On new Catalan identities using Toeplitz–Hessenberg matrices

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The Catalan numbers are a sequence defined directly in terms of binomial coefficients:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, \quad n \geqslant 0,$$

or recursively as follows:

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}, \quad C_0 = 1.$$

The Catalan numbers have a rich history and many unique properties. They count certain types of lattice paths, permutations, binary trees, and many other combinatorial objects (see [1, 3] and the references given there).

Using Trudi's formula [2] for determinants and permanents of Toeplitz–Hessenberg matrices with Catalan entries, we obtain some new identities for the Catalan numbers.

Proposition. *Let $n \geqslant 1$, except when noted otherwise. The following formulas hold*:

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_0^{t_1} C_1^{t_2} \cdots C_{n-1}^{t_n} = \frac{1}{n+1}\binom{2n}{n} = C_n,$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n+1} p_n(t) C_1^{t_1} C_2^{t_2} \cdots C_n^{t_n} = \frac{1}{n}\binom{2n-2}{n-1} = C_{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_1^{t_1} C_2^{t_2} \cdots C_n^{t_n} = \binom{2n-1}{n} = (2n-1)C_{n-1},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n+1} p_n(t) C_2^{t_1} C_4^{t_2} \cdots C_{2n}^{t_n} = \frac{1}{4n-1}\binom{4n}{2n} = \frac{2n+1}{4n-1} C_{2n},$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} p_n(t) C_1^{t_1} C_3^{t_2} \cdots C_{2n-1}^{t_n} = \frac{1}{n}\binom{3n-2}{n-1} \cdot {}_2F_1(1-n, -4n; 2-3n; -1),$$

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{t_1+\cdots+t_n} p_n(t) C_3^{t_1} C_5^{t_2} \cdots C_{2n+1}^{t_n} = \frac{1}{n}\sum_{i=0}^{n+1} 2^i \binom{2n-1+i}{i}\binom{2n-1}{n+1-i}, \quad n \geqslant 2,$$

where the summation is over nonnegative integers satisfying $t_1 + 2t_2 + \cdots + nt_n = n$,

$$p_n(t) = \frac{(t_1 + t_2 + \cdots + t_n)!}{t_1! t_2! \cdots t_n!}$$

is the multinomial coefficient, and ${}_2F_1(a, b; c; -1)$ is the generalized hypergeometric function.

1. Koshy T. Catalan Numbers with Applications. – Oxford: Oxford University Press, 2009, 422 p.
2. Merca M. A note on the determinant of a Toeplitz–Hessenberg matrix. Spec. Matrices, 2013, 1, 10–16.
3. Stanley R. Catalan Numbers. – Cambridge: Cambridge University Press, 2015, 222 p.