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Certain solitons on α-cosymplectic manifolds

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In the present paper, we characterize some solitons such as η-Einstein soliton, η-Yamabe soliton and Ricci-Yamabe soliton on α-cosymplectic manifolds. Furthermore, we characterize 3-dimensional α-cosymplectic manifolds with gradient Ricci-Yamabe soliton. Finally, we construct an example.

Key words and phrases: Einstein soliton, η-Einstein soliton, η-Ricci soliton, gradient η-Ricci soliton, η-Yamabe soliton, α-cosymplectic manifold.

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Introduction

A Ricci soliton in a Riemannian manifold (M, g) is given by

$$\pounds_X g + 2S + 2\lambda g = 0,$$

where $\pounds$ is the Lie-derivative, λ is a constant, S is the Ricci tensor and the vector field X is called the potential vector field. Ricci solitons are the self similar solutions of the Ricci flow equation

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

which was introduced by R.S. Hamilton [15]. Ricci solitons have also been studied by various authors such as [25–27] and many others.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced in the paper [11]. An η-Ricci soliton is given by

$$\pounds_X g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0,$$

where μ is a constant. If $\mu = 0$, then η-Ricci soliton reduces to Ricci soliton and if $\mu \neq 0$, then the η-Ricci soliton is called proper. η-Ricci solitons have been studied by various authors such as [3–6, 12, 17, 19, 22, 24] and many others. Like a Ricci soliton, an η-Ricci soliton is also called shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. Recently, the second named author studied Ricci solitons on generalized Sasakian space forms [23].

Let (M, g) be a Riemannian manifold of dimension $2n + 1$. Then M is called an Einstein soliton [9] if there is a vector field X such that

$$\pounds_X g + 2S + (2\lambda - r)g = 0,$$

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where r is the scalar curvature of the Riemannian metric g. If the scalar curvature r is constant, then the Einstein soliton becomes a Ricci soliton.

As a generalization of Einstein soliton, the η-Einstein soliton is given by

$$\pounds_X g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0. \tag{1}$$

If the scalar curvature r is constant, then the η-Einstein soliton reduces to an η-Ricci soliton.

A Yamabe soliton [2] on a Riemannian or pseudo-Riemannian manifold (M, g) is defined by

$$\frac{1}{2}\pounds_X g = (r - \lambda)g.$$

A Yamabe soliton is said to be expanding, steady or shrinking if $\lambda < 0, \lambda = 0$ or $\lambda > 0$, respectively.

We define the notion of η-Yamabe soliton as

$$\frac{1}{2}\pounds_X g = (r - \lambda)g - \mu\eta \otimes \eta, \tag{2}$$

where λ and μ are constants. If $\mu = 0$, then the η-Yamabe soliton becomes a Yamabe soliton.

A Ricci-Yamabe soliton on Riemannian manifold (M, g) is the structure (g, X, λ, a, b), defined [14] by

$$\pounds_V g + 2aS + (2\lambda - br)g = 0, \tag{3}$$

where $\pounds$ is the Lie-derivative, S is the Ricci tensor, r is the scalar curvature and $\lambda, a, b \in \mathbb{R}$. If X is gradient of a smooth function f on M, then above notion is called gradient Ricci-Yamabe soliton and equation (3) reduces to

$$\nabla^2 f + aS = \left(\lambda - \frac{1}{2}br\right) g, \tag{4}$$

where $\nabla^2 f$ is the Hessian of f.

The Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be expanding, steady or shrinking according as λ is negative, zero or positive, respectively. A Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is called an almost Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) if a, b and λ are smooth functions on M. A Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be a

- Ricci soliton (or gradient Ricci soliton) if $a = 1, b = 0$ (see [15]);
- Yamabe soliton (or gradient Yamabe soliton) if $a = 0, b = 1$ (see [16]);
- Einstein soliton (or gradient Einstein soliton) if $a = 1, b = -1$ (see [9]);
- ρ-Einstein soliton (or gradient ρ-Einstein soliton) if $a = 1, b = -2\rho$ (see [10]).

When $a \neq 0, 1$, Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is proper.

We organize the paper as follows. In Section 2, we consider some solitons on α-cosymplectic manifolds. Next, in Section 3 we study gradient Ricci-Yamabe solitons on α-cosymplectic manifolds. Finally, in Section 4 we construct an example of a 5-dimensional α-cosymplectic manifold to verify our results.

1 Preliminaries

A $(2n+1)$-dimensional Riemannian manifold (M, g) is called an almost contact metric manifold if it admits a $(1,1)$-type tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g such that [8]

$$\phi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1,$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \quad g(\phi U, V) = -g(U, \phi V)$$

for any vector fields U, V of the manifold. The vector field ξ is called the Reeb or characteristic vector field.

An almost contact metric manifold is said to be normal if the Nijenhuis tensor of ϕ [8] vanishes. An almost Kenmotsu manifold is an almost contact metric manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, $\Phi(U, V) = g(U, \phi V)$.

A normal contact metric manifold (M, g) is said to be cosymplectic if the following relations

$$d\eta = 0, \quad d\Phi = 0$$

hold. Similarly,

$$(\nabla_U \phi)V = 0, \quad \nabla_U \xi = 0$$

for any U, V and ∇ is the Levi-Civita connection on M. Also, if

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi$$

are satisfied, then M is called an α-cosymplectic manifold [1, 20], α is a real number. Equivalently,

$$(\nabla_U \phi)V = \alpha\left[g(\phi U, V)\xi - \eta(V)\phi U\right],$$
$$\nabla_U \xi = \alpha\left[U - \eta(U)\xi\right]. \tag{5}$$

In α-cosymplectic manifolds, it is well known [13, 21] that

$$R(U, V)\xi = \alpha^2\left[\eta(U)V - \eta(V)U\right], \quad R(U, \xi)V = \alpha^2\left[g(U, V)\xi - \eta(V)U\right],$$
$$R(U, \xi)\xi = \alpha^2\left[\eta(U)\xi - U\right], \quad S(U, \xi) = -2n\alpha^2\eta(U),$$

where S is the Ricci tensor.

Proposition 1. *In a 3-dimensional α-cosymplectic manifold the Ricci tensor is defined [7] by*

$$S(U, V) = \left(\alpha^2 + \frac{r}{2}\right) g(U, V) - \left(3\alpha^2 + \frac{r}{2}\right) \eta(U)\eta(V). \tag{6}$$

Proposition 2. *In a 3-dimensional α-cosymplectic manifold the following relation*

$$\xi r = -2\alpha\left(6\alpha^2 + r\right) \tag{7}$$

holds.

Proof. Equation (6) implies

$$QU = \left(\alpha^2 + \frac{r}{2}\right) U - \left(3\alpha^2 + \frac{r}{2}\right) \eta(U)\xi.$$

Differentiating (7), after some calculations we obtain

$$(\nabla_V Q)\, U = \frac{1}{2}\left[(Vr)U - (Vr)\eta(U)\xi\right] - \alpha\left(3\alpha^2 + \frac{r}{2}\right)\left[g(U, V)\xi + \eta(U)V - 2\eta(U)\eta(V)\xi\right]. \tag{8}$$

Contracting V in the foregoing equation we have $\xi r = -2\alpha(6\alpha^2 + r)$. This completes the proof. □

2 Solitons on α-cosymplectic manifolds

Let us suppose that the α-cosymplectic manifold admits an η-Einstein soliton (g, ξ). Then (1) implies

$$\left(\pounds_{\xi} g\right)(U, V)+2 S(U, V)+(2 \lambda-r) g(U, V)+\mu \eta(U) \eta(V)=0. \tag{9}$$

Now,

$$\left(\pounds_{\xi} g\right)(U, V)=g\left(\nabla_{U} \xi, V\right)+g\left(U, \nabla_{V} \xi\right).$$

Using (5) in the above equation, we get

$$\left(\pounds_{\xi} g\right)(U, V)=2 \alpha[g(U, V)-\eta(U) \eta(V)]. \tag{10}$$

Using (10) in (9), we obtain

$$S(U, V)=\left(\frac{r}{2}-\alpha-\lambda\right) g(U, V)+(\alpha-\mu) \eta(U) \eta(V). \tag{11}$$

Contracting the above equation, we infer

$$r=\frac{2[2 n \alpha+(2 n+1) \lambda+\mu]}{2 n-1}, \tag{12}$$

which implies r is a constant. Then from (9), it reduces to an η-Ricci soliton. Hence we have the following result.

Theorem 1. *If an α-cosymplectic manifold admits an η-Einstein soliton (g, ξ), then it is an η-Einstein manifold and hence it reduces to an η-Ricci soliton.*

If we take $\alpha=0$ or 1, then from (12) we obtain r is a constant. Thus we get the next assertion.

Corollary 1. *If a cosymplectic or Kenmotsu manifold admits an η-Einstein soliton (g, ξ), then it is an η-Einstein manifold and hence it reduces to an η-Ricci soliton.*

If an α-cosymplectic manifold admits an η-Yamabe soliton (g, ξ), then (2) implies

$$\frac{1}{2}\left(\pounds_{\xi} g\right)(U, V)=(r-\lambda) g(U, V)-\mu \eta(U) \eta(V). \tag{13}$$

Using (10) in (13) we infer

$$\alpha[g(U, V)-\eta(U) \eta(V)]=(r-\lambda) g(U, V)-\mu \eta(U) \eta(V).$$

Contracting the above equation, we get

$$r=\lambda+\frac{2 n \alpha+\mu}{2 n+1},$$

which is a constant. Hence we conclude the following result.

Theorem 2. *If an α-cosymplectic manifold admits an η-Yamabe soliton, then the scalar curvature is constant.*

If we take $\alpha=0$ or 1, then results are similar to the above theorem. Hence we have the next assertion.

Corollary 2. *If a cosymplectic or Kenmotsu manifold admits an η-Yamabe soliton, then the scalar curvature is constant.*

If an α-cosymplectic manifold admits a proper Ricci-Yamabe soliton (g, ξ, λ, a, b), then equation (3) implies

$$(\pounds_\xi g)(U, V) + 2aS(U, V) + (2\lambda - br)g(U, V) = 0.$$

Using (10) in the above equation, we get

$$S(U, V) = \frac{1}{a}\left[\left(\frac{b}{2}r - \lambda - \alpha\right)g(U, V) + \alpha\eta(U)\eta(V)\right]. \tag{14}$$

Hence we conclude the following result.

Theorem 3. *A proper Ricci-Yamabe soliton on an α-cosymplectic manifold is an η-Einstein manifold.*

If we take $\alpha = 0$, then (14) implies

$$S(U, V) = \frac{1}{a}\left(\frac{b}{2}r - \lambda\right)g(U, V).$$

Contracting the above equation, we get

$$\left[b(2n + 1) - 2a\right]r = 2\lambda(2n + 1),$$

which implies the scalar curvature is constant. Hence we have the next assertion.

Corollary 3. *If a cosymplectic manifold admits a proper Ricci-Yamabe soliton, then its scalar curvature is constant.*

Again, if $\alpha = 1$, then (14) implies

$$S(U, V) = \frac{1}{a}\left[\left(\frac{b}{2}r - \lambda - 1\right)g(U, V) + \eta(U)\eta(V)\right].$$

Thus we have the next result.

Corollary 4. *A proper Ricci-Yamabe soliton on a Kenmotsu manifold is an η-Einstein manifold.*

3 Gradient Ricci-Yamabe solitons on 3-dimensional α-cosymplectic manifolds

We assume that an α-cosymplectic manifold admits a gradient Ricci-Yamabe soliton (g, λ, ξ, a, b). Then from (4), we get

$$\nabla_U Df = \left(\lambda - \frac{b}{2}r\right)U - aQU. \tag{15}$$

Differentiating (15) along the vector field V, we get

$$\nabla_V \nabla_U Df = -\frac{b}{2}(Vr)U + \left(\lambda - \frac{b}{2}r\right)\nabla_V U - a\nabla_V QU. \tag{16}$$

Interchanging U and V in the above equation, we infer

$$\nabla_U \nabla_V Df = -\frac{b}{2}(Ur)V + \left(\lambda - \frac{b}{2}r\right) \nabla_U V - a\nabla_U QV. \tag{17}$$

From (15), we get

$$\nabla_{[U,V]} Df = \left(\lambda - \frac{b}{2}r\right) [U,V] - aQ\big([U,V]\big). \tag{18}$$

With the help of (16) – (18), we obtain

$$R(U,V)Df = \frac{b}{2}\big[(Vr)U - (Ur)V\big] - a\left[(\nabla_U Q)\, V - (\nabla_V Q)\, U\right].$$

Using (8) in the above equation, we get

$$\begin{aligned} R(U,V)Df =& \frac{1}{2}(b-a)\big[(Vr)U - (Ur)V\big] \\ &+ a\alpha\left(3\alpha^2 + \frac{r}{2}\right)\big[\eta(V)U - \eta(U)V\big] \\ &+ \frac{a}{2}\big[(Ur)\eta(V)\xi - (Vr)\eta(U)\xi\big]. \end{aligned} \tag{19}$$

Contracting (19), we obtain

$$S(V, Df) = \left(b - \frac{3a}{2}\right)(Vr) + a\alpha\left(6\alpha^2 + r\right)\eta(V) + \frac{a}{2}(\xi r)\eta(V). \tag{20}$$

Using (7) in (20), gives

$$S(V, Df) = \left(b - \frac{3a}{2}\right)(Vr). \tag{21}$$

Replacing U by Df in (6) and comparing with (21), we get

$$\left(\alpha^2 + \frac{r}{2}\right)(Vf) - \left(3\alpha^2 + \frac{r}{2}\right)(\xi f)\eta(V) = \frac{1}{2}(2b - 3a)(Vr). \tag{22}$$

Putting $V = \xi$ in (22) and using (7), we infer that

$$\alpha(\xi f) = \frac{1}{2}(2b - 3a)\left(6\alpha^2 + r\right). \tag{23}$$

Using (23) in (22), we get

$$\alpha\left(\alpha^2 + \frac{r}{2}\right)(Vf) - \frac{1}{2}(2b - 3a)\left[\left(3\alpha^2 + \frac{r}{2}\right)\left(6\alpha^2 + r\right)\eta(V) + \alpha(Vr)\right] = 0.$$

If we take $2b - 3a = 0$. Then the above equation implies

$$\alpha\left(\alpha^2 + \frac{r}{2}\right)(Vf) = 0.$$

The above equation implies either $\alpha = 0$ or $r = -2\alpha^2$ or $Vf = 0$.

Case I. If $\alpha = 0$, then it becomes a cosymplectic manifold.

Case II. If $r = -2\alpha^2$, then scalar curvature is a constant.

Case III. If $Vf = 0$, then f is a constant.

Thus we conclude the following result.

Theorem 4. *If a 3-dimensional α-cosymplectic manifold admits a gradient Ricci-Yamabe soliton, then either it is a cosymplectic manifold or the scalar curvature is constant or the potential function f is constant, provided $2b - 3a = 0$.*

4 Example

Consider the 5-dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z) \in \mathbb{R}^5\}$, where (x_1, x_2, y_1, y_2, z) are the standard coordinates in $\mathbb{R}^5$. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = e^{\alpha z}\frac{\partial}{\partial x_1}, \quad e_2 = e^{\alpha z}\frac{\partial}{\partial x_2}, \quad e_3 = e^{\alpha z}\frac{\partial}{\partial y_1}, \quad e_4 = e^{\alpha z}\frac{\partial}{\partial y_2}, \quad e_5 = -\frac{\partial}{\partial z} = \xi.$$

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 0, & i \neq j, \\ 1, & i = j, \end{cases} \qquad i, j = 1, 2, 3, 4, 5.$$

Let η be the 1-form on M defined by $\eta(U) = g(U, e_5) = g(U, \xi)$ for all $U \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field on M defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = e_1, \quad \phi e_3 = -e_4, \quad \phi e_4 = e_3, \quad \phi e_5 = 0.$$

By applying the linearity of ϕ and g, we have

$$\eta(\xi) = 1, \quad \phi^2 U = -U + \eta(U)\xi, \quad \eta(\phi U) = 0,$$

$$g(U, \xi) = \eta(U), \quad g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for all $U, V \in \chi(M)$. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = \alpha e_1, \quad [e_2, e_5] = \alpha e_2, \quad [e_3, e_5] = \alpha e_3, \quad [e_4, e_5] = \alpha e_4.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{array}{lllll}
\nabla_{e_1} e_1 = -\alpha e_5, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_3 = 0, & \nabla_{e_1} e_4 = 0, & \nabla_{e_1} e_5 = \alpha e_1, \\
\nabla_{e_2} e_1 = 0, & \nabla_{e_2} e_2 = -\alpha e_5, & \nabla_{e_2} e_3 = 0, & \nabla_{e_2} e_4 = 0, & \nabla_{e_2} e_5 = \alpha e_2, \\
\nabla_{e_3} e_1 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_3 = -\alpha e_5, & \nabla_{e_3} e_4 = 0, & \nabla_{e_3} e_5 = \alpha e_3, \\
\nabla_{e_4} e_1 = 0, & \nabla_{e_4} e_2 = 0, & \nabla_{e_4} e_3 = 0, & \nabla_{e_4} e_4 = -\alpha e_5, & \nabla_{e_4} e_5 = \alpha e_4, \\
\nabla_{e_5} e_1 = 0, & \nabla_{e_5} e_2 = 0, & \nabla_{e_5} e_3 = 0, & \nabla_{e_5} e_4 = 0, & \nabla_{e_5} e_5 = 0.
\end{array}$$

It can be easily verified that the manifold satisfies

$$\nabla_U \xi = \alpha[U - \eta(U)\xi] \quad \text{and} \quad (\nabla_U \phi) V = \alpha\left[g(\phi U, V)\xi - \eta(V)\phi U\right]$$

for $\xi = e_5$. Hence the manifold is an α-cosymplectic manifold.

In [18] the authors obtained the expression of the curvature tensor and the Ricci tensor as follows

$$\begin{aligned}
&R(e_1, e_2) e_2 = R(e_1, e_3) e_3 = R(e_1, e_4) e_4 = R(e_1, e_5) e_5 = -\alpha^2 e_1, \\
&R(e_1, e_2) e_1 = \alpha^2 e_2, \\
&R(e_1, e_3) e_1 = R(e_2, e_3) e_2 = R(e_5, e_2) e_5 = \alpha^2 e_3, \\
&R(e_2, e_3) e_3 = R(e_2, e_4) e_4 = R(e_2, e_5) e_5 = -\alpha^2 e_2, \\
&R(e_3, e_4) e_4 = -\alpha^2 e_3, \\
&R(e_1, e_5) e_2 = R(e_1, e_5) e_1 = R(e_4, e_5) e_4 = R(e_3, e_5) e_3 = \alpha^2 e_5, \\
&R(e_1, e_4) e_1 = R(e_2, e_4) e_2 = R(e_3, e_4) e_3 = R(e_5, e_4) e_5 = \alpha^2 e_4,
\end{aligned}$$

and

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4\alpha^2.$$

Therefore,

$$r = S(e_1,e_1) + S(e_2,e_2) + S(e_3,e_3) + S(e_4,e_4) + S(e_5,e_5) = -20\alpha^2,$$

which is a constant. From (11), we obtain

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = \frac{r}{2} - \alpha - \lambda \quad \text{and} \quad S(e_5,e_5) = \frac{r}{2} - \lambda - \mu,$$

which implies $\lambda = -6\alpha^2 - \alpha$ and $\mu = \alpha$. Therefore the data (g, ξ, λ, μ) for $\lambda = -6\alpha^2 - \alpha$ and $\mu = \alpha$ defines an η-Einstein soliton on 5-dimensional α-cosymplectic manifold and the scalar curvature is constant.

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У цій статті ми характеризуємо деякі солітони, такі як η-солітон Айнштайна, η-солітон Ямабе та солітон Річчі-Ямабе на α-косимплектичних многовидах. Крім того, ми характеризуємо 3-вимірні α-косимплектичні многовиди з ґрадієнтним солітоном Річчі-Ямабе. Насамкінець, ми будуємо приклад.

Ключові слова і фрази: солітон Айнштайна, η-солітон Айнштайна, η-солітон Річчі, ґрадієнтний η-солітон Річчі, η-солітон Ямабе, α-косимплектичний многовид.