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Perturbation of an isotropic α-stable stochastic process by a pseudo-gradient with a generalized coefficient

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The article is devoted to the perturbation of an isotropic α-stable stochastic process in a finite-dimensional Euclidean space by a pseudo-gradient operator multiplied by a delta-function on a hypersurface. This is analogous to the construction of some membrane in the phase space. Semigroup of operators on the space of continuous bounded functions is constructed. It has the infinitesimal generator (in some generalized sense) $c\Delta_\alpha + (q\delta_S\nu, \nabla_\beta)$, where c is some positive constant, Δ_α is the fractional Laplacian of the order α, δ_S is the delta-function on the hypersurface S, which has a normal vector ν, q is some continuous bounded function, ∇_β is a fractional gradient (pseudo-gradient), that is the pseudo-differentional operator defined by the symbol $i\lambda|\lambda|^{\beta-1}$. The order of the pseudo-gradient is less than $\alpha - 1$. Some properties of the obtained semigroup are investigated. This semigroup defines a pseudo-process.

Key words and phrases: α-stable stochastic process, perturbation, pseudo-gradient, semigroup of operators, pseudo-process.

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Introduction

Let us fix some constants $d \in \mathbb{N}$, $c > 0$, and $\alpha \in (1, 2]$. Consider the function

$$g(t, x, y) = \frac{1}{(2\pi)^d} \int_{R^d} \exp\left\{i(\lambda, y - x) - ct|\lambda|^\alpha\right\} \mathrm{d}\lambda \tag{1}$$

given for all $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$. At some partial values α, the integral (1) can be calculated explicitly. In particular, for $\alpha = 2$, we have

$$g(t, x, y) = \frac{1}{(4c\pi t)^{\frac{d}{2}}} \exp\left\{-\frac{|y - x|^2}{4ct}\right\}. \tag{2}$$

The function g is the transition probability density of an isotropic α-stable stochastic process; the case of $\alpha = 2$ relates to the Brownian motion.

Let S be some given hypersurface in $\mathbb{R}^d$, i.e. a manifold of dimension $d - 1$. In this paper, we limit ourselves to a hyperplane or a bounded closed surface of the class $H^{1+\gamma}$ with some fixed $\gamma \in (0, 1)$ (see, for example, [5, Ch. III, §8]). Let us denote the unit vector of the outward normal to the surface S at the point $x \in S$ by $\nu(x)$. In the case of a hyperplane this vector is constant, $\nu(x) \equiv \nu$, thus, let us consider that the hyperplane is set by an equation $(x, \nu) = 0$.

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Let a continuous function $q(x)$ be defined at all points of the surface S. Additionally, we will assume its boundedness if S is a hyperplane.

After choosing some $0 < \beta < \alpha$, let us denote the β-order "pseudo-gradient" by ∇_β, i.e. it is the operator which is defined by the symbol $\left(i\lambda|\lambda|^{\beta-1}\right)_{\lambda\in\mathbb{R}^d}$. In the case of $\beta = 1$, it is the ordinary gradient.

Our goal is to construct a perturbation of the isotropic α-stable stochastic process using operator $\left(q\delta_S\nu, \nabla_\beta\right)$, where under δ_S we understand symmetric delta function focused on the surface S. In other words, it is a generalized function that acts on every test function (at least continuous finite function) $\varphi(x)$, $x \in \mathbb{R}^d$, by the formula $\langle\delta_S, \varphi\rangle = \int_S \varphi(x)d\sigma$, where the integral is the surface one of the first kind.

Under the perturbation of an isotropic α-stable stochastic process we understand the construction of an operators semigroup $(T_t)_{t>0}$, given on a set $C_b\left(\mathbb{R}^d\right)$ of continuous bounded functions defined on $\mathbb{R}^d$, with an infinitesimal generator of the form $c\Delta_\alpha + \left(q\delta_S\nu, \nabla_\beta\right)$, where $\Delta_\alpha = -(-\Delta)^{\frac{\alpha}{2}}$ is the α-order fractional Laplacian, i.e. the operator defined by the symbol $\left(-|\lambda|^\alpha\right)_{\lambda\in\mathbb{R}^d}$.

The problem of perturbation of Markov processes was and remains the focus of attention of many researchers. The diffusion case, that is, if $\alpha = 2$, was considered (in much more general situations) by M.I. Portenko (see [13, 17] and referenses there). The perturbation operators have the form $\left(a(\cdot), \nabla\right)$, where the function $a(\cdot)$ belongs to some L_p-space of functions or is some generalized function of the delta function type. The case $\alpha < 2$ was considered in the work [1] by K. Bogdan and T. Jakubowski, who studied such a perturbation with the function a from a Kato class. In [6], T. Jakubowski considered a fundamental solution of the fractional diffusion equation (like written above) with a singular drift, i.e. the case of $\beta = 1$ and some singular function a. S.I. Podolynny and M.I. Portenko [12, 14–16] investigated this perturbation with function from L_p. J.-U. Loebus and M.I. Portenko [7] perturbed the infinitesimal generator of a one-dimensional symmetric α-stable process using the operator $(q\delta_0, \partial_{\alpha-1})$, where $\partial_{\alpha-1}$ is a pseudo-differential operator of order $\alpha - 1$. Results with $\alpha \in (1, 2)$ and perturbation operators of the type $\left(a(\cdot), \nabla_{\alpha-1}\right)$ can be found in [8, 9]. We studied the case of perturbation operators $\left(a(\cdot), \nabla_\beta\right)$ with $a(\cdot)$ with L_p and $0 < \beta < \alpha$ in [2]. In this paper, focusing on the d-dimensional Euclidean space with $d \geq 2$, we consider the cases $1 < \alpha \leq 2$, $0 < \beta < \alpha - 1$ and the function $a(\cdot)$ which is of the delta-function type.

The next section is devoted to solving the perturbation equation. And in the last section, we investigate some properties of the constructed perturbed semigroup of operators defined on the space of continuous bounded functions.

1 Perturbation equations

Let us consider the equation

$$G(t, x, y) = g(t, x, y) + \int_0^t d\tau \int_S g(t-\tau, x, z)\left(\nu(\cdot), \nabla_\beta\right)G(\tau, \cdot, y)(z)q(z)\,\mathrm{d}\sigma_z, \tag{3}$$

with $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, which is called as the perturbation equation (see, for example, [8, 9, 13, 17]). Another perturbation equation

$$G(t, x, y) = g(t, x, y) + \int_0^t d\tau \int_S G(t-\tau, x, z)\left(\nu(\cdot), \nabla_\beta\right)g(\tau, \cdot, y)(z)q(z)\,\mathrm{d}\sigma_z,$$

is conjugate to (3). Solutions to equation (3) can be written as the function

$$G(t,x,y) = g(t,x,y) + \int_0^t d\tau \int_S g(t-\tau,x,z)v(\tau,z,y)q(z)\,\mathrm{d}\sigma_z, \tag{4}$$

given at $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, where the function $v(t,x,y)$ is defined by the equation

$$v(t,x,y) = v_0(t,x,y) + \int_0^t \mathrm{d}\tau \int_S v_0(t-\tau,x,z)v(\tau,z,y)q(z)\mathrm{d}\sigma_z, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{5}$$

in which $v_0(t,x,y) = \big(v(\cdot),\nabla_\beta\big)g(t,\cdot,y)(x)$.

First of all, let us prove the correctness of the function G definition, that is, the existence of a solution to equation (5) and the convergence of the integral in (4). Let us consider the case of $1 < \alpha < 2$.

Lemma 1. *If $0 < \beta < \alpha - 1 < 1$, then there is a unique solution to equation* (5) *in a class of functions satisfying the inequality*

$$\big|v(t,x,y)\big| \le C_T \frac{1}{\big(t^{\frac{1}{\alpha}} + |y-x|\big)^{d+\beta}}, \quad t \in (0;T],\ x \in S,\ y \in \mathbb{R}^d, \tag{6}$$

for each $T > 0$ with some constant $C_T > 0$ possibly depending on T.

Proof. Let us construct the solution to equation (5) using the method of successive approximations. Namely, consider the sequence of functions $\big(v_k(t,x,y)\big)$, $k = 0,1,2,\dots$, given by the recurrence relation

$$v_k(t,x,y) = \int_0^t \mathrm{d}\tau \int_S v_0(t-\tau,x,z)v_{k-1}(\tau,z,y)q(z)\,\mathrm{d}\sigma_z, \quad t > 0,\ x \in S,\ y \in \mathbb{R}^d.$$

The existence of such a sequence follows from a well-known estimation (see [3])

$$\big|v_0(t,x,y)\big| \le \frac{C}{\big(t^{\frac{1}{\alpha}} + |x-y|\big)^{d+\beta}}, \quad t > 0,\ x \in S,\ y \in \mathbb{R}^d, \tag{7}$$

and the inequality (see [11, Lemma 2])

$$\begin{aligned}\int_0^t \mathrm{d}\tau \int_S \frac{(t-\tau)^{\varkappa}}{\big((t-\tau)^{\frac{1}{\alpha}} + |x-z|\big)^{d+k}} \frac{\tau^{\lambda}}{\big(\tau^{\frac{1}{\alpha}} + |z-y|\big)^{d+l}}\,\mathrm{d}\sigma_z \\ \le K\bigg(B\Big(1+\varkappa-\frac{k+1}{\alpha},1+\lambda\Big)\frac{t^{1+\varkappa+\lambda-\frac{k+1}{\alpha}}}{\big(t^{\frac{1}{\alpha}} + |y-x|\big)^{d+l}} \\ + B\Big(1+\lambda-\frac{l+1}{\alpha},1+\varkappa\Big)\frac{t^{1+\varkappa+\lambda-\frac{l+1}{\alpha}}}{\big(t^{\frac{1}{\alpha}} + |y-x|\big)^{d+k}}\bigg),\end{aligned} \tag{8}$$

which is correct for all $k > -1$, $l > -1$, $\varkappa > \frac{k+1}{\alpha} - 1$, $\lambda > \frac{k+1}{\alpha} - 1$ with a constant $K > 0$, which depends only on d, k, l and the surface S.

Indeed, using the method of mathematical induction it is easy to prove that

$$\big|v_k(t,x,y)\big| \le R_k \frac{t^{k\left(1-\frac{1+\beta}{\alpha}\right)}}{\big(t^{\frac{1}{\alpha}} + |x-y|\big)^{d+\beta}}, \quad t > 0,\ x \in S,\ y \in \mathbb{R}^d, \tag{9}$$

where the constants $R_k > 0$ satisfy the recurrence relation ($k = 0,1,2,\dots$)

$$R_k = R_{k-1}C\|q\|K\bigg(B\Big(1-\frac{1+\beta}{\alpha},1+(k-1)\Big(1-\frac{1+\beta}{\alpha}\Big)\Big) + B\Big(k\Big(1-\frac{1+\beta}{\alpha}\Big),1\Big)\bigg)$$

with $R_0 = C$ (constant taken from the estimation v_0), and $\|q\| = \max_{x\in S}|q(x)|$. Inequalities (9) guarantee the good convergence of the series $\sum_{k=0}^{\infty} v_k(t,x,y)$. Namely, this series converges uniformly in $x \in S, y \in \mathbb{R}^d$ and locally uniformly in $t > 0$. Let us denote its sum by $v(t,x,y)$ with $t > 0$, $x \in S$, $y \in \mathbb{R}^d$. It follows from (9) that the function v satisfies estimation (6).

It is obvious that the function v is a solution to equation (5) by its constructing. The uniqueness of this solution in the class of functions that satisfy (6) follows from the fact that the difference w of every two such solutions satisfies the equation

$$w(t,x,y) = \int_0^t \mathrm{d}\tau \int_S v_0(t-\tau,x,z)w(\tau,z,y)q(z)\,\mathrm{d}\sigma_z, \quad t > 0,\ x \in S,\ y \in \mathbb{R}^d.$$

Hence, using (7) and (8), we obtain

$$\begin{aligned}|w(t,x,y)| \le C_T C^k \|q(z)\|^k K^k \Big(& B\Big(1-\frac{1+\beta}{\alpha}, 1+(k-1)\Big(1-\frac{1+\beta}{\alpha}\Big)\Big) \\ & + B\Big(k\Big(1-\frac{1+\beta}{\alpha}\Big),1\Big)\Big)\frac{1}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\beta}},\end{aligned}$$

for all $k \in \mathbb{N}$, $t \in (0,T]$, $T > 0$, $x \in S$, $y \in \mathbb{R}^d$. This proves the identity $w(t,x,y) \equiv 0$. The lemma is proven. □

The convergence of the integral in (4) now follows easily from the statement of Lemma 1, the well-known (see [3]) estimation

$$g(t,x,y) \le \frac{Ct}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\alpha}}, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{10}$$

and inequalities (8).

From equality (4), taking into account estimations (6), (10) and inequality (8), we obtain

$$\begin{aligned}|G(t,x,y)| &\le \frac{Ct}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\alpha}} + C_T\left(\frac{t^{1-\frac{1}{\alpha}}}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\beta}} + \frac{t^{2-\frac{1+\beta}{\alpha}}}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\alpha}}\right) \\ &\le C_T \frac{t^{\frac{\beta}{\alpha}}}{\big(t^{\frac{1}{\alpha}}+|x-y|\big)^{d+\beta}},\end{aligned}$$

which is true for all $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, for each $T > 0$ with some constant $C_T > 0$.

Consider the case of $\alpha = 2$. The function g is now given by equation (2). Then

$$\nabla_\beta g(t,\cdot,y)(x) = \frac{1}{2^{d+1}\pi^{\frac{d}{2}}}\Gamma\Big(\frac{d+\beta+1}{2}\Big)\,{}_1\tilde{F}_1\Big(\frac{d+\beta+1}{2}, 1+\frac{d}{2}; -\frac{|x-y|^2}{4ct}\Big)\frac{y-x}{(ct)^{\frac{d+\beta}{2}}}, \tag{11}$$

where ${}_1\tilde{F}_1(a,b,r) = \frac{1}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(b+k)k!}r^k$ is a regularized confluent hyper-geometric function (see [4]).

It is known that $\lim_{r\to\infty} {}_1\tilde{F}_1(a,b;-r)r^a = \frac{1}{\Gamma(b-a)}$. Hence, taking into account representation (11), we obtain the estimation

$$\big|\nabla_\beta g(t,\cdot,y)(x)\big| \le \frac{Ct^{\frac{1}{2}}}{\big(t^{\frac{1}{2}}+|x-y|\big)^{d+\beta}}, \quad t > 0,\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d, \tag{12}$$

where $C > 0$ is some constant. Estimation (12) is of the same type as (7). Therefore, the statement analogous to Lemma 1 in the case of $\alpha = 2$ is as follows.

Lemma 2. *For $\alpha = 2$, $0 < \beta < 1$, there is a unique solution to equation (5) in the class of functions satisfying the inequality*

$$\big|v(t,x,y)\big| \le C_T \frac{t^{\frac{1}{2}}}{\big(t^{\frac{1}{2}} + |x-y|\big)^{d+\beta}}, \quad t \in (0;T],\ x \in S,\ y \in \mathbb{R}^d, \tag{13}$$

for each $T > 0$ with some constant $C_T > 0$.

Proof. The proof of this statement is absolutely analogous as the proof of Lemma 1. □

Let us use estimation (10), which holds in the case of $\alpha = 2$, also. We obtain the following inequalities

$$\begin{aligned}\big|G(t,x,y)\big| &\le \frac{Ct}{\big(t^{\frac{1}{2}} + |x-y|\big)^{d+2}} + C_T\left(\frac{t}{\big(t^{\frac{1}{2}} + |x-y|\big)^{d+\beta}} + \frac{t^{2-\frac{\beta}{2}}}{\big(t^{\frac{1}{2}} + |x-y|\big)^{d+2}}\right)\\ &\le C_T \frac{t^{\frac{\beta}{2}}}{\big(t^{\frac{1}{2}} + |x-y|\big)^{d+\beta}},\end{aligned}$$

valid for all $t \in (0,T]$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$, and every $T > 0$.

Therefore we can claim that the function $G(t,x,y)$ with $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$ is correctly defined by equality (4). Moreover we have the following estimation

$$\big|G(t,x,y)\big| \le C_T \frac{t^{\frac{\beta}{\alpha}}}{\big(t^{\frac{1}{\alpha}} + |x-y|\big)^{d+\beta}}, \quad t \in (0,T],\ x \in \mathbb{R}^d,\ y \in \mathbb{R}^d,$$

for each $T > 0$ with some constant $C_T > 0$.

2 The operators semigroup

The set $C_b(\mathbb{R}^d)$ of continuous and bounded real-valued functions defined on $\mathbb{R}^d$ is a polish space with respect to the norm $\|\varphi\| = \max_{x\in\mathbb{R}^d}\big|\varphi(x)\big|$. Let us define the family of operators $(T_t)_{t>0}$ by the equality

$$T_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)G(t,x,y)\,\mathrm{d}y, \quad t > 0,\ x \in \mathbb{R}^d,\ \varphi \in C_b(\mathbb{R}^d), \tag{14}$$

where the function G is defined by the formula (4) for the case of $\alpha \in (1,2]$, $0 < \beta < \alpha - 1$. The main properties of the operators family (14) are considered in the following theorem.

Theorem 1. *The following statements are fulfilled.*

1. *The operator T_t is linear and bounded on $C_b\left(\mathbb{R}^d\right)$ for any $t > 0$.*
2. *If $\varphi(x) \equiv 1$, then $T_t\varphi(x) \equiv 1$ for all $t > 0$.*
3. *The equality $T_{s+t} = T_s \circ T_t$ is valid for all $s > 0$ and $t > 0$.*
4. *The equality $\lim_{t\downarrow 0} T_t\varphi(x) = \varphi(x)$ holds for every $\varphi \in C_b\left(\mathbb{R}^d\right)$ and for all $x \in \mathbb{R}^d$.*
5. *The function $U(t,x) = T_t\varphi(x)$ satisfies the equation*

$$\frac{\partial}{\partial t}U(t,x) = c\Delta_\alpha U(t,\cdot)(x) + q(x)\delta_S(x)\big(\nu(x),\nabla_\beta\big)U(t,\cdot)(x), \quad t > 0,\ x \in \mathbb{R}^d, \tag{15}$$

where the equality is understood in the generalized sense and the function $\nu(x)$, $x \in S$, is extended by arbitrary way (keeping the equality $|\nu(x)| \equiv 1$) on $\mathbb{R}^d$.

Proof. The linearity of operator T_t is evident. Let us prove its boundedness. If $\varphi \in C_b\left(\mathbb{R}^d\right)$, then using inequality (12) we can write

$$\left|T_t\varphi(x)\right| \le \int_{\mathbb{R}^d} \left|G(t,x,y)\right| \mathrm{d}y \|\varphi\| \le C_T\|\varphi\| \int_{\mathbb{R}^d} \frac{\mathrm{d}y}{(t^{1/\alpha}+|x-y|)^{d+\beta}} t^{\frac{\beta}{\alpha}}$$
$$\le C_T\|\varphi\| \int_{\mathbb{R}^d} \frac{\mathrm{d}u}{(1+|u|)^{d+\beta}} \le \hat{C}_T\|\varphi\|$$

for all $t \in [0,T]$ and any $T > 0$. Therefore the operators T_t are bounded and statement 1 is proven.

The proof of statement 3 is a verbatim repetition of the corresponding statement proof in [13], where the case of $\alpha = 2$ and $\beta = 1$ is considered in a more general situation. The following two facts are used in that proof. First, the family of operators $\left(T_t^{(0)}\right)_{t\ge 0}$, defined by the equalities $T_t^{(0)}\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)g(t,x,y)\,\mathrm{d}y$, $t > 0$, $x \in \mathbb{R}^d$, $\varphi \in C_b(\mathbb{R}^d)$, is a semigroup. Second, the function $V_t\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)v(t,x,y)\,\mathrm{d}y$, $t > 0$, $x \in \mathbb{R}^d$, is the unique solution to the equation (see (5))

$$V_t\varphi(x) = V_t^{(0)}\varphi(x) + \int_0^t \mathrm{d}\tau \int_S v_0(t-\tau,x,z)q(z)V_\tau\varphi(z)\,\mathrm{d}\sigma_z, \quad t > 0,\ x \in \mathbb{R}^d, \tag{16}$$

for every $\varphi \in C_b\left(\mathbb{R}^d\right)$, where $V_t^{(0)}\varphi(x) = \int_{\mathbb{R}^d} \varphi(y)v_0(t,x,y)\,\mathrm{d}y$. This equation is actually equation (5) which is multiplied by the function $\varphi(y)$ and integrated with respect to $y \in \mathbb{R}^d$. These facts hold true in our case.

If $\varphi(x) \equiv 1$, then the function $V_t\varphi(x)$, $t > 0$, $x \in \mathbb{R}^d$, is the unique solution to equation (16) with $V_t^{(0)}\varphi(x) \equiv 0$. This means that $V_t\varphi(x) \equiv 0$ and $T_t\varphi(x) \equiv T_t^{(0)}\varphi(x) \equiv 1$, i.e. statement 2 is proven.

Let us prove statement 4. We have to take into account that $\lim_{t\downarrow 0} T_t^{(0)}\varphi(x) = \varphi(x)$, $x \in \mathbb{R}^d$, and the following inequality $\left|V_t\varphi(x)\right| \le C_T\|\varphi\| \int_{\mathbb{R}^d} \left(1+|u|\right)^{-d-\beta} \mathrm{d}u \cdot t^{-\frac{\beta}{\alpha}}$ hold for all $\alpha < 2$, $t \in [0,T]$, $x \in \mathbb{R}^d$ and any $T > 0$. The last inequality is a consequence of estimation (6). If $\alpha = 2$ estimation (13) leads us to the inequality $\left|V_t\varphi(x)\right| \le C_T\|\varphi\| \int_{\mathbb{R}^d} \left(1+|u|\right)^{-d-\beta} \mathrm{d}u \cdot t^{\frac{1-\beta}{2}}$. Therefore, using the inequality (see [11, Lemma 1]) $\int_S g(t-\tau,x,z)\,\mathrm{d}\sigma_z \le C_T\, t^{-1/\alpha}$, $t \in (0,T]$, $x \in \mathbb{R}^d$, we obtain

$$\left|\int_0^t \mathrm{d}\tau \int_S g(t-\tau,x,z)q(z)V_\tau\varphi(z)\,\mathrm{d}\sigma_z\right| \le C_T\|\varphi\|\|q\|t^{1-\frac{\beta+1}{\alpha}}, \quad \text{if } \alpha < 2,$$

or

$$\left|\int_0^t \mathrm{d}\tau \int_S g(t-\tau,x,z)q(z)V_\tau\varphi(z)\,\mathrm{d}\sigma_z\right| \le C_T\|\varphi\|\|q\|t^{1-\frac{\beta}{2}}, \quad \text{if } \alpha = 2.$$

Both of these expressions tend to 0 as $t \downarrow 0$. This justifies statement 4.

It is well-known, that the function $u(t,x) = \int_{\mathbb{R}^d} \varphi(y)g(t,x,y)\,\mathrm{d}y$, $t > 0$, $x \in \mathbb{R}^d$, with any $\varphi \in C_b\left(\mathbb{R}^d\right)$ satisfies the equality

$$\frac{\partial}{\partial t}u(t,x) = c\Delta_\alpha u(t,\cdot)(x), \quad t > 0,\ x \in \mathbb{R}^d, \tag{17}$$

and the initial condition $u(0+,x) = \varphi(x)$, $x \in \mathbb{R}^d$. Let us consider the function

$$u_1(t,x) = \int_0^t \mathrm{d}\tau \int_S g(t-\tau,x,z)q(z)V_\tau\varphi(z)\,\mathrm{d}\sigma_z \tag{18}$$

for $\varphi \in C_b\left(\mathbb{R}^d\right)$ and all $t > 0$, $x \in \mathbb{R}^d$.

Since $\left|q(x)V_t\varphi(x)\right| \leq C_T\|q\|\|\varphi\|t^{-\beta/\alpha}$ (or $\left|q(x)V_t\varphi(x)\right| \leq C_T\|q\|\|\varphi\|t^{(1-\beta)/2}$ if $\alpha = 2$), $t \in (0,T]$, $x \in S$ for all $T > 0$ (see above), function (18) satisfies equation (17) for all $t > 0$ and $x \in \mathbb{R}^d \setminus S$ as a single-layer potential (see [10]). Moreover, as we saw above, $u_1(0+,x) \equiv 0$.

Now, for any continuous finite function $\psi(x)$, $x \in \mathbb{R}^d$, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \psi(x)\frac{\partial}{\partial t}u_1(t,x,\varphi)\,\mathrm{d}x &= c\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}x\int_0^t \mathrm{d}\tau\int_S \Delta_\alpha g(t-\tau,\cdot,z)(x)V_\tau\varphi(z)q(z)\,\mathrm{d}\sigma_z\\ &\quad+\lim_{\varepsilon\to 0+}\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}x\int_S g(\varepsilon,x,z)(x)V_\tau\varphi(z)q(z)\,\mathrm{d}\sigma_z\\ &= c\int_{\mathbb{R}^d}\psi(x)\Delta_\alpha\left(\int_0^t\mathrm{d}\tau\int_S g(t-\tau,\cdot,z)V_\tau\varphi(z)q(z)\,\mathrm{d}\sigma_z\right)(x)\,\mathrm{d}x\\ &\quad+\int_S\psi(z)V_t\varphi(z)q(z)\,\mathrm{d}\sigma_z\\ &= c\int_{\mathbb{R}^d}\psi(x)\Delta_\alpha u_1(t,\cdot)(x)\,\mathrm{d}x+\int_S\psi(z)V_t\varphi(z)q(z)\,\mathrm{d}\sigma_z.\end{aligned}$$

This is a consequence of the possibility of entering the operator Δ_α under the integral sign and of the equality $\int_{\mathbb{R}^d} g(\varepsilon,x,z)\varphi(z)\,dz \to \varphi(x)$ as $\varepsilon \to 0+$ for every continuous bounded function φ. But the equality $V_t\varphi(x) = \left(\nu(x),\nabla_\beta\right)U(t,\cdot)(x)$ is true for all $t > 0$, $x \in S$. Therefore, equality (15) is proven. □

Remark 1. *Unfortunately, we cannot assert the non-negativity of the function G. Therefore, it does not generate any stochastic Markov process, but only a pseudo-process.*

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Бойко М.В., Осипчук М.М. *Збурення ізотропного α-стійкого випадкового процесу псевдоградієнтом з узагальненим коефіцієнтом* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 53–60.

Стаття присвячена збуренню ізотропного α-стійкого випадкового процесу в скінченновимірному евклідовому просторі оператором псевдоградієнта, помноженим на дельта-функцію на гіперповерхні. Це аналогічно побудові деякої мембрани у фазовому просторі. Побудовано напівгрупу операторів на просторі неперервних обмежених функцій. Вона має інфінітезимальний генератор (у деякому узагальненому розумінні) $c\Delta_\alpha + (q\delta_S\nu, \nabla_\beta)$, де c — деяка додатна стала, Δ_α — дробовий лапласіан порядку α, δ_S — дельта-функція на гіперповерхні S, яка має нормальний вектор ν, q — деяка неперервна обмежена функція, ∇_β — дробовий градієнт (псевдоградієнт), тобто псевдодиференційний оператор, визначений символом $i\lambda|\lambda|^{\beta-1}$. Порядок псевдоградієнта менший, ніж $\alpha - 1$. Досліджено деякі властивості отриманої напівгрупи. Ця напівгрупа визначає псевдопроцес.

Ключові слова і фрази: α-стійкий випадковий процес, збурення, псевдоградієнт, напівгрупа операторів, псевдопроцес.