ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (2), 593–605
doi:10.15330/cmp.16.2.593-605

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2024, Т.16, №2, С.593–605

Some extremal problems on the Riemannian sphere

Denega I.V., Zabolotnyi Ya.V.

In the paper, the open problem on maximum of the product of inner radii of n domains in the case, when points and domains belong to the unit disk, is investigated. This problem is solved only for $n = 2$ and $n = 3$. No other results are known at present. We obtain the result for all $n \geqslant 2$. Also, we propose an approach that allows to establish evolutionary inequalities for the products of the inner radii of mutually non-overlapping domains.

Key words and phrases: conformal domain radius, inner domain radius, mutually non-overlapping domains, Green function, logarithmic capacity, transfinite diameter, area-minimization theorem.

Institute of Mathematics, National Academy of Sciences of Ukraine, 3 Tereschenkivska str., 01024, Kyiv, Ukraine
E-mail: iradenega@gmail.com (Denega I.V.), yaroslavzabolotnii@gmail.com (Zabolotnyi Ya.V.)

1 Preliminaries

Let $\mathbb{N}$, $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C} \bigcup \{\infty\}$ be its one point compactification, U be the open unit disk in $\mathbb{C}$, $\mathbb{R}^+ = (0, \infty)$.

Let function $f(z)$, meromorphic in a disk $|z| < 1$, maps univalently disk $|z| < 1$ onto the domain $B \subset \overline{\mathbb{C}}$ such that $f(0) = a$, where $a \in B$. Then the value $R(B, a) = \left|f'(0)\right|$ is called conformal radius of the domain B relative to the point $a \in B$. Conformal radius of the domain B with respect to an infinity point is $R(B, \infty) = R\big(\varphi(B), 0\big)$, where $\varphi(z) = 1/z$.

A function $g_B(z, a)$, which is continuous in $\overline{\mathbb{C}}$, harmonic in $B \backslash \{a\}$ apart from z, vanishes outside B, and in the neighborhood of a has the following asymptotic expansion

$$g_B(z, a) = -\ln|z - a| + \gamma + o(1), \qquad o(1) \to 0, \quad z \to a,$$

(if $a = \infty$, then $g_B(z, \infty) = \ln|z| + \gamma + o(1)$, $o(1) \to 0$, $z \to \infty$) is called the (classical) Green function of the domain B with pole at $a \in B$. The inner radius $r(B, a)$ of the domain B with respect to a point a is the quantity e^{γ} (see [1, 12, 15, 23, 25]).

Since the Green function is a conformal invariant, if a function f maps the domain B conformally and univalently onto a domain $f(B)$, then

$$r(B, a)\left|f'(a)\right| = r\big(f(B), f(a)\big)$$

for each $a \in B$. The inner radius increases monotonically with the growth of the domain. Namely, if $B \subset B'$, then

$$r(B, a) \leqslant r(B', a), \quad a \in B.$$

УДК 517.54
2020 *Mathematics Subject Classification:* 30C75.

This work was supported by grants from the Simons Foundation (1030291, 1290607, I.V.D., Y.V.Z.)

It is known [14], that the following inequality $|f'(0)| \leqslant r(B,a)$ holds. For a compact set E, its logarithmic capacity is determined by the equality

$$\operatorname{cap} E := \frac{1}{r\left(\overline{\mathbb{C}}\backslash E, \infty\right)},$$

if the value of $r\left(\overline{\mathbb{C}}\backslash E, \infty\right)$ is finite; otherwise, $\operatorname{cap} E := 0$ (see [1,12]).

Let G be a domain in extended complex plane $\overline{\mathbb{C}}_z$. By a quadratic differential in G we mean the expression

$$Q(z)dz^2, \tag{1}$$

where $Q(z)$ is a meromorphic function in G (see, for example, [1,12,15]).

A finite point $z_0 \in G$ is called a zero or a pole of order n of the differential (1) if it is a zero or a pole, respectively, of the function $Q(z)$.

A circle domain for quadratic differential $Q(z)dz^2$ is called simply connected domain G, containing a unique double pole of the quadratic differential $Q(z)dz^2$ in the point $w = a \in G$, such that for a univalent conformal mapping $w = f(z)$ $(f(a) = 0)$ of the domain G onto the unit circle, the following identity holds

$$Q(z)dz^2 \equiv -k\frac{dw^2}{w^2}, \qquad k \in \mathbb{R}^+.$$

Problem 1. *Find the maximum of the product*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right), \tag{2}$$

where $n \in \mathbb{N}$, $n \geqslant 2$, a_k, $k = \overline{1,n}$, are any different fixed points of $\overline{\mathbb{C}}$, domains B_k, $k = \overline{1,n}$, such that $a_k \in B_k \subset \overline{\mathbb{C}}$ and $B_i \cap B_j = \varnothing$, $1 \leqslant i, j \leqslant n$, $i \neq j$.

For simply-connected domains, Problem 1 was formulated in [13, p. 157]. In the general case, this problem was formulated in [8] (see also [11, Problem 9.4]).

In 1934, M.A. Lavrentiev [22] solved the problem of the maximum of the product of conformal radii of two non-overlapping simply connected domains.

Theorem 1 ([22]). *Let a_1 and a_2 be some fixed points of the complex plane $\mathbb{C}$, B_1, B_2 be any non-overlapping simply connected domains in $\mathbb{C}$ such that $a_k \in B_k$, $k \in \{1, 2\}$. Then the following inequality holds*

$$R\left(B_1, a_1\right) R\left(B_2, a_2\right) \leqslant |a_1 - a_2|^2. \tag{3}$$

The equality in (3) *occurs only in the case, when the domains B_1 and B_2 are two half-planes, the imaginary axis is their common boundary and points a_1, a_2 are symmetric relative to their common boundary.*

In 1951, G.M. Goluzin [13] for $n = 3$ obtained an accurate evaluate

$$\prod_{k=1}^{3} R\left(B_k, a_k\right) \leqslant \frac{64}{81\sqrt{3}} |a_1 - a_2| \cdot |a_1 - a_3| \cdot |a_2 - a_3|.$$

If a_1, a_2, and a_3 are three equidistant points on the unit circle $|z| = 1$, then equality occurs only in the case, when the domains B_1, B_2, and B_3 are bounded by rays emanating from the origin at equal angles to each other and containing a_1, a_2, and a_3 on their bisectors.

In 1980, G.V. Kuzmina [19] showed that the problem of the evaluation of the product (2) for $n = 4$ is reduced to the smallest capacity problem in a certain continuum family and obtained the exact inequality

$$\prod_{k=1}^{4} R\left(B_k, a_k\right) \leqslant \frac{9}{4^{8/3}} \left(\prod_{1\leqslant k<l\leqslant 4} |a_l - a_k| \right)^{\frac{2}{3}}.$$

The equality occurs only in the case, when the points -1, 1, and a form a regular triangle: $a = \pm i\sqrt{3}$.

In the works of V.N. Dubinin [12] and G.V. Kuzmina [20], the Problem 1 for $n = 5$ was solved under the additional assumption on the quintuple $a_1, \ldots, a_5$, two of which are symmetric relative to the straight line or circle passing through the other three

$$\prod_{k=1}^{5} R\left(B_k, a_k\right) \leqslant 4^{\frac{11}{3}} \cdot 3^{-\frac{3}{4}} \cdot 5^{-\frac{25}{6}} \left(\prod_{1\leqslant k<l\leqslant 5} |a_l - a_k| \right)^{\frac{1}{2}}.$$

The equality occurs for points 1, $e^{-\frac{2\pi i}{3}}$, 0, $e^{\frac{2\pi i}{3}}$, ∞, and domains B_k, $k = \overline{1,5}$, which are circular domains of the quadratic differential

$$Q(z)dz^2 = -\frac{z^6 + 7z^3 + 1}{z^2(z^3 - 1)^2}\, dz^2.$$

No other ultimate results related to Problem 1 for $n \geqslant 5$ are known at present. But, in the paper [7], for the product (2) the following theorem is obtained.

Theorem 2 ([7])**.** *Let $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in \mathbb{C}$, $B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, are, respectively, any set of different fixed points and domains of the complex plane such that $a_k \in B_k$, $k = \overline{1,n}$, $B_i \cap B_j = \varnothing$, $i \neq j$. Then the following inequality holds*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1\leqslant p<k\leqslant n} \left|a_p - a_k\right| \right)^{\frac{2}{n-1}}. \tag{4}$$

2 Estimation of the product of the inner radii of non-overlapping domains belonging to the unit disk

Other problems (see, for example, [2, 3, 5, 6, 9, 10, 16, 26, 27]), similar to Problem 1, were also interesting, but with an additional condition on the geometric location of the domains B_k, $k = \overline{1,n}$.

Problem 2. *Find the maximum of the product* (2)*, where $n \in \mathbb{N}$, $n \geqslant 2$, $a_k \in U$, $k = \overline{1,n}$, $U = \{z : |z| < 1\}$, domains $B_k \subset U$, $k = \overline{1,n}$, such that $a_k \in B_k$ and besides $B_i \bigcap B_j = \varnothing$, $1 \leqslant i, j \leqslant n$, $i \neq j$.*

For the case of two non-overlapping domains belonging to the unit disk, the following result is proved.

Theorem 3 ([21]). *For any non-overlapping simply connected domains $D_k \subset \{z : |z| < 1\}$ and points $z_k \in D_k$, $k \in \{1, 2\}$, the following inequality holds*

$$\prod_{k=1}^{2} r(D_k, z_k) \leqslant \frac{4\rho_0^2 \left(1-\rho_0^2\right)^2}{\left(1+\rho_0^2\right)^2}, \tag{5}$$

where $\rho_0^2 = \sqrt{5} - 2$. The equality in (5) *occurs in the case $z_1 + z_2 = 0$, $|z_k| = \rho_0$, D_k are corresponding semi-circles.*

For the case of three mutually non-overlapping domains belonging to the unit disk, the following result is obtained.

Theorem 4 ([17]). *For any three mutually non-overlapping simply connected domains $D_k \subset \{z : |z| < 1\}$ and points $z_k \in D_k$, $k \in \{1, 2, 3\}$, the following inequality holds*

$$\prod_{k=1}^{3} r(D_k, z_k) \leqslant \frac{64}{729}(223 - 70\sqrt{10}). \tag{6}$$

The equality in (6) *is attained only for the sectors $2\pi/3$ and points z_k^* lying on the bisectors and on the circle of the radius $\sqrt[3]{\sqrt{10} - 3}$.*

No other results related to Problem 2 for $n \geqslant 4$ are known at present.

Let $n \in \mathbb{N}$, $n \geqslant 2$. Denote by M_n maximum of the product (2) for all configurations of the domains B_k and points a_k such that $a_k \in U$, $k = \overline{1, n}$, where $U = \{z : |z| < 1\}$, and domains $B_k \subset U$, $k = \overline{1, n}$, such that $a_k \in B_k \subset \overline{\mathbb{C}}$, besides $B_i \bigcap B_j = \varnothing$, $1 \leqslant i, j \leqslant n$, $i \neq j$. Then we obtain the following result.

Theorem 5. *For an arbitrary $n \in \mathbb{N}$, $n \geqslant 2$, the inequality*

$$\left(\frac{4}{n}\right)^n \left(\sqrt{n^2+1} - n\right) \left(\frac{n+1-\sqrt{n^2+1}}{\sqrt{n^2+1} - n + 1}\right)^n \leqslant M_n \leqslant \left(\frac{1}{n}\right)^{\frac{n}{2}} \tag{7}$$

is valid.

Proof. To prove the left side of the inequality (7) it is enough to find the configuration of domains B_k^* and points a_k^* satisfying all conditions of the Theorem 5, for which

$$\prod_{k=1}^{n} r(B_k^*, a_k^*) \geqslant \left(\frac{4}{n}\right)^n \left(\sqrt{n^2+1} - n\right) \left(\frac{n+1-\sqrt{n^2+1}}{\sqrt{n^2+1} - n + 1}\right)^n.$$

The following lemma is true.

Lemma 1. *Let $P_n = \left\{z : |z| < 1; -\frac{\pi}{n} < \arg(z) < \frac{\pi}{n}\right\}$ and $p_n \in \mathbb{R}$, $0 < p_n < 1$. Then,*

$$R(P_n, p_n) = \frac{4p\left(1-p^n\right)}{n\left(1+p^n\right)}. \tag{8}$$

Proof. Consider the function $w_n(z) = \frac{z^n}{(1-z^n)^2}$. The function maps the sector $-\frac{\pi}{n} < \arg(z) < \frac{\pi}{n}$ onto a plane with a cross-cut along the real negative half-axis and $w_n(p) = \frac{p^n}{(1-p^n)^2}$. And the function $w_n^*(z) = \frac{p^n}{(1-p^n)^2}\left(\frac{z-1}{z+1}\right)^2$ maps the unit disk onto a plane with a cross-cut along the real negative half-axis such that $w_n^*(0) = \frac{p^n}{(1-p^n)^2}$. Then the function $w(z) = w_n^{-1}\left(w_n^*(z)\right)$ maps the unit disk onto the sector $-\frac{\pi}{n} < \arg(z) < \frac{\pi}{n}$, besides $w(0) = p$. An inner radius of the sector $-\frac{\pi}{n} < \arg(z) < \frac{\pi}{n}$ in the point p is

$$w'(0) = \frac{4p\left(1-p^n\right)}{n\left(1+p^n\right)}.$$

□

Examining the expression written in the right side of the equation (8), we obtain that the maximum inner radius of the sector $R\left(P_n, p_n\right)$ is attained for the case $p_n = \sqrt[n]{\sqrt{n^2+1}-n}$ and is equal to the following value

$$R_{max}\left(P_n, p_n\right) = \frac{4}{n}\frac{\sqrt[n]{\sqrt{n^2+1}-n}\left(n+1-\sqrt{n^2+1}\right)}{\sqrt{n^2+1}-n+1}.$$

Dividing the unit disk into n sectors with the central angle $\frac{2\pi}{n}$ and taking points a_k on the bisectors of these sectors at a distance $\sqrt[n]{\sqrt{n^2+1}-n}$ from the center, we obtain equality

$$\prod_{k=1}^{n} R\left(P_n, p_n\right) = \left(\frac{4}{n}\right)^n\left(\sqrt{n^2+1}-n\right)\left(\frac{n+1-\sqrt{n^2+1}}{\sqrt{n^2+1}-n+1}\right)^n,$$

which proves the left side of the inequality (7).

Let us prove that $M_n \leqslant \left(\frac{1}{n}\right)^{\frac{n}{2}}$. The area of the domain B_k will be denoted by $S\left(B_k\right) = S_k$. It is clear that $\sum\limits_{k=1}^{n} S_k \leqslant \pi$. Then from the area-minimization theorem [13, p. 30] among domains with the same area, the largest inner radius has a disk relative to the center. Thus it follows that $\pi r^2\left(B_k, a_k\right) \leqslant S_k$ and $\sum\limits_{k=1}^{n} r^2\left(B_k, a_k\right) \leqslant 1$. From the Cauchy inequality of arithmetic and geometric means, we obtain the following relationship

$$\frac{\sum\limits_{k=1}^{n} r^2\left(B_k, a_k\right)}{n} \geqslant \sqrt[n]{\prod_{k=1}^{n} r^2\left(B_k, a_k\right)}.$$

Hence

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{\sum\limits_{k=1}^{n} r^2\left(B_k, a_k\right)}{n}\right)^{\frac{n}{2}} \leqslant \left(\frac{1}{n}\right)^{\frac{n}{2}}.$$

□

For example, if $n = 4$, then the estimates $0,04575 \leqslant M_4 \leqslant 0,0625$ are true, that is, the error in estimating M_4 is relatively small.

Theorem 6. *For any set of different points a_k, $a_k \in \overline{\mathbb{C}} \setminus [-1,1]$, $k = \overline{1,n}$, and for any collection of mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1,1]$, $k = \overline{1,n}$, the following inequality holds*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant \left(\frac{1}{n}\right)^{\frac{n}{2}} \prod_{k=1}^{n} \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right|. \tag{9}$$

Proof. Let B_k^* be the image of the domain B_k by the mapping $w = z - \sqrt{z^2-1}$. Consider the branch of the root for which $\sqrt{1} = 1$. Taking into account the invariance of the Green function under conformal and univalent mappings, we get

$$g_{B_k}\left(z, a_k\right) = g_{B_k^*}\left(w, a_k^*\right) = \ln \frac{1}{\left|w - a_k^*\right|} + \ln r\left(B_k^*, a_k^*\right) + o(1). \tag{10}$$

Note that

$$\begin{aligned}
\ln \frac{1}{\left|w - a_k^*\right|} &= \ln \left| \frac{1}{z - \sqrt{z^2-1} - a_k + \sqrt{a_k^2-1}} \right| \\
&= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{1}{1 - \frac{\sqrt{z^2-1} - \sqrt{a_k^2-1}}{z - a_k}} \right| \\
&= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{1}{1 - \frac{\left(\sqrt{z^2-1} - \sqrt{a_k^2-1}\right)\left(\sqrt{z^2-1} + \sqrt{a_k^2-1}\right)}{(z-a_k)\left(\sqrt{z^2-1} + \sqrt{a_k^2-1}\right)}} \right| \\
&= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{1}{1 - \frac{z+a_k}{\sqrt{z^2-1} + \sqrt{a_k^2-1}}} \right| \\
&= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{\sqrt{z^2-1} + \sqrt{a_k^2-1}}{\sqrt{z^2-1} + \sqrt{a_k^2-1} - z - a_k} \right| \\
&= \ln \frac{1}{|z - a_k|} + \ln \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right| + \ln \left| \frac{\sqrt{\frac{z^2-1}{a_k^2-1}} + 1}{1 + \frac{\sqrt{z^2-1} - z}{\sqrt{a_k^2-1} - a_k}} \right|.
\end{aligned}$$

Substituting this expression in (10) and taking into account that

$$\ln \left| \frac{\sqrt{\frac{z^2-1}{a_k^2-1}} + 1}{1 + \frac{\sqrt{z^2-1} - z}{\sqrt{a_k^2-1} - a_k}} \right| \to 0 \quad \text{as} \quad z \to a_k,$$

the following equality is true

$$g_{B_k}\left(z, a_k\right) = \ln \frac{1}{|z - a_k|} + \ln \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right| r\left(B_k^*, a_k^*\right) + o(1).$$

Hence,

$$r\left(B_k, a_k\right) = \left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right| r\left(B_k^*, a_k^*\right).$$

And thus, from above-posed considerations, the equality

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) = \prod_{k=1}^{n} r\left(B_k^*, a_k^*\right) \prod_{k=1}^{n}\left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right| \tag{11}$$

follows.

The function $w = z - \sqrt{z^2-1}$ maps the points a_k onto the points a_k^*, which lie in the unit circle, and the domains B_k, that contain, respectively, the points a_k, onto the domains $B_k^* \subset U$, which contain, respectively, the points a_k^*. Therefore, all conditions of the Theorem 6 are satisfied for them and the inequality

$$\prod_{k=1}^{n} r\left(B_k^*, a_k^*\right) \leqslant \left(\frac{1}{n}\right)^{\frac{n}{2}}$$

is true. Combining the last inequality and the inequality (11), we obtain (9). □

3 Evolutionary inequalities for the products of the inner radii

This section is devoted to obtaining evolutionary inequalities for the functionals of the following type:

$$I_n(1) = r\left(B_0, 0\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$Y_n(1) = r\left(B_\infty, \infty\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

$$J_n(1) = r\left(B_0, 0\right) r\left(B_\infty, \infty\right) \prod_{k=1}^{n} r\left(B_k, a_k\right),$$

where $n \in \mathbb{N}$, $A_n = \{a_k\}_{k=1}^n$ is an arbitrary fixed system of points of the complex plane $\mathbb{C}\backslash\{0\}$, B_0, B_∞ and $\{B_k\}_{k=1}^n$ is an arbitrary system of mutually non-overlapping domains such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$.

The method, proposed in this paper, originates from the papers [4, 6, 18]. The following results are valid.

Theorem 7. *Let $n \in \mathbb{N}$, $\tau \in (0,1)$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ and for any collection of mutually non-overlapping domains B_k, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{0,n}$, $a_0 = 0$, the following inequality holds*

$$I_n(1) \leqslant n^{-\frac{1-\tau}{2}} I_n(\tau)\big(I_n(0)\big)^{-\frac{1-\tau}{n}} \left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2(1-\tau)}{n}}, \tag{12}$$

where $I_n(\tau) := r^\tau(B_0, 0) \prod_{k=1}^{n} r\left(B_k, a_k\right)$, $I_n(0) = \prod_{k=1}^{n} r\left(B_k, a_k\right)$.

Proof. Let $d(E)$ be the transfinite diameter of the compact set $E \subset \mathbb{C}$. It is known [1,12,13], that

$$\operatorname{cap} E = d(E) = \frac{1}{r(\overline{\mathbb{C}} \backslash E, \infty)}.$$

Then from Theorem 2 [6] it follows that the following relationships hold

$$r\left(B_0, 0\right) = r\left(B_0^+, \infty\right) = \frac{1}{d\left(\overline{\mathbb{C}} \setminus B_0^+\right)}, \quad B^+ = \left\{z : \frac{1}{z} \in B\right\}. \tag{13}$$

Further (see [6]), taking into account the Pólya theorem [24], monotonicity and additivity of the Lebesgue measure and the area-minimization theorem [13], we get the inequality

$$r\left(B_0, 0\right) \leqslant \frac{1}{\left[\sum\limits_{k=1}^{n} r^2\left(B_k^+, a_k^+\right)\right]^{\frac{1}{2}}}.$$

Taking advantage of the Green's function invariance at conformal and single-leaf mapping, we have

$$g_{B_k}\left(z, a_k\right) = g_{B_k^+}\left(w^+, a_k^+\right), \quad w^+ = \frac{1}{z}.$$

Then

$$g_{B_k^+}\left(w^+, a_k^+\right) = g_{B_k^+}\left(\frac{1}{z}, \frac{1}{a_k}\right) = \ln \frac{1}{\left|\frac{1}{z} - a_k^+\right|} + \ln r\left(B_k^+, a_k^+\right) + o(1).$$

Using simple transformations, we obtain

$$g_{B_k^+}\left(w^+, a_k^+\right) = \ln \frac{|z|}{\left|1 - z a_k^+\right|} + \ln r\left(B_k^+, a_k^+\right) + o(1) = \ln \frac{1}{\left|z - a_k\right|} + \ln \left|a_k\right|^2 r\left(B_k^+, a_k^+\right) + o(1).$$

Hence,

$$r\left(B_k^+, a_k^+\right) = \frac{r\left(B_k, a_k\right)}{\left|a_k\right|^2}$$

and we arrive at the following inequality

$$r\left(B_0, 0\right) \leqslant \frac{1}{\left[\sum\limits_{k=1}^{n} \dfrac{r^2\left(B_k, a_k\right)}{\left|a_k\right|^4}\right]^{\frac{1}{2}}}.$$

Taking it into consideration, we have

$$I_n(1) \leqslant \frac{\prod\limits_{k=1}^{n} r\left(B_k, a_k\right)}{\left[\sum\limits_{k=1}^{n} \dfrac{r^2\left(B_k, a_k\right)}{\left|a_k\right|^4}\right]^{\frac{1}{2}}}.$$

From the Cauchy inequality of arithmetic and geometric means, the following relationship holds

$$\frac{1}{n} \sum_{k=1}^{n} \frac{r^2\left(B_k, a_k\right)}{\left|a_k\right|^4} \geqslant \left(\prod_{k=1}^{n} \frac{r^2\left(B_k, a_k\right)}{\left|a_k\right|^4}\right)^{\frac{1}{n}}.$$

Whence it is easy to obtain that

$$\left(\sum_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right)^{\frac{1}{2}} \geqslant \left(n\left(\prod_{k=1}^{n}\frac{r^2\left(B_k,a_k\right)}{|a_k|^4}\right)^{\frac{1}{n}}\right)^{\frac{1}{2}} \geqslant n^{\frac{1}{2}}\left(\prod_{k=1}^{n}\frac{r\left(B_k,a_k\right)}{|a_k|^2}\right)^{\frac{1}{n}}.$$

From the above arguments it follows that

$$I_n(1) \leqslant n^{-\frac{1}{2}}\left(\prod_{k=1}^{n} r\left(B_k,a_k\right)\right)^{1-\frac{1}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2}{n}}. \tag{14}$$

It is clear that

$$I_n(1) = r^{\tau}\left(B_0,0\right)\left(r^{1-\tau}\left(B_0,0\right)\prod_{k=1}^{n} r\left(B_k,a_k\right)\right).$$

Combining the last equality and the previous inequality, we obtain

$$I_n(1) \leqslant r^{\tau}(B_0,0)\left(n^{-\frac{1-\tau}{2}}\left(\prod_{k=1}^{n} r\left(B_k,a_k\right)\right)^{1-\frac{1-\tau}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2(1-\tau)}{n}}\right).$$

And after some transformations, we get

$$\begin{aligned} I_n(1) &\leqslant r^{\tau}\left(B_0,0\right)\left(n^{-\frac{1-\tau}{2}}\prod_{k=1}^{n} r\left(B_k,a_k\right)\left(\prod_{k=1}^{n} r\left(B_k,a_k\right)\right)^{-\frac{1-\tau}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2(1-\tau)}{n}}\right) \\ &= r^{\tau}(B_0,0)\prod_{k=1}^{n} r\left(B_k,a_k\right) n^{-\frac{1-\tau}{2}}\left(\prod_{k=1}^{n} r\left(B_k,a_k\right)\right)^{-\frac{1-\tau}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2(1-\tau)}{n}}. \end{aligned}$$

Whence inequality (12) follows. □

Using Theorem 6 and inequality (14), we obtain the following result.

Corollary 1. *Let* $n \in \mathbb{N}$*,* $n \geqslant 2$*. Then for any fixed system of different points* $A_n = \{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash[-1,1]$ *and for any collection of mutually non-overlapping domains* B_0*,* B_k*,* $a_0 = 0 \in B_k \subset \overline{\mathbb{C}}$*,* $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1,1]$*,* $k = \overline{1,n}$*, the following inequality holds*

$$I_n(1) \leqslant n^{-\frac{n}{2}}\left(\prod_{k=1}^{n}\left|\frac{\sqrt{a_k^2-1}}{a_k-\sqrt{a_k^2-1}}\right|\right)^{1-\frac{1}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2}{n}}.$$

Taking into account Theorem 2 and inequality (14), from Theorem 7 we obtain the following result.

Corollary 2. *Let* $n \in \mathbb{N}$*,* $n \geqslant 2$*. Then for any fixed system of different points* $A_n = \{a_k\}_{k=1}^{n} \in \mathbb{C}\backslash\{0\}$ *and for any collection of mutually non-overlapping domains* B_k*,* $a_k \in B_k \subset \overline{\mathbb{C}}$*,* $k = \overline{0,n}$*,* $a_0 = 0$*, the following inequality holds*

$$I_n(1) \leqslant n^{-\frac{1}{2}}\left(n-1\right)^{-\frac{n-1}{4}}\left(\prod_{1\leqslant p<k\leqslant n}|a_p-a_k|\right)^{\frac{2}{n}}\left(\prod_{k=1}^{n}|a_k|\right)^{\frac{2}{n}}.$$

Theorem 8. *Let $n \in \mathbb{N}$, $\tau \in (0,1)$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}$ and for any collection of mutually non-overlapping domains B_∞, B_k, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds*

$$Y_n(1) \leqslant n^{-\frac{1-\tau}{2}} Y_n(\tau) \left(Y_n(0)\right)^{-\frac{1-\tau}{n}},$$

where $Y_n(\tau) := r^\tau(B_\infty,\infty) \prod\limits_{k=1}^n r(B_k,a_k)$, $Y_n(0) = \prod\limits_{k=1}^n r(B_k,a_k)$.

The proof of Theorem 8 is similar to that of Theorem 7, so we have chosen to omit the analogous details.

Corollary 3. *Let $n \in \mathbb{N}$, $n \geqslant 2$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}\backslash[-1,1]$ and for any collection of mutually non-overlapping domains B_∞, B_k, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}} \setminus [-1,1]$, $k = \overline{1,n}$, the following inequality holds*

$$Y_n(1) \leqslant n^{-\frac{n}{2}} \left(\prod_{k=1}^n \left| \frac{\sqrt{a_k^2-1}}{a_k - \sqrt{a_k^2-1}} \right| \right)^{1-\frac{1}{n}}.$$

Taking into account Theorem 2, from Theorem 8 we obtain the following result.

Corollary 4. *Let $n \in \mathbb{N}$, $n \geqslant 2$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}$ and for any collection of mutually non-overlapping domains B_∞, B_k, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds*

$$Y_n(1) \leqslant n^{-\frac{1}{2}} (n-1)^{-\frac{n-1}{4}} \left(\prod_{1 \leqslant p < k \leqslant n} |a_p - a_k| \right)^{\frac{2}{n}}.$$

Theorem 9. *Let $n \in \mathbb{N}$, $\tau \in (0,1)$. Then for any fixed system of different points $A_n = \{a_k\}_{k=1}^n \in \mathbb{C}\backslash\{0\}$ and for any collection of mutually non-overlapping domains B_0, B_∞, B_k, $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $\infty \in B_\infty \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality holds*

$$J_n(1) \leqslant (n+1)^{-(1-\tau)\frac{n+1}{n+2}} J_n(\tau) \left[J_n(0)\right]^{-\frac{2(1-\tau)}{n+2}} \prod_{k=1}^n |a_k|^{\frac{2(1-\tau)}{n+2}}, \tag{15}$$

where $J_n(\tau) := \Big[r(B_0,0)\, r(B_\infty,\infty) \Big]^\tau \prod\limits_{k=1}^n r(B_k,a_k)$, $J_n(0) = \prod\limits_{k=1}^n r(B_k,a_k)$.

Proof. Using the constructions given in the proof of Theorem 7, we have

$$r(B_0,0) \leqslant \left(r^2(B_\infty,\infty) + \sum_{k=1}^n r^2\left(B_k^+, a_k^+\right) \right)^{-\frac{1}{2}}.$$

By applying the relationship

$$r\left(B_k^+, a_k^+\right) = \frac{r(B_k,a_k)}{|a_k|^2},$$

we obtain the inequality

$$r(B_0,0) \leqslant \left[r^2(B_\infty,\infty) + \sum_{k=1}^n \frac{r^2(B_k,a_k)}{|a_k|^4} \right]^{-\frac{1}{2}}.$$

Similarly,

$$r\left(B_{\infty},\infty\right)\leqslant\left[r^{2}\left(B_{0},0\right)+\sum_{k=1}^{n}r^{2}\left(B_{k},a_{k}\right)\right]^{-\frac{1}{2}}.$$

Further, by using the Cauchy inequality of arithmetic and geometric means and by performing simple transformations, we deduce the estimate

$$r\left(B_{0},0\right)r\left(B_{\infty},\infty\right)\leqslant\frac{\left(\prod\limits_{k=1}^{n}|a_{k}|\right)^{\frac{2}{n+2}}}{(n+1)^{\frac{n+1}{n+2}}\left(\prod\limits_{k=1}^{n}r\left(B_{k},a_{k}\right)\right)^{\frac{2}{n+2}}}.$$

Whence it follows that

$$J_{n}\left(1\right)\leqslant(n+1)^{-\frac{n+1}{n+2}}\left(\prod_{k=1}^{n}r\left(B_{k},a_{k}\right)\right)^{1-\frac{2}{n+2}}\left(\prod_{k=1}^{n}|a_{k}|\right)^{\frac{2}{n+2}}. \tag{16}$$

Obviously,

$$r(B_{0},0)r(B_{\infty},\infty)\prod_{k=1}^{n}r\left(B_{k},a_{k}\right)=\left[r\left(B_{0},0\right)r\left(B_{\infty},\infty\right)\right]^{\tau}\left(\left[r\left(B_{0},0\right)r\left(B_{\infty},\infty\right)\right]^{1-\tau}\prod_{k=1}^{n}r\left(B_{k},a_{k}\right)\right).$$

Combining this with inequality (16), we conclude that

$$\begin{aligned}J_{n}(1)\leqslant&\left[r\left(B_{0},0\right)r\left(B_{\infty},\infty\right)\right]^{\tau}\left(\prod_{k=1}^{n}r\left(B_{k},a_{k}\right)\right)\\&\times\left((n+1)^{-\frac{(1-\tau)(n+1)}{n+2}}\left(\prod_{k=1}^{n}r\left(B_{k},a_{k}\right)\right)^{-\frac{2(1-\tau)}{n+2}}\left(\prod_{k=1}^{n}|a_{k}|\right)^{\frac{2(1-\tau)}{n+2}}\right).\end{aligned}$$

Whence inequality (15) follows. □

As a consequence of Theorem 9 and Theorem 6, we obtain the following result.

Corollary 5. *Let $n\in\mathbb{N}$, $n\geqslant 2$. Then for any fixed system of different points $A_n=\{a_k\}_{k=1}^{n}\in\mathbb{C}\backslash[-1,1]$ and for any collection of mutually non-overlapping domains B_0, B_∞, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}\setminus[-1,1]$, $k=\overline{1,n}$, the following inequality holds*

$$J_{n}\left(1\right)\leqslant(n+1)^{-\frac{n+1}{n+2}}\left((n)^{-\frac{n}{2}}\prod_{k=1}^{n}\left|\frac{\sqrt{a_{k}^{2}-1}}{a_{k}-\sqrt{a_{k}^{2}-1}}\right|\right)^{1-\frac{2}{n+2}}\left(\prod_{k=1}^{n}|a_{k}|\right)^{\frac{2}{n+2}}.$$

Using Theorem 2 and inequality (16), we have the following result.

Corollary 6. *Let $n\in\mathbb{N}$, $n\geqslant 2$. Then for any fixed system of different points $A_n=\{a_k\}_{k=1}^{n}\in\mathbb{C}\backslash\{0\}$ and for any collection of mutually non-overlapping domains B_0, B_∞, B_k, $a_0=0\in B_0\subset\overline{\mathbb{C}}$, $\infty\in B_\infty\subset\overline{\mathbb{C}}$, $a_k\in B_k\subset\overline{\mathbb{C}}$, $k=\overline{1,n}$, the following inequality holds*

$$J_{n}\left(1\right)\leqslant(n+1)^{-\frac{n+1}{n+2}}\left((n-1)^{-\frac{n}{4}}\left(\prod_{1\leqslant p<k\leqslant n}|a_{p}-a_{k}|\right)^{\frac{2}{n-1}}\right)^{1-\frac{2}{n+2}}\left(\prod_{k=1}^{n}|a_{k}|\right)^{\frac{2}{n+2}}.$$

4 Acknowledgements

The authors thank the anonymous reviewers for very careful analysis of this work and remarks helped improve and clarify this manuscript.

References

[1] Bakhtin A.K., Bakhtina G.P., Zelinskii Yu.B. Topological-algebraic structures and geometric methods in complex analysis. Zb. prats of the Inst. of Math. of NASU, 2008.

[2] Bakhtin A. *Extremal decomposition of the complex plane with restrictions for free poles*. J. Math. Sci. (N.Y.) 2018, **231** (1), 1–15. doi:10.1007/s10958-018-3801-5

[3] Bakhtin A. *Separating transformation and extremal problems on nonoverlapping simply connected domains*. J. Math. Sci. (N.Y.) 2018, **234** (1), 1–13. doi:10.1007/s10958-018-3976-9

[4] Bakhtin A.K., Denega I.V. *Inequalities for the inner radii of nonoverlapping domains*. Ukrainian Math. J. 2019, **71** (7), 1138–1145. doi:10.1007/s11253-019-01703-x

[5] Bakhtin A.K., Denega I.V. *Extremal decomposition of the complex plane with free poles*. J. Math. Sci. (N.Y.) 2020, **246** (3), 1–17. doi:10.1007/s10958-020-04718-z

[6] Bakhtin A.K., Denega I.V. *Generalized M.A. Lavrentiev's inequality*. J. Math. Sci. (N.Y.) 2022, **262** (2), 138–153. doi:10.1007/s10958-022-05806-y

[7] Bakhtin A.K., Zabolotnii Ya.V. *Estimates of the products on inner radii for multiconnected domains*. Ukrainian Math. J. 2021, **73** (1), 6–21. doi:10.37863/umzh.v73i1.6200

[8] Bakhtina G.P. Variational methods and quadratic differentials in problems of non-overlapping domains. Abstract of the Dissertation. Candidate of physico-mathematical sciences. Kyiv, 1975.

[9] Denega I.V., Zabolotnii Ya.V. *Estimates of products of inner radii of non-overlapping domains in the complex plane*. Complex Var. Elliptic Equ. 2017, **62** (11), 1611–1618. doi:10.1080/17476933.2016.1265952

[10] Denega I., Zabolotnii Ya. *Problem on extremal decomposition of the complex plane*. An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. 2019, **27** (1), 61–77.

[11] Dubinin V.N. *Symmetrization in the geometric theory of functions of a complex variable*. Russian Math. Surveys 1994, **49** (1), 1–79.

[12] Dubinin V.N. Condenser capacities and symmetrization in geometric function theory. Birkhäuser/Springer, Basel, 2014.

[13] Goluzin G.M. Geometric theory of functions of a complex variable. In: Translations of Mathematical Monographs, 26. Amer. Math. Soc. Providence, Rhode Island, 1969.

[14] Hayman W.K. Multivalent functions. In: Bertoin J., Fulton W. (Eds.) Cambridge Tracts in Mathematics, 110. Cambridge University Press, London, 1994.

[15] Jenkins J.A. Univalent functions and conformal mapping. Publishing House of Foreign Literature, Moscow, 1962. (in Russian)

[16] Kolbina L.I. *Conformal mapping of the unit circle onto non-overlapping domains*. Bulletin of Leningrad University 1955, **5**, 37–43.

[17] Kostyuchenko E.V. *The solution of one problem of extremal decomposition*. Dal'nevost. Mat. Zh. 2001, **2** (1), 3–15.

[18] Kovalev L.V. *On three disjoint domains*. Dal'nevost. Mat. Zh. 2000, **1** (1), 3–7.

[19] Kuz'mina G.V. *Methods of geometric function theory, II*. St. Petersbg. Math. J. 1998, **9** (5), 889–930.

[20] Kuz'mina G.V. *Problems of extremal decomposition of the Riemann sphere*. J. Math. Sci. (N.Y.) 2003, **118** (1), 4880–4894. doi:10.1023/A:1025580802209

[21] Kyfarev P.P. *On the question of conformal mappings of additional domains*. Doklady Akademii Nauk SSSR 1950, **73** (5), 881–884.

[22] Lavrentiev M.A. *On the theory of conformal mappings*. Tr. Sci. Inst. An. USSR 1934, **5**, 159–245.

[23] Lebedev N.A. The area principle in the theory of univalent functions. Moscow, Nauka, 1975.

[24] Pólya G., Szegö G. Isoperimetric inequalities in mathematical physics. In: Annals of Mathematics Studies, 27. Princeton Univ. Press, Princeton, 1951.

[25] Tamrazov P.M. *Extremal conformal mappings and poles of quadratic differentials*. Izv. Akad. Nauk SSSR Ser. Mat. 1968, **32** (5), 1033–1043.

[26] Zabolotnii Ya., Denega I. *On conformal radii of non-overlapping simply connected domains*. Int. J. Adv. Res. Math. 2018, **11**, 1–7.

[27] Zabolotnii Ya.V., Denega I.V. *An extremal problem on non-overlapping domains containing ellipse points*. Eurasian Math. J. 2021, **12** (4), 82–91.

Received 01.03.2023
Revised 03.03.2024

Денега І.В., Заболотний Я.В. *Деякі екстремальні задачі на рімановій сфері* // Карпатські матем. публ. — 2024. — Т.16, №2. — С. 593–605.

У даній роботі розглянуто відкриту проблему про максимум добутку внутрішніх радіусів n областей у випадку, коли точки та області містяться в одиничному крузі. Ця проблема розв'язана лише для $n = 2$ і $n = 3$. На даний час авторам невідомо про інші результати. Ми отримали нерівність для всіх $n \geqslant 2$. Крім того, у статті запропоновано підхід, який дозволяє встановити нерівності еволюційного типу для добутків внутрішніх радіусів областей, що не перетинаються між собою.

Ключові слова і фрази: конформний радіус області, внутрішній радіус області, взаємно неперетинні області, функція Гріна, логарифмічна ємність, трансфінітний діаметр, теорема про мінімізацію площі.