrt{8x+5}}\left(\left(\frac{1+\sqrt{8x+5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\det(1; j_0(x), \ldots, j_{2n-2}(x)) = \frac{1}{\sqrt{8x-3}}\left(\left(\frac{-4x-1-\sqrt{8x-3}}{2}\right)^{n-1} - \left(\frac{-4x-1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\det(1; j_0(x), \ldots, j_{2n-2}(x)) = \frac{1}{\sqrt{8x+5}}\left(\left(\frac{4x+1+\sqrt{8x+5}}{2}\right)^{n-1} - \left(\frac{4x+1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\det(1; j_1(x), \ldots, j_n(x)) = 2^{\frac{n-3}{2}} x^{\frac{n-1}{2}} \big(1-(-1)^n\big); \tag{4}$$

$$\det(-1; j_1(x), \ldots, j_n(x)) = \frac{\big(1+\sqrt{2x+1}\big)^n - \big(1-\sqrt{2x+1}\big)^n}{2\sqrt{2x+1}};$$

$$\det(1; j_1(x), \ldots, j_{2n-1}(x)) = \frac{\big(-2x-\sqrt{2x}\big)^{n-1} + \big(-2x+\sqrt{2x}\big)^{n-1}}{2};$$

$$\det(1; j_2(x), \ldots, j_{n+1}(x)) = 0, \qquad n \geq 3;$$

$$\det(-1; j_2(x), \ldots, j_{n+1}(x)) = \frac{\big(1+\sqrt{4x+1}\big)^{n+1} - \big(1-\sqrt{4x+1}\big)^{n+1}}{4\sqrt{4x+1}};$$

$$\det(1; j_2(x), \ldots, j_{2n}(x)) = n(-2x)^{n-1};$$

$$\det(-1; j_2(x), \ldots, j_{2n}(x)) = \frac{1}{2\sqrt{4x+1}}\Big(\big(2x+1+\sqrt{4x+1}\big)^n - \big(2x+1-\sqrt{4x+1}\big)^n\Big);$$

$$\det(1; j_3(x), \ldots, j_{n+2}(x)) = (2x)^n, \qquad n \geq 2;$$

$$\det(-1; j_3(x), \ldots, j_{2n+1}(x)) = 2^{n-1}\sum_{k=0}^{n}\sum_{i=0}^{n}\binom{n-i}{k}\binom{n+k}{i}x^{n-k};$$

$$\det(1; j_3(x), \ldots, j_{2n+1}(x)) = (-2x)^{n-1}, \qquad n \geq 2;$$

$$\det(1; j_4(x), \ldots, j_{n+3}(x)) = (2x)^{n-1}\big(2x(n+1)+1\big);$$

$$\det(1; j_4(x), \ldots, j_{2n+2}(x)) = 0, \qquad n \geq 3.$$

Proof. We will prove formula (4) using induction on n. The other proofs follows similarly, so we omit them for the sake of brevity. Clearly, formula (4) works when $n = 1$ and $n = 2$. Suppose it is true for all $k \leq n-1$, where $n \geq 3$.

Let $D_n = \det(1; j_1(x), \ldots, j_n(x))$. Using recurrences (2) and relation (1), we then obtain

$$\begin{aligned}
D_n &= \sum_{s=1}^{n}(-1)^{s-1}j_s(x)D_{n-s} \\
&= j_1(x)D_{n-1} + \sum_{s=2}^{n}(-1)^{s-1}\big(j_{s-1}(x) + 2xj_{s-2}(x)\big)D_{n-s} \\
&= D_{n-1} + \sum_{s=2}^{n}(-1)^{s-1}j_{s-1}(x)D_{n-s} + 2x\sum_{s=2}^{n}(-1)^{s-1}j_{s-2}(x)D_{n-s} \\
&= D_{n-1} + \sum_{s=1}^{n-1}(-1)^{s}j_s(x)D_{n-1-s} + 2x\sum_{s=0}^{n-2}(-1)^{s+1}j_s(x)D_{n-2-s} \\
&= 2xD_{n-2}.
\end{aligned}$$

Using the induction hypothesis we have

$$\begin{aligned}
D_n &= 2x \cdot 2^{\frac{n-5}{2}}x^{\frac{n-3}{2}}\big(1-(-1)^{n-2}\big) \\
&= 2^{\frac{n-3}{2}}x^{\frac{n-1}{2}}\big(1-(-1)^{n}\big).
\end{aligned}$$

Consequently, the formula (4) is true for n. Therefore, by induction, the formula works for all integers $n > 0$. $\square$

3. New formulas with multinomial coefficients for Jacobsthal polynomials

Next we focus on multinomial extension of Theorem 1. Trudi's formula (3), taken together with Theorem 1, yield the following result.

Theorem 2. *Let $n \geq 1$, except when noted otherwise. Then*

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_0^{s_1}(x)\cdots j_{n-1}^{s_n}(x)=$$

$$\frac{1}{\sqrt{8x-3}}\left(\left(\frac{1-\sqrt{8x-3}}{2}\right)^{n-1}-\left(\frac{1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\sum_{\sigma_n=n}p_n(s)j_0^{s_1}(x)\cdots j_{n-1}^{s_n}(x)=$$

$$\frac{1}{\sqrt{8x+5}}\left(\left(\frac{1+\sqrt{8x+5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_0^{s_1}(x)\cdots j_{2n-2}^{s_n}(x)=$$

$$\frac{1}{\sqrt{8x-3}}\left(\left(\frac{4x+1-\sqrt{8x-3}}{2}\right)^{n-1}-\left(\frac{4x+1+\sqrt{8x-3}}{2}\right)^{n-1}\right);$$

$$\sum_{\sigma_n=n}p_n(s)j_0^{s_1}(x)\cdots j_{2n-2}^{s_n}(x)=$$

$$\frac{1}{\sqrt{8x+5}}\left(\left(\frac{4x+1+\sqrt{8x+5}}{2}\right)^{n-1}-\left(\frac{4x+1-\sqrt{8x+5}}{2}\right)^{n-1}\right);$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_1^{s_1}(x)\cdots j_n^{s_n}(x)=(2x)^{\frac{n-3}{2}}x\big((-1)^n-1\big);$$

$$\sum_{\sigma_n=n}p_n(s)j_1^{s_1}(x)\cdots j_n^{s_n}(x)=\frac{\left(1+\sqrt{2x+1}\right)^n-\left(1-\sqrt{2x+1}\right)^n}{2\sqrt{2x+1}};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_1^{s_1}(x)\cdots j_{2n-1}^{s_n}(x)=\frac{\left(2x+\sqrt{2x}\right)^{n-1}+\left(2x-\sqrt{2x}\right)^{n-1}}{-2};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)j_2^{s_1}(x)\cdots j_{n+1}^{s_n}(x)=0,\qquad n\geq 3;$$

$$\sum_{\sigma_n=n}p_n(s)j_3^{s_1}(x)\cdots j_{2n+1}^{s_n}(x)=2^{n-1}\sum_{k=0}^{n}\sum_{i=0}^{n}\binom{n-i}{k}\binom{n+k}{i}x^{n-k};$$

$$\sum_{\sigma_n=n} p_n(s) j_2^{s_1}(x)\cdots j_{n+1}^{s_n}(x) = \frac{\big(1+\sqrt{4x+1}\big)^{n+1} - \big(1-\sqrt{4x+1}\big)^{n+1}}{4\sqrt{4x+1}};$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_2^{s_1}(x)\cdots j_{2n}^{s_n}(x) = -n(2x)^{n-1};$$

$$\sum_{\sigma_n=n} p_n(s) j_2^{s_1}(x)\cdots j_{2n}^{s_n}(x) = \frac{\big(2x+1+\sqrt{4x+1}\big)^{n} - \big(2x+1-\sqrt{4x+1}\big)^{n}}{2\sqrt{4x+1}};$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_3^{s_1}(x)\cdots j_{n+2}^{s_n}(x) = (-2x)^n, \qquad n \ge 2;$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_3^{s_1}(x)\cdots j_{2n+1}^{s_n}(x) = -(2x)^{n-1}, \qquad n \ge 2;$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_4^{s_1}(x)\cdots j_{n+3}^{s_n}(x) = -(-2x)^{n-1}\big(2x(n+1)+1\big);$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) j_4^{s_1}(x)\cdots j_{2n+2}^{s_n}(x) = 0, \qquad n \ge 3,$$

where $\sigma_n = s_1 + 2s_2 + \cdots + ns_n$, $|s| = s_1 + \cdots + s_n$, $p_n(s) = \frac{|s|!}{s_1!\cdots s_n!}$, *and the summation is over integers* $s_i \ge 0$ *satisfying* $\sigma_n = n$.

Corollary. *Since* $j_n(1) = J_n$ *and* $j_n(1/2) = F_n$, *we can obtain multinomial identities for Jacobsthal and Fibonacci numbers. For example, for* $n \ge 1$, *the following formulas hold*:

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) F_0^{s_1} F_1^{s_2}\cdots F_{n-1}^{s_n} = -1;$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) J_0^{s_1} J_1^{s_2}\cdots J_{n-1}^{s_n} = -F_n;$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) F_1^{s_1} F_2^{s_2}\cdots F_{n}^{s_n} = \frac{1-(-1)^n}{2};$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) J_1^{s_1} J_2^{s_2}\cdots J_{n}^{s_n} = 2^{\frac{n-3}{2}}\left((-1)^n - 1\right);$$

$$\sum_{\sigma_n=n} (-1)^{|s|} p_n(s) F_2^{s_1} F_4^{s_2}\cdots F_{2n}^{s_n} = -n;$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_2^{s_1}J_4^{s_2}\cdots J_{2n}^{s_n}=-n2^{n-1};$$

$$\sum_{\sigma_n=n}(-1)^{|s|}p_n(s)J_4^{s_1}J_5^{s_2}\cdots J_{n+3}^{s_n}=-(-2)^{n-1}(2n+3).$$

References

[1] C. Bender, *Fibonacci transmission lines*, Fibonacci Quart. **31** (1993), no. 3, 227–238.

[2] P. Catarino and M. L. Morgado, *On Generalized Jacobsthal and Jacobsthal-Lucas polynomials*, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat. **24** (2016), no. 3, 61–78.

[3] G. B. Djordjević and G. V. Milovanović, Special Classes of Polynomials, University of Nis, Leskovac, 2014.

[4] L. Fortuna and M. Frasca, *Generating passive systems from recursively defined polynomials*, Int. J. Circuits Syst. Signal Process **6** (2012), no. 2, 179–188.

[5] T. Goy, *On Jacobsthal and Jacobsthal-Lucas identities with multinomial coefficients*, Proc. Int. Conf. "Actual Problems of Pure and Applied Mathematics", Almaty, Kazakhstan (2017), 61–64.

[6] T. Goy, *On new identities for Mersenne numbers*, Appl. Math. E-Notes **18** (2018), 100–105.

[7] T. Goy, *Some families of identities for Padovan numbers*, Proc. Jangjeon Math. Soc. **21** (2018), no. 10, 413–419.

[8] A. F. Horadam, *Convolutions for Jacobsthal-type polynomials* Fibonacci Quart., **40** (2002), no. 3, 212–222.

[9] A. Horadam, *Jacobsthal representation polynomials*, Fibonacci Quart. **35** (1997), no. 2, 137–148.

[10] T. Koshy, *Fibonacci and Lucas Numbers and Applications*, Wiley, New York, 2017.

[11] T. Koshy, *Pell and Pell-Lucas Numbers with Applications*, Springer, New York, 2014.

[12] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Spec. Matrices **1** (2013), 10–16.

[13] T. Muir, *The Theory of Determinants in the Historical Order of Development*, Vol. 3, Dover Publications, New York, 1960.

[14] M. H. Reyhani, S. Alikhani and R. Hasni, *On the independence polynomials of certain molecular graphs*, Digest J. Nanomater. Biostruct. **7** (2012), no. 1, 193–197.

[15] N. J. A. Sloane, editor. The On-Line Encyclopedia of Integer Sequences. Available at https://oeis.org.

[16] A. Tereszkiewicz and I. Wawreniuk, *Generalized Jacobsthal polynomials and special points for them*, Appl. Math. Comput. **268** (2015), 806–814.

Faculty of Mathematics and Computer Science, Vasyl Stefanyk Precarpathian National University, 57 Shevchenko St., Ivano-Frankivsk, 76018, Ukraine

E-mail address: tarasgoy@yahoo.com