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A new modification of the finite difference method for solving transmission problems for two-interval differential equations

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In this paper, we consider a new type of boundary value problems, the main feature of which is the nature of the differential equation being solved and the imposed boundary conditions. Namely, the differential equation under consideration is given on two non-intersecting segments with a common and, on which additional interaction conditions are imposed, the so-called transmission conditions.

As is known, classical analytical and numerical methods are designed to solve one-interval differential equations without transmission conditions. The main purpose of this work is to develop a new modification of the classical finite difference method for solving two-interval boundary value transmission problems.

Key words and phrases: finite difference method, singular point, transmission condition, two-interval boundary value problem.

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1 Introduction

Boundary value problems for differential equations arise as mathematical models of many problems in physics, engineering and other branches of natural sciences. Evidently, not all problems can be solved analytically. In some cases the given differential equation can be solved analytically, but the implicit form of the exact solution may take such a complex form that it is useless to use as an important tool to find approximate solutions of many types of linear and nonlinear differential equations with unknown exact solutions. The finite difference method (FDM, for short) is one of the simple but effective methods for solving various type ordinary and partial differential equations [7, 18]. This method was used by L. Euler in solving ordinary differential equations and was extended to partial differential equations by C. Runge. Since the early 1950s years the FDM has been used to numerically solve some problems in physics. The main idea of this method is that it replace ordinary and partial derivaties included in the differential equations by algebraic relations, the so-called finite differences that approximate them.

Further development of this method was stimulated by the emergence of computers, which offered a convenient basis for solving complex problems of mathematical physics [8, 19]. Later many important theoretical results were obtained regarding the convergence, accuracy and

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efficiency of FDM [1,2,10,13,14,20]. M.M. Chawla and C.P. Katti [4] investigate the convergence of FDM for a class of singular boundary value problems (BVP's, for short). E. Uğurlu and K. Tas [21] developed a new modification of the finite difference scheme and demonstrated its applications to double singular BVP's. M. Kumar [12] examined the applicability of three-point FDM and 2nd order spline FDM to singular two-point BVP's.

Two-interval BVP's with singular points and additional transmission conditions arise in solving of many mathematical physics problems, including those related to vibration of loaded strings, diffraction, electric circuits, heat and mass transfer problems, thermal conduction for a thin laminated plate, hydraulic fracturing problems etc. (see [9, 11] and references cited therein).

O.S. Mukhtarov et. al. studied some theoretical aspects of BVP's involving additional transmission conditions at some interval singular points [15–17]. S. Çavuşoğlu et. al. suggested a new generalization of FDM to solve a new type BVP's consisting of two-interval differential equations and boundary-transmission conditions [3, 5, 6].

Note that classical FDM cannot be directly applied to two-interval BVP's with additional transmission conditions. The main goal of this study is to develop a new modification of classical FDM, which can also be applied to two-interval BVP's with singular points and additional transmission conditions.

2 Finite difference method for second-order ordinary differential equations

The idea of the numerical method, called FDM, is based on replacing derivatives in the considered differential equation with finite differences. As a result, the considered differential equation is reduced to a computer-solvable linear algebraic equation system as follows.

Consider a two-order linear differential equation

$$y'' + p(x)y' + q(x)y = f(x), \quad x \in [a, b], \tag{1}$$

subject to the separated boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta, \tag{2}$$

where the functions $p(x)$, $q(x)$ and $f(x)$ are continuous on the interval $[a, b]$, α, β are real numbers. To discretize the boundary value problem (1), (2) the definition range $[a, b]$ is divided into finite number intervals $[x_{j-1}, x_j]$, $j = 1, \dots, N$, where

$$x_j = a + jh, \quad j = 0, 1, \dots, N, \ \ h = \frac{b-a}{N}.$$

Based on Taylor's expansion, we obtain

$$y(x_j + h) = y(x_j) + y'(x_j)h + o\big(h^2\big), \quad j = 0, 1, \dots, N-1. \tag{3}$$

From the last asymptotic equation it follows immediately that if h is sufficiently small, then

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{h} \qquad \text{and} \qquad y'(x_j) \approx \frac{y(x_j) - y(x_{j-1})}{h}.$$

Based on Taylor's expansion, we obtain

$$y(x_j - h) = y(x_j) - y'(x_j)h + o\big(h^2\big), \quad j = 1, \dots, N. \tag{4}$$

From (3) and (4) it follows that if h is sufficiently small, then

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_{j-1})}{2h}, \quad j = 1, \dots, N-1.$$

Definition 1. *The finite differences*

$$D_+y(x) := \frac{y(x+h) - y(x)}{h}, \qquad D_-y(x) := \frac{y(x) - y(x-h)}{h}$$

and

$$D_0y(x) := \frac{y(x+h) - y(x-h)}{2h}$$

are called forward finite difference, backward finite difference and centered finite difference of the function $y(x)$, respectively.

By using the similar technique, the first and second derivatives can be expressed by the relations

$$y'(x_j) \approx \frac{1}{2}\Big(D_+y(x_j) + D_-y(x_j)\Big) = \frac{1}{2h}\Big(y(x_j + h) - y(x_j - h)\Big) \tag{5}$$

and

$$y''(x_j) \approx \frac{1}{h}\Big(D_+y(x_j) - D_-y(x_j)\Big) = \frac{1}{h^2}\Big(y(x_j + 1) - 2y(x_j) + y(x_j - 1)\Big). \tag{6}$$

Substituting these finite differences in the boundary value problem (1)–(2), we get the following finite difference approximation

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + p_j\frac{y_{j+1} - y_{j-1}}{2h} + q_jy_j = f_j, \quad j = 0, 1, \dots, N,$$

where the notations y_j, p_j, q_j, f_j are used for $y(x_j)$, $p(x_j)$, $q(x_j)$ and $f(x_j)$, respectively.

Thus we have the following linear system of algebraic equations

$$\big(-p_jh + 2\big)y_{j-1} + \big(2q_jh^2 - 4\big)y_j + \big(p_jh + 2\big)y_{j+1} = 2h^2f_j, \quad j = 0, 1, 2, \dots, N,$$

with respect to the variables $y_0, y_1, \dots, y_N$. Boundary conditions (2) give $y_0 = \alpha$ and $y_N = \beta$.

Now we can write (1)–(2) in the matrix-equation form

$$AY \approx B, \tag{7}$$

where Y and B are the component vectors, given by

$$Y = \big(y_1, y_2, \dots, y_{N-2}, y_{N-1}\big)^T, \qquad B = \big(b_1, b_2, \dots, b_{N-2}, b_{N-1}\big)^T$$

and $A = (a_{ij})$, $i, j = 0, 1, \dots, N-1$, is the $(N-1) \times (N-1)$ tridiagonal matrix. Here

$$a_{ij} = \begin{cases} 2q_ih^2 - 4, & \text{for } i = j, \\ p_ih + 2, & \text{for } i = j - 1, \\ -p_ih + 2, & \text{for } i = j + 1, \\ 0, & \text{for } |i - j| \geq 2, \end{cases}$$

$$Y = \big(y_1, y_2, \dots, y_{N-2}, y_{N-1}\big)^T,$$

$$b_j = \begin{cases} 2h^2f(x_j) + (p_ih - 2)\alpha, & \text{for } i = 1, \\ 2h^2f(x_j), & \text{for } 1 < i \leq N-2, \\ 2h^2f(x_j) + (p_ih - 2)\beta, & \text{for } i = N-1. \end{cases}$$

Since the linear system (7) of algebraic equations is tridiagonal, it can be solved efficiently by the Crout or Cholesky algoritm [2].

3 Solution of boundary-value transmission problems by using modified finite difference method

Now, consider a boundary value transmission problem, consisting of the two-interval differential equation

$$y'' + p(x)y' + q(x)y = f(x), \quad x \in [a, c) \cup (c, b]\,, \tag{8}$$

separable boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta \tag{9}$$

and additional interaction conditions at the common end $x = c$, the so-called transmission conditions

$$y(c-0) = \theta y(c+0) \tag{10}$$

and

$$y'(c-0) = \gamma y'(c+0), \tag{11}$$

where p, q and f are continuous on each of intervals $[a, c)$ and $(c, b]$ with the finite one-hand limit values $p(c \mp 0)$, $q(c \mp 0)$ and $f(c \mp 0)$, respectively, α, β, θ and γ are real constants.

Let us divide the domain of definition $[a, b]$ into $N+1$ equal subintervals by the grid points $x_j = a + jh$, $j = 0, 1, \ldots, N+1$, where $h = \frac{b-a}{N+1}$ is the mesh width, the distance between successive grid points. Denote by p_j, q_j and f_j the values of the data functions p, q and f at the grid point x_j, respectively, and by y_j the approximation to the solution $y(x)$ at the grid point x_j.

Now we will calculate the grid solution of the boundary-value-transmission problem (8)–(11) consisting of values $y_0, y_1, \ldots, y_N, y_{N+1}$. From the boundary conditions (9) we obtain that $y_0 = \alpha$, $y_{N+1} = \beta$. Therefore, we have N unknown values $y_1, y_2, \ldots, y_N$ to compute.

If we apply the centered finite difference (CFD, for short) approximations to equation (8), we get a set of algebraic equations

$$\frac{1}{h^2}\left(y_{j-1} - 2y_j + y_{j+1}\right) + \frac{1}{2h} p_j \left(y_{j+1} - y_{j-1}\right) + q_j y_j = f_j, \quad j = 1, \ldots, N. \tag{12}$$

Consider the case when $x_j \neq c$ for all $j = 1, \ldots, N$. Then there is an unique $k \in \{0, 1, \ldots, N\}$ such that $x_k < c < x_{k+1}$. At first we apply the CFD approximation to the equation (8) on the left interval $[x_0, x_k] = [a, x_k]$, then we obtain a set of algebraic equations (12) for $j = 1, \ldots, k-1$. Similarly, applying CFD to the same equation on the right interval $[x_{k+1}, x_{N+1}] = [x_{k+1}, b]$, we get a set of algebraic equations (12) for $j = k+2, k+3, \ldots, N-1$. Thus we get $N-2$ linear algebraic equations for N unknown $y_1, y_2, \ldots, y_N$. Taking in view the fact that

$$\begin{aligned} y(c-0) &\approx y(x_k), & y(c+0) &\approx y(x_{k+1}), \\ y'(c-0) &\approx y'(x_k), & y'(c+0) &\approx y'(x_{k+1}) \end{aligned}$$

for sufficiently large N and using finite difference approximation at the points x_k and x_{k+1}, the transmission conditions (9) take the form

$$y_k = \theta y_{k+1}, \qquad \frac{1}{h}\left(y_k - y_{k-1}\right) = \frac{\theta}{h}\left(y_{k+2} - y_{k+1}\right).$$

Consequently, for the unknowns $y_1, y_2, \ldots, y_N$ we have the following N linear algebraic equations

$$\begin{cases} \left(1 - \frac{p_j h}{2}\right) y_{j-1} + \left(-2 + q_j h^2\right) y_j + \left(1 + \frac{p_j h}{2}\right) y_{j+1} = h^2 f_j, & \text{for } j = 1, 2, \ldots, k-1, \\ y_j - \theta y_{j+1} = 0, & \text{for } j = k, \\ -y_{j-1} + y_j - \gamma y_{j+1} - \gamma y_{j+2} = 0, & \text{for } j = k+1, \\ \left(1 - \frac{p_j h}{2}\right) y_{j-1} + \left(-2 + q_j h^2\right) y_j + \left(1 + \frac{p_j h}{2}\right) y_{j+1} = h^2 f_j, & \text{for } j = k+2, \ldots, N. \end{cases}$$

The solution of this linear system of algebraic equations can be found by using MATLAB, Octave or Mathematica. The case $c = x_j$ for some $j = 1, \ldots, N$ can be investigated similarly.

4 Convergence of the method and error estimates of the approximate solution

When the approximate method is applied to solve a differential equation, it is very essential to know how accurate the discrete values of the numerical solution relative to the true solution.

Definition 2. *Let $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ denote the finite difference solution. Let the values of the exact solution at the grid points $x_1, x_2, \ldots, x_n$ be denoted by $\tilde{y} = (y(x_1), y(x_2), \ldots, y(x_n))$. Then the vector*

$$\tilde{E} = \big(y_1 - y(x_1), y_2 - y(x_2), \ldots, y_n - y(x_n)\big) = \tilde{Y} - \tilde{y}$$

is said to be the global error vector.

Our goal is to find an admissible upper bound for this error with respect to the infinite norm (so-called maximum norm), defined by $\|\tilde{E}\|_\infty = \max\limits_{1 \le i \le n} \big|y_i - y(x_i)\big|$.

Denote $h := \max\limits_{1 \le i \le n} (x_{i+1} - x_i)$. If $\|\tilde{E}\|_\infty$ converges to zero as $h \to \infty$, then a finite difference method is called convergent.

Moreover, if there is $c \ge 0$ such that $\|\tilde{E}\|_\infty \le Ch^q$, $q > 0$, we say that the finite difference method is qth order accurate.

Definition 3. *If $\lim\limits_{h \to 0} \|\tilde{E}\|_\infty = 0$, then we say that the FDM is convergent with respect to the norm $\|\tilde{E}\|_\infty$.*

Definition 4. *The vector $\tau = A\tilde{y} - F$ is said to be the local truncation error (LTE, for short), that is $\tau = A(\tilde{y} - \tilde{Y})$, where $\tilde{y}$ and $\tilde{Y}$ are the values of the exact solution and FDM solution at the grid points.*

Remark 1. *Actually, the LTE is defined by replacing finite difference solutions y_j by the exact solution $y(x_i)$ in the finite difference approximation of the original differential equation.*

We will show that the finite difference solution converges to the exact solution of the BVP (1)–(2) as h converges to zero. Using formulas (5) and (6), one can show that the exact solution $\tilde{y} = \big(y(x_1), y(x_2), \ldots, y(x_n)\big)$ satisfies the following linear system of equations

$$\frac{y(x_{j+1}) - 2y(x_j) + y(x_{j-1})}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_j) + p_j \frac{y(x_{j+1}) - y(x_{j-1})}{2h} - \frac{h^2}{6} y^{(3)}(\eta_j) + q_j y(x_j) = f(x_j),$$
$$1 \le j \le n.$$

On the other hand the FDM solution $\tilde{Y} = (y_1, y_2, \ldots, y_n)$ satisfies the following linear system of equations

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + p_j \frac{y_{j+1} - y_{j-1}}{2h} + q_j y_j = f_j, \quad 1 \le j \le n.$$

Subtracting these equations one from the other, we get

$$\frac{e_{j+1} - 2e_j + e_{j-1}}{h^2} + p_j \frac{e_{j+1} - e_{j-1}}{2h} + q_j e_j = h^2 f_j, \quad 1 \le j \le n, \tag{13}$$

where $e_j := y(x_j) - y_j$ is the global error, $h^2 f_j$ is the local truncation error at the grid point x_j and

$$f_j = \frac{1}{12} y^{(4)}(\xi_j) - \frac{1}{6} y^{(3)}(\eta_j).$$

After multiplying both sides of (13) by h^2 and then collecting the corresponding terms, we obtain

$$\left(1 - \frac{h}{2} p_j\right) e_{j-1} + \left(-2 + h^2 q_j\right) e_j + \left(1 + \frac{h}{2} p_j\right) e_{j+1} = h^4 f_j. \tag{14}$$

To estimate the magnitude of the error vector $\tilde{e} = (e_0, e_1, \ldots, e_{N+1})$, it is necessary to use an infinite norm $\|\tilde{e}\|_\infty$, because it is used to measure grid functions.

The equation (14) can be written as

$$\left(2 + h^2 q_j\right) e_i = \left(1 - \frac{h}{2} p_j\right) e_{ij+1} - \left(1 + \frac{h}{2} p_j\right) e_{ij} + h^4 f_j.$$

Consequently

$$\begin{aligned} \left|2 + h^2 q_j\right| |e_j| &\le \left|1 - \frac{h}{2} p_j\right| |e_{j+1}| + \left|1 + \frac{h}{2} p_j\right| |e_j| + h^4 |f_j| \\ &\le \left|1 - \frac{h}{2} p_j\right| \|\tilde{e}\|_\infty + \left|1 + \frac{h}{2} p_j\right| \|\tilde{e}\|_\infty + h^4 \|\tilde{f}\|_\infty, \end{aligned}$$

where $\|\tilde{f}\|_\infty = \max\limits_{1 \le j \le n} |f_j|$.

From the latter inequality it follows immediately that

$$\left|2 + h^2 q_j\right| \|\tilde{e}\|_\infty \le \left(\left|1 - \frac{h}{2} p_j\right| + \left|1 + \frac{h}{2} p_j\right|\right) \|\tilde{e}\|_\infty + h^4 \|\tilde{f}\|_\infty. \tag{15}$$

Since $q(x) < 0$, one can choose $h > 0$ small enough to satisfy

$$\left|1 - \frac{h}{2} p_j\right| + \left|1 + \frac{h}{2} p_j\right| = 2 \quad \text{and} \quad \left|2 + h^2 q_j\right| = 2 + h^2 |q_j|$$

for all $j = 1, 2, \ldots, N$. Consequently, for sufficiently small $h > 0$ we have from (15) that

$$|q_j| \, \|\tilde{e}\|_\infty \le h^2 \|\tilde{f}\|_\infty.$$

Denoting $C = \frac{\|\tilde{f}\|_\infty}{\min\limits_{1 \le j \le n} |q_j|}$, we obtain

$$\|\tilde{e}\|_\infty \le C h^2.$$

Hence, the FDM is convergent and 2-order accurate.

5 Numerical example

Consider the following boundary value transmission problem, consisting of the two-interval differential equation

$$y'' + xy' + \left(\frac{1}{4}x^2 + \frac{1}{2}\right) y = 0, \quad x \in [-2,0) \cup (0,2], \tag{16}$$

subject to the boundary conditions at the endpoints $x = -2$ and $x = 2$, given by

$$y(-2) = 0, \quad y(2) = 3, \tag{17}$$

together with additional transmission conditions across the common endpoint $x = 0$, given by

$$y(0^-) = 3y(0^+), \tag{18}$$

$$y'(0^-) = 2y'(0^+). \tag{19}$$

At first we will consider the problem (16)–(19) without transmission conditions (18)–(19). It is easy to verify that the function

$$y = \frac{3}{4}e^{\frac{1-x^2}{4}}(x+2) \tag{20}$$

satisfies the equation (16) on whole $[-2,0) \cup (0,2]$ and both boundary conditions (17). For simplicity we will use the uniform cartesian grid $x_i = -2 + ih$, $i = 0,1,\ldots,50$, for $h = 0{,}08$. In particular we have $x_0 = -2$, $x_{50} = 2$.

The CFD approximation of the derivatives y' and y'' are defined by

$$y'(x) \approx \frac{1}{2}\big(D_+y(x) + D_-y(x)\big) \qquad \text{and} \qquad y''(x) \approx \frac{1}{h}\big(D_+y(x) - D_-y(x)\big),$$

where $D_+y(x)$ and $D_-y(x)$ denote the forward finite difference and backward finite difference of $y(x)$, respectively.

By applying the CFD to the differential equation (16) at a typical grid point x_i and denoting $y_i = y(x_i)$, we obtain the following finite difference equations

$$(4 - 2hx_i)\, y_{i-1} + \big(-8 + \big(x_i^2 + 2\big)h^2\big)y_i + (4 + 2hx_i)\, y_{i+1} = 0, \quad i = 1,2,\ldots,49. \tag{21}$$

That is we have the linear algebraic system of equations with respect to the variables $y_1, y_2, \ldots, y_{49}$. The system of linear algebraic equations (21) can be written in a tridiagonal matrix-vector form $Ay = b$, where $y = (y_1, y_2, \ldots, y_{48}, y_{49})^T$, $b = (0, 0, \ldots, 0, -3(4 + 2hx_{49}))^T$ and

$$A = \begin{pmatrix} -8 + \left(x_1^2 + 2\right) h^2 & 4 + 2hx_1 & \cdots & 0 & 0 \\ 4 + 2hx_2 & -8 + \left(x_2^2 + 2\right) h^2 & \cdots & 0 & 0 \\ 0 & 4 + 2hx_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -8 + \left(x_{48}^2 + 2\right) h^2 & 4 + 2hx_{48} \\ 0 & 0 & \cdots & 4 + 2hx_{49} & -8 + \left(x_{49}^2 + 2\right) h^2 \end{pmatrix}.$$

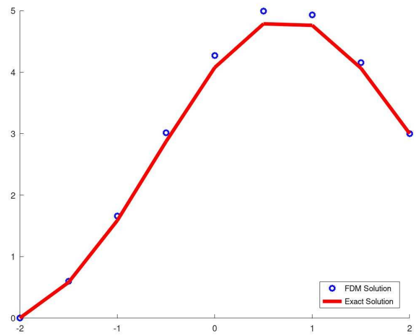

Figure 1. Graph of the FDM solution and exact solution for the problem (16)–(17) for $N = 8$

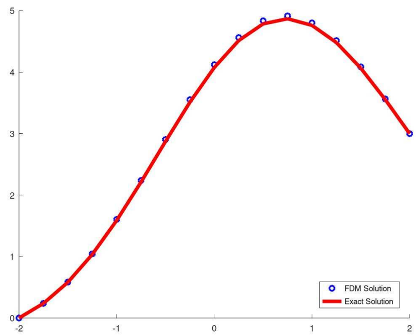

Figure 2. Graph of the FDM solution and exact solution for the problem (16)–(17) for $N = 16$

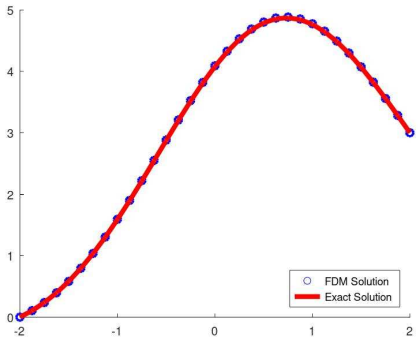

Figure 3. Graph of the FDM solution and exact solution for the problem (16)–(17) for $N = 32$

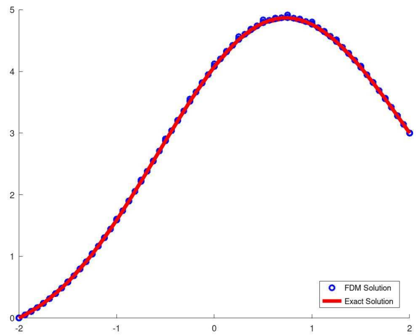

Figure 4. Graph of the FDM solution and exact solution for the problem (16)–(17) for $N = 64$

The solution of this system can be found by using MATLAB/Octave. The obtained numerical finite difference method solutions are graphically compared with the exact solution (20) (see Figures 1, 2, 3 and 4).

N	h	$\|\|E\|\|_\infty$	N	h	$\|\|E\|\|_\infty$
4	$\frac{3}{4}$	1.0654	128	$\frac{3}{128}$	0.00075157
8	$\frac{3}{8}$	0.20712	256	$\frac{3}{256}$	0.00018786
16	$\frac{3}{16}$	0.048747	512	$\frac{3}{512}$	0.000046963
32	$\frac{3}{32}$	0.012079	1024	$\frac{3}{1024}$	0.000011741
64	$\frac{3}{64}$	0.0030087	2048	$\frac{3}{2048}$	0.0000029352

Table 1. Maximum absolute error

Remark 2. *It can be seen from these graphical illustrations that the error between the finite difference method solutions and the exact solution decreases as the number of grid points increases.*

6 Solution of transmission problem by modification of FDM

Now we will investigate the boundary value transmission problem (16)–(19). It is easy to show that the exact solution of this problem is

$$Y = \begin{cases} \frac{9}{5}e^{1-\frac{x^2}{4}}\left(2+x\right), \\ \frac{3}{10}e^{1-\frac{x^2}{4}}\left(4+3x\right). \end{cases}$$

Now let us solve the boundary value transition problem (16)–(19) with the modified finite difference method. If we select $N = 32$, then we can discretize the first transmission condition by

$$y_{16} = 3y_{17}.$$

Discretization of the second transmission conditions (18) gives

$$\frac{y_{16} - y_{14}}{h^2} = 2\frac{y_{19} - y_{17}}{h^2}.$$

Thus we have two additional algebraic equations

$$y_{16} - 3y_{17} = 0, \tag{22}$$

$$y_{14} - y_{16} - 2y_{17} + 2y_{19} = 0. \tag{23}$$

Note that each equation of this system involves solution values at three nodal points x_{i-1}, x_i and x_{i+1}.

By adding equations (22) and (23) to the system of equations (21) writing for $1 \le i \le 14$ and for $17 \le i \le 31$ we have the following linear system of algebraic equations

$$\begin{cases} y_0 = 0, \\ \left(1 - \frac{1}{2}hp_i\right)y_{i-1} + (-2 + h^2q_i)y_i + \left(1 + \frac{1}{2}hp_i\right)y_{i+1} = h^2 f(x_i), \quad 1 \le i \le 14, \\ y_{16} - 3y_{17} = 0, \\ y_{14} - y_{16} - 2y_{17} + 2y_{19} = 0, \\ \left(1 - \frac{1}{2}hp_i\right)y_{i-1} + (-2 + h^2q_i)y_i + \left(1 + \frac{1}{2}hp_i\right)y_{i+1} = h^2 f(x_i), \quad 17 \le i \le 31, \\ y_{32} = 3. \end{cases}$$

The solution of this system of equations can be found by using MATLAB/Octave. The obtained numerical solutions are graphically compared with the exact solution (see Figures 5, 6, 7 and 8).

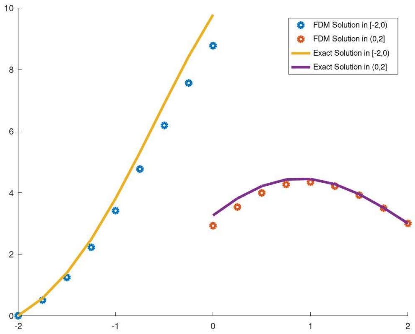

Figure 5. Graph of the numerical solution and exact solution for the problem (16)–(19) for $N = 16$

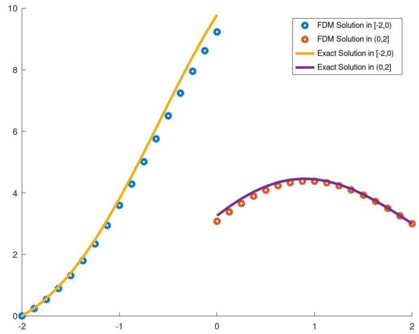

Figure 6. Graph of the numerical solution and exact solution for the problem (16)–(19) for $N = 32$

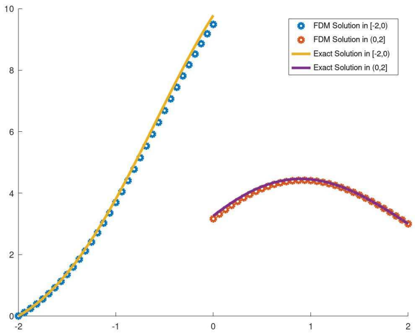

Figure 7. Graph of the numerical solution and exact solution for the problem (16)–(19) for $N = 64$

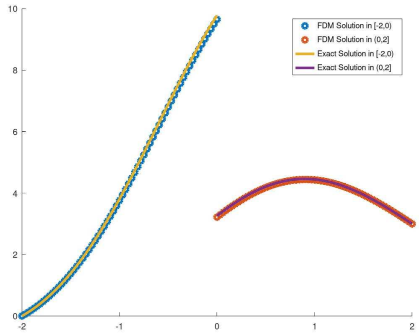

Figure 8. Graph of the numerical solution and exact solution for the problem (16)–(19) for $N = 128$

N	h	$\|E\|_\infty$	N	h	$\|E\|_\infty$
4	$\frac{3}{4}$	2.5858	32	$\frac{3}{32}$	0.55334
8	$\frac{3}{8}$	1.7106	64	$\frac{3}{64}$	0.29347
16	$\frac{3}{16}$	1.0080	128	$\frac{3}{128}$	0.12946143

Table 2. Maximum absolute error for the transmission problem (16)–(19)

7 Conclusion

We know that the classical finite difference method is intended for solving one-interval differential equations, and it is not clear how to apply this technique to two-interval differential equations that satisfy additional conditions of interaction between these two-intervals, the so-called transmission conditions. In this study, we have adapted this method to two-interval differential equations satisfying not only boundary value conditions, but also additional transmission conditions. The modified FDM was applied to one illustrative two-interval boundary-transmission problem. To demonstrate the applicability and efficiency of the proposed modification of FDM the obtained finite-difference solution is compared graphically with the exact solution, and the corresponding error analysis is also presented in tables. The obtained results show the applicability, efficiency and reliability of the proposed modified FDM, developed for the first time in this study.

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Чавушоглу С., Мухтаров Ф.С., Мухтаров О.Ш. *Нова модифікація методу скінченних різниць для розв'язання задач передачі для диференціальних рівнянь на двох інтервалах* // Карпатські матем. публ. — 2024. — Т.16, №2. — C. 575–586.

У цій статті ми розглядаємо новий тип крайових задач, основною особливістю яких є характер диференціального рівняння, що розв'язується, та накладені крайові умови. Зокрема, диференціальне рівняння розглядається на двох неперетинних відрізках із спільною точкою, на яку накладаються додаткові умови взаємодії, так звані умови передачі.

Як відомо, класичні аналітичні та чисельні методи призначені для розв'язування диференціальних рівнянь на одному відрізку без умов передачі. Основною метою цієї роботи є розробка нової модифікації класичного методу скінченних різниць для розв'язування крайових задач передачі на двох відрізках.

Ключові слова і фрази: метод скінченних різниць, сингулярна точка, умова передачі, крайова задача на двох інтервалах.