NEW IDENTITIES FOR PADOVAN NUMBERS VIA THE DETERMINANTS OF THE TOEPLITZ-HESSENBERG MATRICES

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Abstract. In this paper, we study some families of Toeplitz-Hessenberg determinants and permanents the entries of which are Padovan numbers. These studies have led us to discover new identities for these numbers.

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1. Determinants of the Toeplitz-Hessenberg matrices

A lower *Toeplitz-Hessenberg matrix* is a square matrix of the order n in the form

$$M_n(a_0, a_1, \dots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{pmatrix}, \tag{1}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$. This class of matrices is encountered in various application (see [4] and the references given there).

Lemma 1.1. *Let matrix $M_n = M_n(a_0, a_1, \dots, a_n)$ defined by (1). Then*

$$\det(M_n) = \sum_{t_1+2t_2+\cdots+nt_n=n} (-a_0)^{n-(t_1+t_2+\cdots+t_n)} p_n(t) a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}, \tag{2}$$

where the summation is over nonnegative integers satisfying

$$t_1 + 2t_2 + \cdots + nt_n = n,$$

and $p_n(t) = \frac{(t_1+t_2+\cdots+t_n)!}{t_1!t_2!\cdots t_n!}$ is the multinomial coefficient.

Note, that formula (2) is known as Trudi's formula ([5], p. 214). The case $a_0 = 1$ of this formula is known as Brioschi's formula ([5], p. 208).

There are a lot of relations between determinants or permanents of matrices and number sequences. For example, Yılmaz and Bozkurt [9] obtained some relations between Padovan sequence and permanents of one type of Hessenberg matrix. Kiliç [3] established some relations between the tribonacci sequence and permanents of one type of Hessenberg matrix. Using Hessenberg matrices, Cereceda [2] provided some determinantal representations of the general terms of second-order and third-order linear recurrence sequences with arbitrary initial values, including the Padovan, Perrin, and tribonacci numbers.

2. Determinants of the Toeplitz-Hessenberg matrices whose entries are Padovan numbers

The Padovan sequence is the sequence of integers P_n defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation $P_{n+3} = P_{n+1} + P_n$. The above definition is the one given by Stewart [7]. Other sources may start the sequence at a different place, in which case some of the identities in this paper must be adjusted with appropriate offsets. The first few terms are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21\ldots$ (sequence A000931 in Sloane's OEIS).

The Padovan numbers and their properties have been studied by some authors, see for example, [1, 6, 8].

Proposition 2.1. *For all $n \geq 1$, the following formulae hold:*

$$\det(1, P_1, P_2, \ldots, P_n) = 1 - \delta_{n,2},$$

$$\det(1, P_2, P_3, \ldots, P_{n+1}) = \frac{(-1)^{\lfloor \frac{n+1}{3} \rfloor} + (-1)^{\lfloor \frac{n+2}{3} \rfloor}}{2} + \delta_{n,1},$$

$$\det(1, P_3, P_4, \ldots, P_{n+2}) = n + \delta_{n,1},$$

$$\det(1, P_4, P_5, \ldots, P_{n+3}) = \frac{2 + (-1)^{\lfloor \frac{n}{3} \rfloor} + (-1)^{\lfloor \frac{n+1}{3} \rfloor}}{2},$$

$$\det(1, P_5, P_6, \ldots, P_{n+4}) = (n^2 + n + 4)/2,$$

$$\det(1, P_0, P_2, \ldots, P_{2n-2}) = (-1)^{n-1} + \delta_{n,2},$$

$$\det(1, P_2, P_4, \ldots, P_{2n}) = \frac{(-1)^{\lfloor \frac{2n}{3} \rfloor} + (-1)^{\lfloor \frac{2n+1}{3} \rfloor}}{2} + \delta_{n,1},$$

$$\det(1, P_4, P_6, \ldots, P_{2n+2}) = \frac{(-1)^{\lfloor n/2 \rfloor} + (-1)^{n+\lfloor n/2 \rfloor}}{2} + \delta_{n,1},$$

$$\det(1, P_1, P_3, \ldots, P_{2n-1}) = \frac{(-1)^n}{2} \left((-1)^{\left\lfloor \frac{2n+1-(-1)^n}{4} \right\rfloor} - 1 \right),$$

$$\det(1, P_3, P_5, \ldots, P_{2n+1}) = \begin{cases} 3 - n + \delta_{n,3}, & \text{if } 1 \leq n \leq 3; \\ 0, & \text{if } n \geq 4, \end{cases}$$

$$\det(1, P_5, P_7, \ldots, P_{2n+3}) = \sum_{i=0}^{\lfloor n/3 \rfloor + 1} \binom{n+3-2i}{i},$$

where $\delta_{n,k}$ is the Kronecker symbol, $\lfloor s \rfloor$ is the floor function of s.

3. Main formulae

Using (2) for determinants in Proposition 2.1, after obviously transformations we obtain the following identities for the Padovan numbers with successive, even, and odd indices.

Proposition 3.1. *The following formulae hold:*

$$\sum_{\tau_n = n} (-1)^{T_n} p_n(t) P_1^{t_1} P_2^{t_2} \cdots P_n^{t_n} = (-1)^n, \quad (n \geq 3),$$

$$\sum_{\tau_n = n} (-1)^{T_n} p_n(t) P_2^{t_1} P_3^{t_2} \cdots P_{n+1}^{t_n} = \frac{(-1)^{\lfloor \frac{n+1}{3} \rfloor + n} + (-1)^{\lfloor \frac{n+2}{3} \rfloor + n}}{2}, \quad (n \geq 3),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_3^{t_1}P_4^{t_2}\cdots P_{n+2}^{t_n}=(-1)^n n,\quad (n\geq 2),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_4^{t_1}P_5^{t_2}\cdots P_{n+3}^{t_n}=\frac{2+(-1)^{\lfloor\frac{n}{3}\rfloor}+(-1)^{\lfloor\frac{n+1}{3}\rfloor}}{2(-1)^n},\quad (n\geq 1),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_5^{t_1}P_6^{t_2}\cdots P_{n+4}^{t_n}=\frac{(-1)^n}{2}(n^2+n+4),\quad (n\geq 1),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_0^{t_1}P_2^{t_2}\cdots P_{2n-2}^{t_n}=-1,\quad (n\geq 3),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_2^{t_1}P_4^{t_2}\cdots P_{2n}^{t_n}=\frac{(-1)^{\lfloor\frac{2n}{3}\rfloor}+(-1)^{\lfloor\frac{2n+1}{3}\rfloor}}{2(-1)^n},\quad (n\geq 2),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_4^{t_1}P_6^{t_2}\cdots P_{2n+2}^{t_n}=\frac{(-1)^{\lfloor n/2\rfloor}+(-1)^{\lfloor n/2\rfloor+n}}{2},\quad (n\geq 2),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_1^{t_1}P_3^{t_2}\cdots P_{2n-1}^{t_n}=\frac{(-1)^{\left\lfloor\frac{2n+1-(-1)^n}{4}\right\rfloor}-1}{2},\quad (n\geq 1),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_3^{t_1}P_5^{t_2}\cdots P_{2n+1}^{t_n}=0,\quad (n\geq 4),$$

$$\sum_{\tau_n=n}(-1)^{T_n}p_n(t)P_5^{t_1}P_7^{t_2}\cdots P_{2n+3}^{t_n}=(-1)^n\cdot\sum_{i=0}^{\lfloor n/3\rfloor+1}\binom{n+3-2i}{i},\quad (n\geq 1),$$

where $p_n(t)=\binom{t_1+\cdots+t_n}{t_1,\ldots,t_n}$ *is the multinomial coefficient,* $T_n=t_1+\cdots+t_n$*,* $\tau_n=t_1+2t_2+\cdots+nt_n$*, and the summation is over nonnegative integers satisfying* $\tau_n=n$*.*

References

[1] C. Ballantine, M. Merca, *Padovan numbers as sums over partitions into odd parts*, J. Inequal. Appl. (2016), 2016:1.

[2] J. L. Cereceda, *Determinantal represantations for generalized Fibonacci and tribonacci numbers*, Int. J. Contemp. Math. Sci. 9 (2014), no. 6, 269-285.

[3] E. Kiliç, *Tribonacci sequences with certain indices and their sums*, Ars Comb. 86 (2008), 13-22.

[4] M. Merca, *A note on the determinant of a Toeplitz-Hessenberg matrix*, Spec. Matrices, 1 (2013), 10-16.

[5] T. Muir, *The Theory of Determinants in the Historical Order of Development.* Vol. 3, Dover Publications, New York, 1960.

[6] K. Sokhuma, *Matrices formula for Padovan and Perrin sequences*, Appl. Math. Sci. 7 (2013), no. 142, 7093-7096.

[7] I. Stewart, *Tales of a neglected number*, Sci. Am. 274 (1996), 102-103.

[8] S. Taş, E. Karaduman, *The Padovan sequences in finite groups*, Chiang Mai J. Sci. 41 (2014), no. 2, 456-462.

[9] F. Yılmaz, D. Bozkurt, *Some properties of Padovan sequence by matrix methods*, Ars Comb. 104 (2012), 149-160.

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