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Stability of a fractional heat equation with memory

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Of concern is a fractional differential problem of order between zero and one. The model generalizes an existing well-known problem in heat conduction theory with memory. First, we justify the replacement of the first order derivative by a fractional one. Then, we establish a Mittag-Leffler stability result for a class of heat flux relaxation functions. We will combine the energy method with some properties from fractional calculus.

Key words and phrases: Caputo fractional derivative, heat-conduction, memory term, Mittag-Leffler stability, multiplier technique.

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Introduction

The problem we want to address here is the following:

$$\begin{cases} {}^{C}D^{\alpha}\theta(t,x) = \Delta\theta(t,x) - b\displaystyle\int_0^t \phi(t-s)\Delta\theta(s,x)\,ds, & t>0,\ x\in\Omega, \\ \theta(t,x)=0, & t>0,\ x\in\partial\Omega, \\ \theta(0,x)=\theta_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where $0<\alpha<1$, ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order α defined below, the kernel ϕ in the memory term is a nonnegative function, $\theta_0(x)$ is the initial data of the state and Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and b is a positive constant.

This model is the fractional version (and multi-dimensional case) of a problem which appears in the theory of rigid heat-conductors with memory corresponding to the case $\alpha=1$ (and $n=1$)

$$\begin{cases} \theta'(t,x) = b_0\theta_{xx}(t,x) - b\displaystyle\int_0^t \phi(t-s)\theta_{xx}(s,x)\,ds, & t>0,\ x\in(0,1), \\ \theta(t,0)=0,\ \theta(t,1)=0, & t>0, \\ \theta(0,x)=\theta_0(x), & x\in(0,1). \end{cases} \tag{2}$$

Model (2) describes the temperature distribution in a homogeneous bar of unit length while the ends of the bar are kept at zero temperature. The bar is made of a material with memory and therefore the diffusion is not a “normal” one. According to the theory developed in

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[11, 26, 30] this model is derived by assuming that the temperature $\theta(t,x)$, the internal energy $\epsilon(t,x)$ and the heat flux $q(t,x)$ satisfy the relations

$$\epsilon(t,x) = \theta(t,x), \qquad q(t,x) = -b_0\theta_x(t,x) + b\int_0^t \phi(t-s)\theta_x(s,x)\,ds,$$

where ϕ is heat flux relaxation function. When augmented by the balance of heat equation

$$\epsilon_t = -q_x + \sigma,$$

where σ denotes possibly the external heat supply, we obtain

$$\frac{\partial}{\partial t}\theta(t,x) = b_0\theta_{xx}(t,x) - b\int_0^t \phi(t-s)\theta_{xx}(s,x)\,ds + \sigma(t,x).$$

An important class of heat flux relaxation functions ϕ in applications is finite linear combinations of decaying exponentials (see [9, 10])

$$\phi(t) = \sum_{k=1}^{m} d_k e^{-\lambda_k t}, \quad d_k, \lambda_k > 0,\ k = 1,\dots,m,$$

or even just one of them ($m = 1$). These functions are in line with the concept of fading memory. They have been extended to the class of completely monotone functions and other classes. This problem has been studied extensively since the sixties (see [8–13, 15, 23–26, 28–30, 33] and references therein). The well-posedness (see [4] and references therein), asymptotic behavior and stability issues have been discussed in the abstract setting

$$\vartheta'(t) = \mathcal{A}\vartheta(t) + \int_0^t \Phi(t-s)\vartheta(s)\,ds$$

and also generalized to the multi-dimensional case. Here $\mathcal{A} : D(A) \subset X \to X$ is a linear operator satisfying certain conditions ensuring that it generates an analytic semigroup $e^{t\mathcal{A}}$ in X. The operator kernel $\Phi(\cdot)$ belongs to $L\big(0,+\infty; L(D(\mathcal{A}),X)\big)$ with a certain condition on its Laplace transform $\widehat{\Phi}(\cdot)$. A forcing term $h(t)$ can also be added. In [12,13,24,25], it is proved that there exists a resolvent operator $R(t)$, $t \geq 0$, such that $t \to R(t)$ is analytic in $(0,+\infty)$ with values in $L\big(X, D(\mathcal{A})\big)$ and

$$R'(t) = \mathcal{A}R(t) + \int_0^t \Phi(t-s)R(s)\,ds, \quad t > 0,$$

$$\lim_{t\to 0} R(t)x = x \quad \text{for all } \ x \in \overline{D(\mathcal{A})}.$$

The resolvent operator $R(t)$ is given by

$$R(t) = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{\lambda t}\big(\lambda - \mathcal{A} - \widehat{\Phi}(\lambda)\big)^{-1} d\lambda, \quad t > 0, \qquad R(0) = 1,$$

where $\mathcal{C}$ is an appropriate curve ensuring the existence of $\big(\lambda - \mathcal{A} - \widehat{\Phi}(\lambda)\big)^{-1}$ for $\lambda \in \mathcal{C}$.

When studying transport of fluids in a porous media, the above model come to light through the use of Darcy's law. However, it has been observed that the movement of the fluid may cause reduction of the size of the pores thus affecting the permeability of the media. In [6, 7], the author modified Darcy's law by introducing a memory term. This resulted in a fractional derivative, which is able to better describe the decrease in the permeability. The above considerations have motivated the study of problem (1). We also refer the reader to [3,18,27].

In general, for complex media it has been observed that the mean square displacement is not linear. It is proportional to t^α, $\alpha > 0$. The case $0 < \alpha < 1$ corresponds to the subdiffusion case, where particles move slower than in the normal case. An interesting derivation of the model (for a special kernel) from the Rayleigh-Stokes problem, which describes the flow of some non-Newtonian fluids

$$\begin{cases} \partial_t w - \left(1 + \gamma \partial_t^\beta\right) \Delta w = g \in \Omega, \quad 0 < t \leq T \\ w = 0 \text{ on } \partial\Omega, \quad 0 < t \leq T \\ w(\cdot, 0) = h \text{ in } \Omega, \end{cases} \tag{3}$$

can be found in [19], see also [16,32].

The well-posedness in suitable spaces, for more general and abstract problems, has been already treated in some papers [5,14,19–22]. The results will be discussed briefly below. We shall rather investigate the asymptotic behavior of solutions for a class of kernels. We recall that, in the integer case ($\alpha = 1$), kernels of the form $e^{-\gamma t}$, $\gamma > 0$, (or a finite sum of such functions) have been first considered and then generalized to the class of functions satisfying the differential inequality

$$\phi'(t) \leq -\gamma \phi(t), \quad t > 0, \; \gamma > 0.$$

This motivated us to look at kernels satisfying the fractional differential inequality

$$^{RL}D^\alpha \phi(t) \leq -\gamma \phi(t), \quad t > 0, \; \gamma > 0. \tag{4}$$

The cases of regular and singular kernels will be treated separately. We prove Mittag-Leffler stability of solutions. We mention here that some power-type decays of solutions in some Sobolev spaces have been shown in [5]. Namely, solutions of (3) behave like $1/t$ at infinity in a certain Sobolev space. In [22], the author proved an asymptotic stability result for locally Lipschitz nonlinearities $g(w)$ and a stability result of order $t^{\alpha-1}$, when $g(w)$ satisfies a condition of the form

$$\limsup_{\|u\| \to 0} \frac{\|g(u)\|}{\|u\|} = a \in [0, \lambda].$$

The rest of the paper is organized as follows. After some preliminaries presented in the next section, we present the available results on existence and uniqueness of different kinds of solutions in Section 2. Section 3 is devoted to the stability in case of regular kernels satisfying (4). A singular special case is discussed in the last section under a regularity assumption of solutions.

1 Preliminaries

In this section, we prepare some material which we shall use to prove our results in the next section. Namely, few definitions and propositions are in order. For material on fractional calculus we refer, for instance, to [17,31].

Definition 1. *The Riemann-Liouville fractional integral of order* $\alpha > 0$ *of a function* w *is defined by*

$$I^{\alpha}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(s)\,ds, \quad \alpha > 0,$$

where $\Gamma(\alpha)$ *is the usual Gamma function. Here* w *is any measurable function, provided that the right hand side exists.*

Definition 2. *The fractional derivative of order* α, $n-1<\alpha<n$, $n\in\mathbb{Z}^{+}$, *in the sense of Caputo is defined by*

$$^{C}D^{\alpha}w(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{-\alpha+n-1}w^{(n)}(s)\,ds, \quad t>0.$$

Clearly

$$^{C}D^{\alpha}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}w'(s)\,ds, \quad 0<\alpha<1,\ t>0.$$

The Riemann-Liouville fractional derivative of order α in the sense of Riemann-Liouville is defined by

$$^{RL}D^{\alpha}w(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}w(s)\,ds, \quad 0<\alpha<1,\ t>0,$$

as long as the integral makes sense.

The passage from one derivative to the other obeys the relationship

$$^{RL}D^{\alpha}w(t) = \frac{w(0)t^{-\alpha}}{\Gamma(1-\alpha)} + {}^{C}D^{\alpha}w(t), \quad 0<\alpha<1,\ t>0. \tag{5}$$

We shall need this relationship in particular to use Proposition 3 below on the differentiation under the integral sign.

We recall the one-parametric and two-parametric Mittag-Leffler functions

$$E_{\alpha}(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}, \quad \mathrm{Re}(\alpha)>0,$$

and

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)}, \quad \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0,$$

respectively. It is useful to notice that $E_{\alpha,1}(z)\equiv E_{\alpha}(z)$.

Proposition 1 ([34]). *In case the relation*

$$^{C}D^{\alpha}w(t) \le -\gamma w(t), \quad 0<\alpha<1,$$

holds true for a differentiable function $w(t)$ *for some* $\gamma>0$, *then*

$$w(t) \le w(0)E_{\alpha}\left(-\gamma t^{\alpha}\right), \quad t\ge 0.$$

The decay takes the form $t^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma t^{\alpha}\right)$, *when the derivative is rather of Riemann-Liouville type.*

Proposition 2 ([17, p. 61])**.** *For $\mu, \alpha, \beta > 0$, we have*

$$\frac{1}{\Gamma(\mu)}\int_0^z (z-t)^{\mu-1}E_{\alpha,\beta}\left(\lambda t^{\alpha}\right)t^{\beta-1}\,dt = z^{\mu+\beta-1}E_{\alpha,\mu+\beta}\left(\lambda z^{\alpha}\right), \quad z>0.$$

The following proposition is about the fractional differentiation under the integral sign.

Proposition 3 ([31, p. 99])**.** *The permutation of the Riemann-Liouville fractional derivative and the integral sign*

$${}^{RL}D^{\alpha}\int_0^t w(t-s)\phi(s)\,ds = \int_0^t \phi(s)\,{}^{RL}D^{\alpha}w(t-s)\,ds + \phi(t)\lim_{t\to 0^+} I^{1-\alpha}w(t), \quad t>0$$

is valid provided $\phi(t)$ is a continuous function and $I^{1-\alpha}w(t)\in C^1\left([0,\infty)\right)$, $0<\alpha<1$.

We end this section with the Caputo fractional derivative of the product of two functions.

Proposition 4 ([2])**.** *Let $u(t)$ and $v(t)$ be absolutely continuous functions on $[0,T]$, $T>0$. Then, for $0<\alpha<1$, we have*

$$\begin{aligned} u(t)^C D^{\alpha}v(t) &+ v(t)^C D^{\alpha}u(t) \\ &= {}^C D^{\alpha}\left(uv(t)\right) + \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{u'(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{v'(s)ds}{(t-s)^{\alpha}}, \quad t\in[0,T]. \end{aligned}$$

In particular

$${}^C D^{\alpha}\left(u^2(t)\right) = 2u(t)^C D^{\alpha}u(t) - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 \le 2u(t)^C D^{\alpha}u(t).$$

For notation convenience we shall drop the superscript "C" from ${}^C D^{\alpha}$. Moreover we denote

$$L(u) := \frac{\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{u'(\eta)d\eta}{(t-\eta)^{\alpha}}\right)^2 dx \tag{6}$$

and

$$L(v) := \frac{\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left|\int_0^{\xi}\frac{v'(\eta)d\eta}{(t-\eta)^{\alpha}}\right|^2 dx \tag{7}$$

for vectors.

2 Existence and uniqueness

Let $\left(X,\|\cdot\|\right)$ be a Banach space and $\mathcal{P}$, $\left(\mathcal{S}(t)\right)_{t\ge 0}$ be closed linear operators defined on domains $D(\mathcal{P})$ and $D\left(\mathcal{S}(t)\right)\supseteq D(\mathcal{P})$ dense in X, respectively. We denote by $\|\cdot\|_1$ the graph norm in $D(\mathcal{P})$, $R(\nu,\mathcal{P}) := (\nu I-\mathcal{P})^{-1}$ and $\rho(\mathcal{P})$ the resolvent set of $\mathcal{P}$.

In [14], the author proved the existence and uniqueness of a mild solution for the abstract problem

$$\begin{cases} {}^C D^{\gamma}U(t,x) = \mathcal{P}U(t,x) + \int_0^t \mathcal{S}(t-s)U(s,x)\,ds + f\left(t,U(t)\right), & 0<\gamma<1, \\ U(0,x) = U_0(x)\in X, \end{cases} \tag{8}$$

(see [1] for the case $1<\gamma<2$) under the above assumptions and the following three.

(A1) For some $\pi/2 < \varpi < \pi$, there exists a constant $C = C(\varpi) > 0$ such that

$$\Sigma_{0,\varpi} := \left\{ \nu \in \mathbf{C} : \left| \arg(\nu) \right| < \varpi \right\} \subset \rho(\mathcal{P})$$

and $\left\| R(\nu, \mathcal{P}) \right\| < C / |\nu|$, $\nu \in \Sigma_{0,\varpi}$.

(A2) For each $\vartheta \in D(\mathcal{P})$, $\mathcal{S}(\cdot)\vartheta$ is strongly measurable on $(0, \infty)$. There exists a locally integrable function $f(t)$ with Laplace transform $\widehat{f}(\nu)$, $\mathrm{Re}(\nu) > 0$, and $\left\| \mathcal{S}(t)\vartheta \right\| \leq f(t) \left\| \vartheta \right\|_1$, $t > 0$, $\vartheta \in D(\mathcal{P})$. In addition, $\mathcal{S}^* : \Sigma_{0,\pi/2} \to \mathcal{L}\big(D(\mathcal{P}); X\big)$, where $\mathcal{L}\big(D(\mathcal{P}); X\big)$ is the space of bounded linear operators from $D(\mathcal{P})$ into X, has an analytic extension $\tilde{\mathcal{S}}$ to $\Sigma_{0,\delta}$ verifying $\left\| \tilde{\mathcal{S}}(\nu)\vartheta \right\| \leq \left\| \tilde{\mathcal{S}}(\nu) \right\| \left\| \vartheta \right\|_1$, $\vartheta \in D(\mathcal{P})$ and $\left\| \tilde{\mathcal{S}}(\nu) \right\| = O\left(\frac{1}{|\nu|} \right)$ as $|\nu| \to \infty$.

(A3) There exist a subspace F dense in $\big(D(\mathcal{P}), \left\| \cdot \right\|_1\big)$ and a constant $\tilde{C} > 0$ such that $\mathcal{P}(F) \subseteq D(\mathcal{P})$, $\mathcal{S}^*(\nu)(F) \subseteq D(\mathcal{P})$, $\left\| \mathcal{P}\mathcal{S}^*(\nu)\vartheta \right\| \leq \tilde{C} \left\| \vartheta \right\|$, $\vartheta \in F$ and $\nu \in \Sigma_{0,\delta}$.

The author used the resolvent operator notion.

Definition. *The family of bounded linear operators* $\big(R_\gamma(t)\big)_{t \geq 0}$ *determines a* γ*-resolvent for* (8) *if*

(a) the mapping $R_\gamma(t) : [0, \infty) \to \mathcal{L}(X) := \mathcal{L}(X; X)$ *is strongly continuous and* $R_\gamma(0) = I$*;*

(b) for all $\vartheta \in D(\mathcal{P})$*,* $R_\gamma(\cdot)\vartheta \in C\big([0, \infty); D(\mathcal{P})\big) \cap C^\gamma\big((0, \infty); X\big)$*, where* $C^\gamma\big((0, \infty); X\big)$ *is the space of continuous functions* ϑ *for which* ${}^C D^\gamma \vartheta$ *exists and is continuous, and*

$$ {}^C D^\gamma R_\gamma(t)\vartheta = \mathcal{P} R_\gamma(t)\vartheta + \int_0^t \mathcal{S}(t-s) R_\gamma(s)\vartheta \, ds = R_\gamma(t)\mathcal{P}\vartheta + \int_0^t R_\gamma(t-s)\mathcal{S}(s)\vartheta \, ds $$

for $t \geq 0$.

Then he showed that the family

$$R_\gamma(t) := \frac{1}{2\pi i} \int_\Gamma \mu^{\gamma-1} e^{\mu t} \big(\mu^\gamma I - \mathcal{P} - \mathcal{S}\hat{}(\preceq)\big)^{-1} d\mu, \quad t \geq 0,$$

is a γ-resolvent for (8) for an appropriate path

$$\Gamma := \left\{ te^{i\omega} : t \geq r \right\} \cup \left\{ re^{i\zeta} : -\omega \leq \zeta \leq \omega \right\} \cup \left\{ te^{-i\omega} : t \geq r \right\}$$

oriented counterclockwise, where $\pi/2 < \omega < \varpi$ and $r > r_1$ (a positive number determined in the proofs).

Theorem. *For* $U_0 \in D(\mathcal{P})$ *and* $f\big(t, U(t)\big) \equiv 0$*, the function*

$$U(t) := R_\gamma(t) U_0 \in C\big([0, \infty); D(\mathcal{P})\big) \cap C^\gamma\big((0, \infty); X\big)$$

is a mild solution.

The existence and uniqueness of a global classical solution in fractional Hölder spaces for the equation

$$ {}^C D^\alpha v = L_1 v + \int_0^t \phi(t-s) L_2 v(s, \cdot) \, ds + f(t, x, v) + g(t, x)$$

with different boundary conditions, has been established in [19] for sub-power type kernels. We refer the reader also to [20,21]. For the problem (3), the existence, uniqueness and regularity of a solution is proved in Sobolev spaces in [5]. More recently, it was proved in [22], that in case of Lipschitzian nonlinearities $g(w)$, mild solutions are in fact classical.

In this paper, we shall assume the existence and uniqueness of a classical solution

$$\theta \in L^2\big((0, +\infty); W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)\big), \qquad {}^C D^\alpha \theta \in L^2\big((0, +\infty); L^2(\Omega)\big)$$

to our problem for which all the computation below are justified.

3 Stability: the case ${}^{RL}D^{\alpha}\phi(t) \leq -\gamma\phi(t)$

In this section, we consider continuous relaxation functions $\phi(t)$ on $[0,\infty)$, which satisfy the fractional differential inequality ${}^{RL}D^{\alpha}\phi(t) \leq -\gamma\phi(t)$. This is in analogy with the differential inequality $\phi'(t) \leq -\gamma\phi(t)$ used in the integer case.

We shall assume that our initial data satisfies $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and

$$\bar{\phi} := \int_0^\infty \phi(s)\,ds < 1/b. \tag{9}$$

The last assumption (9) is justified in [10]. Roughly, it guarantees the negativity of the equilibrium flux and consequently the "forward" heat flow at equilibrium as well as the "forward" heat flow for all $t \geq 0$. More precisely, by Proposition 2, it is clear that such kernels are summable and

$$\int_0^t \phi(s)\,ds = \phi_0 \int_0^t s^{\alpha-1}E_{\alpha,\alpha}(-\gamma s^\alpha)\,ds = \phi_0 t^\alpha E_{\alpha,\alpha+1}(-\gamma t^\alpha) \leq \phi_0/\gamma, \quad \phi_0 > 0,\ t > 0. \tag{10}$$

A multiplication of the equation in (1) by θ followed by an integration over Ω gives

$$\int_\Omega \theta D^\alpha \theta\,dx = -\|\nabla\theta\|^2 - b\int_\Omega \theta \int_0^t \phi(t-s)\Delta\theta(s)\,ds\,dx.$$

By Proposition 4 (and (6)), we have

$$\begin{aligned}\int_\Omega \theta D^\alpha \theta\,dx &= \frac{1}{2}D^\alpha \|\theta\|^2 + \frac{\alpha}{2\Gamma(1-\alpha)}\int_\Omega \int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^\xi \frac{\theta'(\eta)d\eta}{(t-\eta)^\alpha}\right)^2 dx \\ &= \frac{1}{2}D^\alpha \|\theta\|^2 + \frac{1}{2}L(\theta), \quad t \in [0,T],\ T > 0.\end{aligned}$$

Therefore

$$\frac{1}{2}D^\alpha \|\theta\|^2 + \frac{1}{2}L(\theta) = -\|\nabla\theta\|^2 + b\int_\Omega \nabla\theta. \int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx.$$

Young inequality implies

$$\begin{aligned}&\int_\Omega \nabla\theta. \int_0^t \phi(t-s)\,\nabla\theta(s)\,ds\,dx \\ &\qquad \leq \delta_1\|\nabla\theta\|^2 + \frac{1}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right)\int_\Omega\int_0^t \phi(t-s)\left|\nabla\theta(s)\right|^2 ds\,dx, \quad \delta_1 > 0.\end{aligned}$$

Hence

$$\frac{1}{2}D^\alpha \|\theta\|^2 \leq (b\delta_1 - 1)\|\nabla\theta\|^2 + \frac{b}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right)\int_0^t \phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds - \frac{1}{2}L(\theta). \tag{11}$$

We introduce the functional

$$H(t) := \int_0^t \phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds, \quad t \geq 0.$$

Lemma 1. *The functional $H(t)$ satisfies the following inequality*

$$D^{\alpha}H(t) \le -\gamma H(t), \quad t \ge 0.$$

Proof. In view of our hypotheses on the kernel $\phi(t)$, notice that both derivatives below exist. Therefore, by the relation (5) and Proposition 3, we infer

$$\begin{aligned} D^{\alpha}H(t) &= {}^{RL}D^{\alpha}\int_0^t \phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds \\ &= \int_0^t {}^{RL}D^{\alpha}\,\phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds + \Big(\lim_{t\to 0^+} I^{1-\alpha}\phi(t)\Big)\left\|\nabla\theta\right\|^2 \end{aligned}$$

and therefore the claim is proved. □

We shall examine the functional

$$E_1(t) = \frac{1}{2}\left\|\theta\right\|^2 + \kappa H(t), \quad \kappa > 0.$$

Theorem 1. *The problem* (1) *under the above hypotheses on the kernel $\phi(t)$, is Mittag-Leffler stable. That is, there exist positive constants B and ν such that*

$$\left\|\theta\right\|^2 \le BE_{\alpha}\left(-\nu t^{\alpha}\right), \quad t \ge 0.$$

Proof. The fractional derivative of $E_1(t)$ may be deduced from Lemma 1 and (11) as follows

$$\begin{aligned} D^{\alpha}E_1(t) &\le (b\delta_1 - 1)\left\|\nabla\theta\right\|^2 + \frac{b}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right)\int_0^t \phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds - \frac{1}{2}L(\theta) - \kappa\gamma H(t) \\ &\le (b\delta_1 - 1)\left\|\nabla\theta\right\|^2 + \left[\frac{b}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right) - \kappa\gamma\right]H(t), \quad t \in [0,T]. \end{aligned}$$

Picking $\delta_1 = 1/2b$, $\kappa = b^2\bar{\phi}/\gamma$ and using Poincaré inequality, we obtain

$$D^{\alpha}E_1(t) \le -CE_1(t), \quad t \in [0,T]$$

for some positive constant C, independent on T. The conclusion follows from Proposition 1. □

4 Stability: singular kernel but regular solution

In this section, we consider the special (singular) kernel $\phi(t) = \phi_0 t^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma t^{\alpha}\right)$, $\phi_0 > 0$. In this case, we request a minimum regularity on the solution (see [22] for the Rayleigh-Stokes problem). In particular, we need $D^{\alpha}\nabla\theta$ and $D^{\alpha}\left(\int_0^t \phi(s)\nabla\theta(t-s)\,ds\right)$ to make sense. Here, we find from the previous argument

$$\begin{aligned} D^{\alpha}E_1(t) &\le (b\delta_1 - 1)\left\|\nabla\theta\right\|^2 + \left[\frac{b}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right) - \kappa\gamma\right]H(t) + \kappa\Big(\lim_{t\to 0^+} I^{1-\alpha}\phi(t)\Big)\left\|\nabla\theta\right\|^2 \\ &\le \left[b\delta_1 + \kappa\Big(\lim_{t\to 0^+} I^{1-\alpha}\phi(t)\Big) - 1\right]\left\|\nabla\theta\right\|^2 + \left[\frac{b}{4\delta_1}\left(\int_0^t \phi(s)\,ds\right) - \kappa\gamma\right]H(t), \quad t \in [0,T]. \end{aligned}$$

Having in mind Proposition 2 and

$$I^{1-\alpha}\phi(t) = \frac{\phi_0}{\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma(t-s)^{\alpha}\right)s^{-\alpha}\,ds = \phi_0 E_{\alpha}\left(-\gamma t^{\alpha}\right),$$

we may request

$$b\delta_1 + \kappa\phi_0 < 1 \qquad \text{and} \qquad \frac{b\phi_0}{4\gamma\delta_1} < \kappa\gamma.$$

These conditions are fulfilled provided that $\phi_0/\gamma < 1/b$. This is a reasonable condition (see (9), (10) and also the definition of $E(t)$ below). Consequently, we reach a Mittag-Leffler decay of $E_1(t)$ again.

Profiting from the assumed regularity of the solutions, it is possible to have in fact a Mittag-Leffler decay in the H^1-norm of the solution.

We denote by

$$(\phi\Box\nabla\theta)(t) := \int_\Omega \int_0^t \phi(t-s)\left|\nabla\theta(t) - \nabla\theta(s)\right|^2 ds\,dx, \quad t \geq 0.$$

Next, we multiply the equation (1) by $D^\alpha\theta$ and integrate. We get over Ω

$$\|D^\alpha\theta\|^2 = -\int_\Omega \nabla\theta.D^\alpha\nabla\theta\,dx + b\int_\Omega D^\alpha\nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx.$$

As

$$\int_\Omega \nabla\theta.D^\alpha\nabla\theta\,dx = \frac{1}{2}D^\alpha\|\nabla\theta\|^2 + \frac{1}{2}L(\nabla\theta),$$

we see that

$$\frac{1}{2}D^\alpha\|\nabla\theta\|^2 \leq -\|D^\alpha\theta\|^2 + b\int_\Omega D^\alpha\nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx - \frac{1}{2}L(\nabla\theta). \tag{12}$$

The term

$$\int_\Omega D^\alpha\nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx$$

in (12) is estimated in the next proposition.

Proposition 5. *For our kernel $\phi(t)$, the following identity holds*

$$\begin{aligned}
\int_\Omega D^\alpha\nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)dsdx &= \frac{1}{2}\left\{\left({}^{RL}D^\alpha\phi\Box\nabla\theta\right) + \left(\int_0^t \phi(s)\,ds\right)D^\alpha\|\nabla\theta\|^2\right\} \\
&- \frac{\alpha}{2\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^\xi \frac{\phi(\eta)d\eta}{(t-\eta)^\alpha}\int_0^\xi \frac{[\|\nabla\theta(s)\|^2]'ds}{(t-s)^\alpha} \\
&- \frac{1}{2}D^\alpha(\phi\Box\nabla\theta)(t) + \frac{\alpha}{\Gamma(1-\alpha)}\int_\Omega\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^\xi \frac{[(\nabla\theta)(\eta)]'d\eta}{(t-\eta)^\alpha} \\
&\times\left(\int_0^\xi (t-s)^{-\alpha}\left(\int_0^s \phi(s-\tau)\nabla\theta(\tau)\,d\tau\right)' ds\right)dx.
\end{aligned}$$

Proof. Using Proposition 4 and the identity

$$\begin{aligned}
(\phi\Box\nabla\theta)(t) &:= \int_\Omega\int_0^t \phi(t-s)\left|\nabla\theta(t) - \nabla\theta(s)\right|^2 ds\,dx \\
&= \|\nabla\theta\|^2\int_0^t \phi(t-s)\,ds + \int_0^t \phi(t-s)\left\|\nabla\theta(s)\right\|^2 ds \\
&\quad - 2\int_\Omega \nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx, \quad t \in [0,T],
\end{aligned}$$

we find

$$
\begin{aligned}
D^{\alpha}(\phi\Box\nabla\theta)(t) = & \left(D^{\alpha}\int_0^t \phi(t-s)\,ds\right)\|\nabla\theta\|^2 \\
& + \left(\int_0^t \phi(t-s)\,ds\right) D^{\alpha}\|\nabla\theta\|^2 \\
& - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{\left[\|\nabla\theta(s)\|^2\right]' ds}{(t-s)^{\alpha}} \\
& + D^{\alpha}\int_0^t \phi(t-s)\big\|\nabla\theta(s)\big\|^2\,ds \\
& - 2\int_{\Omega} D^{\alpha}\nabla\theta.\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx \\
& - 2\int_{\Omega}\nabla\theta.D^{\alpha}\int_0^t \phi(t-s)\nabla\theta(s)\,ds\,dx \\
& + \frac{2\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]' d\eta}{(t-\eta)^{\alpha}} \\
& \times\left(\int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s \phi(s-\tau)\nabla\theta(\tau)d\tau\right)' ds\right) dx, \quad t\in[0,T].
\end{aligned}
\tag{13}
$$

Clearly, from the definition of the quadratic form $(\phi\Box\nabla\theta)\,(t)$ we get

$$
\begin{aligned}
\left({}^{RL}D^{\alpha}\phi\Box\nabla\theta\right)(t) = \|\nabla\theta\|^2\int_0^t {}^{RL}D^{\alpha}\phi(t-s)\,ds + \int_0^t {}^{RL}D^{\alpha}\phi(t-s)\big\|\nabla\theta(s)\big\|^2\,ds \\
- 2\int_{\Omega}\nabla\theta.\int_0^t {}^{RL}D^{\alpha}\phi(t-s)\;\nabla\theta(s)\,ds\,dx
\end{aligned}
\tag{14}
$$

and in virtue of Proposition 3, we entail

$$
D^{\alpha}\int_0^t \phi(t-s)\,ds = \int_0^t {}^{RL}D^{\alpha}\phi(t-s)\,ds + \lim_{t\to 0^+} I^{1-\alpha}\phi(t), \quad t>0. \tag{15}
$$

Again, by Proposition 3 and the summability of ϕ, for $t\in[0,T]$ we have

$$
\begin{aligned}
D^{\alpha}\int_0^t \phi(t-s)\big\|\nabla\theta(s)\big\|^2\,ds = & {}^{RL}D^{\alpha}\int_0^t \phi(t-s)\big\|\nabla\theta(s)\big\|^2\,ds \\
& - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\left(\int_0^t \phi(t-s)\big\|\nabla\theta(s)\big\|^2 ds\right)\Bigg|_{t=0} \\
= & \int_0^t {}^{RL}D^{\alpha}\phi(t-s)\;\big\|\nabla\theta(s)\big\|^2 ds + \big\|\nabla\theta(t)\big\|^2\lim_{t\to 0^+} I^{1-\alpha}\phi(t).
\end{aligned}
\tag{16}
$$

Moreover,

$$
\begin{aligned}
\int_{\Omega}\nabla\theta.D^{\alpha}\int_0^t \phi(t-s)\nabla\theta(s)dsdx = & \int_{\Omega}\nabla\theta\int_0^t {}^{RL}D^{\alpha}\phi(t-s)\;\nabla\theta(s)\,ds\,dx \\
& + \big\|\nabla\theta(t)\big\|^2\lim_{t\to 0^+} I^{1-\alpha}\phi(t), \quad t\in[0,T].
\end{aligned}
\tag{17}
$$

Gathering the previous relations (16) and (17) in (13), we find

$$
\begin{aligned}
D^{\alpha}(\phi\Box\nabla\theta)(t) &= \|\nabla\theta\|^{2}\left[\int_{0}^{t}{}^{RL}D^{\alpha}\phi(t-s)\,ds+\lim_{t\to 0^{+}}I^{1-\alpha}\phi(t)\right]+\left(\int_{0}^{t}\phi(t-s)\,ds\right)D^{\alpha}\|\nabla\theta\|^{2}\\
&-\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_{0}^{\xi}\frac{\left[\|\nabla\theta(s)\|^{2}\right]'ds}{(t-s)^{\alpha}}\\
&+\int_{0}^{t}{}^{RL}D^{\alpha}\phi(t-s)\left\|\nabla\theta(s)\right\|^{2}ds+\left\|\nabla\theta(t)\right\|^{2}\lim_{t\to 0^{+}}I^{1-\alpha}\phi(t)\\
&-2\int_{\Omega}D^{\alpha}\nabla\theta.\int_{0}^{t}\phi(t-s)\nabla\theta(s)\,ds\,dx\\
&-2\int_{\Omega}\nabla\theta.\int_{0}^{t}{}^{RL}D^{\alpha}\,\phi(t-s)\nabla\theta(s)\,ds\,dx-2\left\|\nabla\theta(t)\right\|^{2}\lim_{t\to 0^{+}}I^{1-\alpha}\phi(t)\\
&+\frac{2\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]'d\eta}{(t-\eta)^{\alpha}}\\
&\times\left(\int_{0}^{\xi}(t-s)^{-\alpha}\left(\int_{0}^{s}\phi(s-\tau)\nabla\theta(\tau)d\tau\right)'ds\right)dx,
\end{aligned}
$$

or

$$
\begin{aligned}
D^{\alpha}(\phi\Box\nabla\theta)(t) &= \|\nabla\theta\|^{2}\int_{0}^{t}{}^{RL}D^{\alpha}\,\phi(t-s)ds+\left(\int_{0}^{t}\phi(t-s)ds\right)D^{\alpha}\|\nabla\theta\|^{2}\\
&-\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_{0}^{\xi}\frac{\left[\|\nabla\theta(s)\|^{2}\right]'ds}{(t-s)^{\alpha}}\\
&+\int_{0}^{t}{}^{RL}D^{\alpha}\,\phi(t-s)\left\|\nabla\theta(s)\right\|^{2}ds-2\int_{\Omega}D^{\alpha}\nabla\theta.\int_{0}^{t}\phi(t-s)\nabla\theta(s)\,ds\,dx\\
&-2\int_{\Omega}\nabla\theta.\int_{0}^{t}{}^{RL}D^{\alpha}\,\phi(t-s)\nabla\theta(s)\,ds\,dx\\
&+\frac{2\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]'d\eta}{(t-\eta)^{\alpha}}\\
&\times\left(\int_{0}^{\xi}(t-s)^{-\alpha}\left(\int_{0}^{s}\phi(s-\tau)\nabla\theta(\tau)d\tau\right)'ds\right)dx.
\end{aligned}
$$

Therefore, in view of the relations (14) and (15), we obtain

$$
\begin{aligned}
D^{\alpha}(\phi\Box\nabla\theta)(t) &= \left({}^{RL}D^{\alpha}\,\phi\Box\nabla\theta\right)+\left(\int_{0}^{t}\phi(s)ds\right)D^{\alpha}\|\nabla\theta\|^{2}\\
&-\frac{\alpha}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_{0}^{\xi}\frac{\left[\|\nabla\theta(s)\|^{2}\right]'ds}{(t-s)^{\alpha}}\\
&-2\int_{\Omega}D^{\alpha}\nabla\theta.\int_{0}^{t}\phi(t-s)\nabla\theta(s)\,ds\,dx\\
&+\frac{2\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_{0}^{t}\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_{0}^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]'d\eta}{(t-\eta)^{\alpha}}\\
&\times\left(\int_{0}^{\xi}(t-s)^{-\alpha}\left(\int_{0}^{s}\phi(s-\tau)\nabla\theta(\tau)d\tau\right)'ds\right)dx
\end{aligned}
$$

for $t\in[0,T]$. This finishes the proof. □

We deduce from this Proposition 5 and the inequality (12) that

$$\begin{aligned}\frac{1}{2}D^{\alpha}\|\nabla\theta\|^{2} \leq &-\|D^{\alpha}\theta\|^{2}-\frac{1}{2}L(\nabla\theta)+\frac{b}{2}\left({}^{RL}D^{\alpha}\phi\Box\nabla\theta\right)+\frac{b}{2}\left(\int_0^t\phi(s)\,ds\right)D^{\alpha}\|\nabla\theta\|^{2}\\ &-\frac{\alpha b}{2\Gamma(1-\alpha)}\int_0^t\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{\left[\|\nabla\theta(s)\|^2\right]' ds}{(t-s)^{\alpha}}-\frac{b}{2}D^{\alpha}(\phi\Box\nabla\theta)(t)\\ &+\frac{\alpha b}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]' d\eta}{(t-\eta)^{\alpha}}\\ &\times\left(\int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s\phi(s-\tau)\nabla\theta(\tau)d\tau\right)' ds\right)dx\end{aligned}$$

for $t\in[0,T]$, which may be written as

$$\begin{aligned}\frac{1}{2}\left(1-b\int_0^t\phi(s)\,ds\right)&D^{\alpha}\|\nabla\theta\|^{2}+\frac{b}{2}D^{\alpha}(\phi\Box\nabla\theta)(t)\\ &\leq-\|D^{\alpha}\theta\|^{2}-\frac{1}{2}L(\nabla\theta)+\frac{b}{2}\left({}^{RL}D^{\alpha}\phi\Box\nabla\theta\right)\\ &\quad-\frac{\alpha b}{2\Gamma(1-\alpha)}\int_0^t\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{\left[\|\nabla\theta(s)\|^2\right]' ds}{(t-s)^{\alpha}}\\ &\quad+\frac{\alpha b}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]' d\eta}{(t-\eta)^{\alpha}}\\ &\quad\times\left(\int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s\phi(s-\tau)\nabla\theta(\tau)d\tau\right)' ds\right)dx.\end{aligned}\tag{18}$$

The relations (11) and (18) triggered the introduction of the "energy" functional

$$E(t)=\frac{1}{2}\left\{\|\theta\|^{2}+\left(1-b\int_0^t\phi(s)\,ds\right)\|\nabla\theta\|^{2}+b\,(\phi\Box\nabla\theta)\,(t)\right\},\quad t\geq 0.$$

We recall the assumption (9). The above considerations lead to the following assertion.

Lemma 2. *The functional $E(t)$ along solutions of* (1) *satisfies*

$$\begin{aligned}D^{\alpha}E(t)\leq&-\|D^{\alpha}\theta\|^{2}+\left[b\delta_1-\left(1-b\int_0^t\phi(s)\,ds\right)-\frac{b}{2}I^{1-\alpha}\phi(t)\right]\|\nabla\theta\|^{2}-\frac{1}{2}L(\theta)\\ &+\left(b\delta_2-\frac{1}{2}\right)L(\nabla\theta)+\frac{b}{2}\left(\frac{\bar{\phi}}{2\delta_1}-\gamma\right)(\phi\Box\nabla\theta)(t)\\ &+\frac{b}{4\delta_2}L\left(\int_0^t\phi(t-\tau)\nabla\theta(\tau)\,d\tau\right)\end{aligned}$$

for $\delta_1,\delta_2>0$, $t\geq 0$.

Proof. Clearly, a direct fractional differentiation of $E(t)$, using Proposition 4, yields

$$\begin{aligned}D^{\alpha}E(t)=&\frac{1}{2}D^{\alpha}\left\{\|\theta\|^{2}+\left(1-b\int_0^t\phi(s)\,ds\right)\|\nabla\theta\|^{2}+b\,(\phi\Box\nabla\theta)\,(t)\right\}\\ =&\frac{1}{2}D^{\alpha}\|\theta\|^{2}+\frac{1}{2}D^{\alpha}\left(1-b\int_0^t\phi(s)\,ds\right)\|\nabla\theta\|^{2}+\frac{1}{2}\left(1-b\int_0^t\phi(s)\,ds\right)D^{\alpha}\|\nabla\theta\|^{2}\\ &+\frac{\alpha b}{2\Gamma(1-\alpha)}\int_{\Omega}\int_0^t\frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\left(\int_0^{\xi}(t-s)^{-\alpha}\left(\|\nabla\theta(s)\|^{2}\right)' ds\right)dx\\ &+\frac{b}{2}D^{\alpha}\,(\phi\Box\nabla\theta)\,(t)\end{aligned}$$

or (see (7))

$$
\begin{aligned}
D^{\alpha}E(t) \leq & \left[b\delta_1 - \left(1 - b\int_0^t \phi(s)\,ds \right) \right] \|\nabla\theta\|^2 - \frac{1}{2}L(\theta) \\
& + \frac{b}{4\delta_1}\left(\int_0^t \phi(s)ds \right)(\phi\Box\nabla\theta)(t) - \frac{b}{2}\left(D^{\alpha}\int_0^t \phi(s)\,ds \right)\|\nabla\theta\|^2 - \|D^{\alpha}\theta\|^2 \\
& - \frac{\alpha b}{2\Gamma(1-\alpha)}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}}\int_0^{\xi}\frac{\left[\|\nabla\theta(s)\|^2\right]'ds}{(t-s)^{\alpha}} - \frac{1}{2}L(\nabla\theta) \\
& + \frac{b}{2}\left({}^{RL}D^{\alpha}\phi\Box\nabla\theta \right)(t) + \frac{\alpha b}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\left[(\nabla\theta)(\eta)\right]'d\eta}{(t-\eta)^{\alpha}} \\
& \qquad \times \left(\int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s \phi(s-\tau)\nabla\theta(\tau)d\tau \right)' ds \right) dx \\
& + \frac{\alpha b}{2\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\int_0^{\xi}\frac{\phi(\eta)d\eta}{(t-\eta)^{\alpha}} \\
& \qquad \times \left(\int_0^{\xi}(t-s)^{-\alpha}\left(\|\nabla\theta\|^2 \right)' ds \right) dx, \quad \delta_1 > 0.
\end{aligned} \tag{19}
$$

Because

$$
-D^{\alpha}\int_0^t \phi(s)ds = -I^{1-\alpha}D\int_0^t \phi(s)ds = -I^{1-\alpha}\phi(t),
$$

the last relation (19) is reduced to

$$
\begin{aligned}
D^{\alpha}E(t) \leq -\|D^{\alpha}\theta\|^2 + & \left[b\delta_1 - \left(1 - b\int_0^t \phi(s)ds \right) - \frac{b}{2}I^{1-\alpha}\phi(t) \right] \|\nabla\theta\|^2 \\
& - \frac{1}{2}L(\theta) + \left(b\delta_2 - \frac{1}{2} \right) L(\nabla\theta) + \frac{b}{2}\left(\frac{\bar{\phi}}{2\delta_1} - \gamma \right)(\phi\Box\nabla\theta)(t) \\
& + \frac{b}{4\delta_2}L\left(\int_0^t \phi(t-\tau)\nabla\theta(\tau)d\tau \right),
\end{aligned}
$$

where we used

$$
\begin{aligned}
& \int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left(\int_0^{\xi}\frac{\left[\nabla\theta(\eta)\right]'d\eta}{(t-\eta)^{\alpha}} \right)\left(\int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s \phi(s-\tau)\nabla\theta(\tau)d\tau \right)' ds \right) dx \\
& \quad \leq \delta_2\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left| \int_0^{\xi}\frac{(\nabla\theta)'(\eta)d\eta}{(t-\eta)^{\alpha}} \right|^2 dx \\
& \quad + \frac{1}{4\delta_2}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left| \int_0^{\xi}(t-s)^{-\alpha}\left(\int_0^s \phi(s-\tau)\nabla\theta(\tau)d\tau \right)' ds \right|^2 dx, \quad \delta_2 > 0,
\end{aligned}
$$

and the form of the kernel ϕ. This ends the proof. □

Observe that the last term in the estimation of $D^{\alpha}E(t)$ is positive and there is no clear way to control it. To this end we introduce the functional

$$
G(t) := \int_{\Omega}\left| \int_0^t \phi(t-s)\nabla\theta(s)\,ds \right|^2 dx, \quad t \geq 0.
$$

Lemma 3. *The fractional derivative of the functional $G(t)$ is evaluated by*

$$D^{\alpha}G(t) \leq -2\gamma G(t) + \phi_0\left(\delta_3 + 2\phi\right)\|\nabla\theta\|^2 + \frac{\bar{\phi}\phi_0}{\delta_3}(\phi\Box\nabla\theta)(t) - L\bigg(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\bigg), \quad \delta_3 > 0,\ t \geq 0.$$

Proof. First, we take the Caputo derivative of order α of $G(t)$ and use Proposition 4, namely

$$D^{\alpha}G(t) = 2\int_{\Omega}\left(\int_0^t \phi(t-s)\nabla\theta(s)ds\right)D^{\alpha}\left(\int_0^t \phi(t-s)\nabla\theta(s)ds\right)dx - \frac{\alpha}{\Gamma(1-\alpha)}\int_{\Omega}\int_0^t \frac{d\xi}{(t-\xi)^{1-\alpha}}\left|\int_0^{\xi}(t-\eta)^{-\alpha}\left(\int_0^{\eta}\phi(\eta-\tau)\nabla\theta(\tau)d\tau\right)'d\eta\right|^2 dx$$

for $t \in [0,T]$. Second, the relation (5) gives

$$D^{\alpha}G(t) = 2\int_{\Omega}\left(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\right) \times \left[{}^{RL}D^{\alpha}\left(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\right)dx - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}\bigg(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\bigg)\bigg|_{t=0}\right]dx - L\bigg(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\bigg), \quad t \in [0,T].$$

Then, the summability of ϕ and Proposition 3 allows us to write

$$D^{\alpha}G(t) = 2\int_{\Omega}\left(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\right) \times \left[\int_0^t {}^{RL}D^{\alpha}\phi(t-s)\nabla\theta(s)\,ds + \nabla\theta(t)\lim_{t\to 0^+} I^{1-\alpha}\phi(t)\right]dx - L\left(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\right),$$

and by the nature of our kernel ϕ we get

$$D^{\alpha}G(t) = -2\gamma G(t) + 2\lim_{t\to 0^+} I^{1-\alpha}\phi(t)\int_{\Omega}\nabla\theta(t).\left(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\right)dx - L\bigg(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\bigg), \quad t \in [0,T].$$

Next, in virtue of Proposition 1 and Proposition 2 with $\mu = 1-\alpha$ and $\beta = \alpha$ we have

$$I^{1-\alpha}\phi(t) = \frac{\phi_0}{\Gamma(1-\alpha)}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\left(-\gamma(t-s)^{\alpha}\right)s^{-\alpha}\,ds = \phi_0 E_{\alpha}\left(-\gamma t^{\alpha}\right),$$

and

$$\int_{\Omega}\nabla\theta(t).\bigg(\int_0^t \phi(t-s)\nabla\theta(s)\,ds\bigg)dx = \int_{\Omega}\nabla\theta(t).\bigg(\int_0^t \phi(t-s)\left[\nabla\theta(s) - \nabla\theta(t)\right]ds\bigg)dx + \left(\int_0^t \phi(s)\,ds\right)\|\nabla\theta\|^2.$$

We end up with

$$D^{\alpha}G(t) \leq -2\gamma G(t) + \phi_0\left(\delta_3 + 2\bar{\phi}\right)\|\nabla\theta\|^2 + \frac{\bar{\phi}\phi_0}{\delta_3}(\phi\Box\nabla\theta)(t) - L\bigg(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\bigg), \quad \delta_3 > 0,\ t \in [0,T].$$

This completes the proof. □

Now, we introduce the functional

$$E_2(t) = E(t) + \rho G(t), \quad \rho > 0,$$

and the number

$$U := \frac{1}{2b} \min \left\{ \frac{\gamma}{2 + 5b^2\phi_0^2}, \frac{1}{1 + 2b\phi_0} \right\}.$$

Theorem 2. *Under the above assumptions, problem* (1) *is Mittag-Leffler stable (in the* H^1*-norm). That is, there exist positive constants* B *and* ν *such that*

$$E(t) \leq BE_\alpha\left(-\nu t^\alpha\right), \quad t \geq 0,$$

provided that ϕ *satisfies* $\bar{\phi} \leq U$.

Proof. The fractional derivative of $E_2(t)$ verifies (see Lemma 2 and Lemma 3)

$$\begin{aligned} D^\alpha E_2(t) \leq & -\|D^\alpha \theta\|^2 + \left[b\delta_1 - \left(1 - b\int_0^t \phi(s)\,ds\right) - \frac{b}{2} I^{1-\alpha}\phi(t) \right] \|\nabla\theta\|^2 \\ & - \frac{L(\theta)}{2} + \left(b\delta_2 - \frac{1}{2}\right) L(\nabla\theta) + \frac{b}{2}\left(\frac{\bar{\phi}}{2\delta_1} - \gamma\right)(\phi\Box\nabla\theta)(t) \\ & + \frac{b}{4\delta_2} L\left(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\right) - 2\rho\gamma G(t) + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right)\|\nabla\theta\|^2 \\ & + \frac{\rho\phi_0\bar{\phi}}{\delta_3}(\phi\Box\nabla\theta)(t) - \rho L\left(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\right). \end{aligned}$$

Therefore

$$\begin{aligned} D^\alpha E_2(t) \leq & -\frac{L(\theta)}{2} + \left(b\delta_2 - \frac{1}{2}\right) L(\nabla\theta) - \|D^\alpha\theta\|^2 \\ & + \left[\frac{b}{2}\left(\frac{\bar{\phi}}{2\delta_1} - \gamma\right) + \rho\frac{\bar{\phi}\phi_0}{\delta_3}\right](\phi\Box\nabla\theta)(t) - 2\rho\gamma G(t) \\ & + \left[b\delta_1 - (1 - b\bar{\phi}) + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right)\right]\|\nabla\theta\|^2 \\ & + \left(\frac{b}{4\delta_2} - \rho\right) L\left(\int_0^t \phi(t-\tau)\nabla\theta(\tau)\,d\tau\right). \end{aligned} \tag{20}$$

Now, we start selecting the parameters in such a way that all the coefficients in the right hand side of (20) be negative. The strategy is to get $-CE_2(t)$ in the right hand side or an equivalent expression. By Poincaré inequality with constant C_p, if

$$b\delta_1 + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right) \leq \frac{1 - b\bar{\phi}}{2},$$

then

$$b\delta_1 + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right) - (1 - b\bar{\phi}) \leq \frac{1 - b\bar{\phi}}{2} - (1 - b\bar{\phi}) = -\frac{1 - b\bar{\phi}}{2}$$

and

$$\left[b\delta_1 + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right) - (1 - b\bar{\phi})\right]\|\nabla\theta\|^2 \leq -\frac{1 - b\bar{\phi}}{2C_p}\|\theta\|^2.$$

Therefore, we need

$$\begin{cases} b\delta_1 + \rho\phi_0\left(\delta_3 + 2\bar{\phi}\right) < \dfrac{1 - b\bar{\phi}}{2}, \\ b\delta_2 = \dfrac{1}{2}, \\ \dfrac{b\bar{\phi}}{4\delta_1} + \rho\dfrac{\bar{\phi}\phi_0}{\delta_3} < \dfrac{b\gamma}{2}, \\ \dfrac{b}{4\delta_2} = \rho. \end{cases}$$

Let $\delta_2 = \dfrac{1}{2b}$, $\rho = \dfrac{b^2}{2}$ and to fix ideas $\bar{\phi}b\left(1 + 2b\phi_0\right) < \dfrac{5}{8}$, then

$$\begin{cases} b\delta_1 + \dfrac{b^2}{2}\delta_3\phi_0 < \dfrac{1 - b\bar{\phi} - 2\bar{\phi}\phi_0 b^2}{2}, \\ \dfrac{\bar{\phi}}{2\delta_1} + \dfrac{b\bar{\phi}\phi_0}{\delta_3} < \gamma \end{cases}$$

or

$$\begin{cases} b\delta_1 + \dfrac{b^2}{2}\delta_3\phi_0 < \dfrac{3}{16}, \\ \dfrac{\bar{\phi}}{2\delta_1} + \dfrac{b\bar{\phi}\phi_0}{\delta_3} < \gamma. \end{cases} \tag{21}$$

Picking $\delta_1 = \dfrac{1}{8b}$, $\delta_3 = \dfrac{1}{10\phi_0 b^2}$, we see that these last relations (22) are verified under our condition $\bar{\phi} \leq U$, which in turn requires that either γ is large or ϕ_0 is small. It appears that

$$D^\alpha E_2(t) \leq -CE_2(t), \quad t \geq 0,$$

for some positive constant C. The conclusion follows from Proposition 3. □

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Розглядається дробова диференціальна задача порядку, що знаходиться між нулем і одиницею. Модель узагальнює існуючу відому проблему теорії теплопровідності з пам'яттю. Спочатку ми обґрунтовуємо заміну похідної першого порядку на дробову. Після цього, ми встановлюємо результат стійкості Міттаг-Леффлера для класу функцій релаксації теплового потоку. Ми поєднуємо метод енергії з деякими властивостями дробового числення.

Ключові слова і фрази: дробова похідна Капуто, теплопровідність, термін пам'яті, стабільність Міттаг-Леффлера, метод множника.