ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2024, **16** (1), 16–31
doi:10.15330/cmp.16.1.16-31

https://journals.pnu.edu.ua/index.php/cmp
Карпатськi матем. публ. 2024, Т.16, №1, С.16–31

Convergence sets and relative stability to perturbations of a branched continued fraction with positive elements

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In the paper, the problems of convergence and relative stability to perturbations of a branched continued fraction with positive elements and a fixed number of branching branches are investigated. The conditions under which the sets of elements

$$\Omega_0 = (0, \mu_0^{(2)}] \times [\nu_0^{(1)}, +\infty), \quad \Omega_{i(k)} = [\mu_k^{(1)}, \mu_k^{(2)}] \times [\nu_k^{(1)}, \nu_k^{(2)}], \quad i(k) \in I_k, \quad k = 1, 2, \ldots,$$

where $\nu_0^{(1)} > 0$, $0 < \mu_k^{(1)} < \mu_k^{(2)}$, $0 < \nu_k^{(1)} < \nu_k^{(2)}$, $k = 1, 2, \ldots$, are a sequence of sets of convergence and relative stability to perturbations of the branched continued fraction

$$\frac{a_0}{b_0} + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}} + \sum_{i_2=1}^{N} \frac{a_{i(2)}}{b_{i(2)}} + \cdots + \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}} + \cdots$$

have been established. The obtained conditions require the boundedness or convergence of the sequences whose members depend on the values $\mu_k^{(j)}$, $\nu_k^{(j)}$, $j = 1, 2$. If the sets of elements of the branched continued fraction are sets $\Omega_{i(k)} = (0, \mu_k] \times [\nu_k, +\infty)$, $i(k) \in I_k$, $k = 0, 1, \ldots$, where $\mu_k > 0$, $\nu_k > 0$, $k = 0, 1, \ldots$, then the conditions of convergence and stability to perturbations are formulated through the convergence of series whose terms depend on the values μ_k, ν_k. The conditions of relative resistance to perturbations of the branched continued fraction are also established if the partial numerators on the even floors of the fraction are perturbed by a shortage and on the odd ones by an excess, i.e. under the condition that the relative errors of the partial numerators alternate in sign. In all cases, we obtained estimates of the relative errors of the approximants that arise as a result of perturbation of the elements of the branched continued fraction.

Key words and phrases: branched continued fraction, convergence, stability to perturbations, convergence set, stability set to perturbations.

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Introduction

Continued fractions and their multidimensional generalizations, branched continued fractions (BCF), are effectively used in various fields of mathematics, applied mathematics, physics and engineering, quantum mechanics and computer science. Artificial intelligence has used continuous fractions, particularly machine learning, to approximate objective functions and model complex dependencies between data [18, 26–29, 31]. In cryptography, continued frac-

УДК 517.526
2020 *Mathematics Subject Classification:* 11A55, 11J70, 30B70, 40A15, 33C65, 41A20.

tions are used in RSA-like cryptosystems [9, 20, 21, 30]. In signal and image processing, continued fractions are used for data compression, approximation and reconstruction of signals and images, and noise detection and filtering [32]. The use of continued fractions and BCF in the theory of functions is especially effective for constructing rational approximations of special functions (see [1–5] and also [11–13, 16, 17, 25]).

Current problems in the analytical theory of continued and branched continued fractions are their convergence and stability to perturbations [6, 10, 23, 24, 33]. The problems of convergence of continued fractions with positive partial numerators and denominators is completely solved by the Seidel-Stern criterion [23, 24, 33]. For BCF with positive partial denominators and numerators equal to one, the necessary convergence conditions [6, Theorems 3.2–3.3], the sufficient convergence conditions [6, Theorems 3.4–3.6] are obtained. At the same time, the Theorem 3.6 in [6] is a multidimensional analogue of the sufficiency of the Seidel-Stern criterion for continued fractions with positive partial denominators and numerators equal to one. An analogue of the Seidel-Stern criterion for two-dimensional continued fractions [22] and BCF of a special form [7, Theorem 2] is established. However, the necessary and sufficient condition of the Seidel-Stern criterion for the BCF of the general form of the multidimensional analogue has not been established.

Continued fractions have the property of stability – non-accumulation or limited accumulation of errors arising in the process of their calculations [19]. This property provides prospects for the applications of this mathematical tool in various fields. When studying the stability of the BCF, the errors of the approximate fractions that arise as a result of the perturbation of the elements of the fraction are studied [6]. This task was called the study of stability to perturbations and is interpreted as a continuous dependence of the BCF on its elements [14].

The analysis of error estimates of continued fractions and BCF shows that they depend not only on the errors of the elements but also on the elements themselves. Therefore, the problems of studying the conditions on the elements under which continued and branched continued fractions will have stability to perturbations and establishing sets of stability to perturbations are relevant. The problem of stability to perturbations of BCF with positive partial denominators and numerators equal to one is completely solved. It is established that the domain $G = (0; +\infty)$ is the domain of stability to perturbations of the mentioned BCF [6, Theorem 3.28]. In [8, Theorem 1], some sufficient conditions for stability to perturbations of BCF of the general form with positive elements are established. This work aims to investigate further the convergence and stability of BCF of the general form with positive elements and to establish new sets of convergence and stability.

We consider the BCF

$$a_0\left(b_0+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{b_{i(k)}}\right)^{-1} := \cfrac{a_0}{b_0+\sum\limits_{i_1=1}^{N}\cfrac{a_{i(1)}}{b_{i(1)}+\sum\limits_{i_2=1}^{N}\cfrac{a_{i(2)}}{b_{i(2)}+\ddots}}}, \tag{1}$$

where N is the number of branching branches on the fraction floor,

$$I_0 = \{0\}, \quad I_k = \{i(k) = (i_1, i_2, \ldots, i_k) : \ i_p = \overline{1,N}, \ p = \overline{1,k}\}, \quad k = 1, 2, \ldots,$$

is a sequence of multiindex sets, $i(0) = 0$, $a_{i(k)}$, $b_{i(k)}$, $i(k) \in I_k$, $k = 0, 1, \ldots$, are the partial numerators and denominators of BCF, respectively. For convenience, BCF (1) is also written in the form

$$\frac{a_0}{b_0+} \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}+} \sum_{i_2=1}^{N} \frac{a_{i(2)}}{b_{i(2)}+} \cdots .$$

Finite branched continued fractions

$$f^{(0)} = \frac{a_0}{b_0}, \quad f^{(s)} = a_0 \left(b_0 + \mathop{\mathrm{D}}_{k=1}^{s} \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}} \right)^{-1}, \quad s = 1,2,\ldots,$$

are called approximants of (1). The values

$$Q_{i(k)}^{(s)} = b_{i(k)} + \sum_{i_{k+1}=1}^{N} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(s)}}, \quad i(k) \in I_k, \quad k = \overline{0, s-1}, \quad s = 1,2,\ldots, \tag{2}$$

with initial conditions $Q_{i(s)}^{(s)} = b_{i(s)}$, $i(s) \in I_s$, $s = 0,1,\ldots$, are called tails of sth approximants of BCF (1). We set

$$g_{i(k)}^{(s)} = \frac{a_{i(k)}}{Q_{i(k-1)}^{(s)} Q_{i(k)}^{(s)}}, \quad i(k) \in I_k, \quad k = \overline{1, s}, \quad s = 1,2,\ldots . \tag{3}$$

A BCF (1) is called convergent if there is a finite limit of the sequence of its approximants. The value of this limit is called the BCF value.

In what follows, when studying the convergence of the BCF (1), we will use the formula for the difference of two approximants, which, taking into account the form of the studied fraction, will be written as follows

$$f^{(n)} - f^{(m)} = (-1)^{m+1} \sum_{i_1, i_2, \ldots, i_{m+1}=1}^{N} \frac{a_{i(m+1)}}{Q_{i(m+1)}^{(n)}} \prod_{k=0}^{m} \frac{a_{i(k)}}{Q_{i(k)}^{(n)} Q_{i(k)}^{(m)}}, \quad n > m, \tag{4}$$

under the assumption that $Q_{i(k)}^{(n)} \neq 0$, $i(k) \in I_k$, $k = \overline{0, n}$, $Q_{i(k)}^{(m)} \neq 0$, $i(k) \in I_k$, $k = \overline{0, m}$.

For the BCF (1) with positive elements, the formula (4) implies the fork property, which is expressed by a system of inequalities

$$f^{(2m-1)} < f^{(2m+1)} < f^{(2n)} < f^{(2n-2)}, \quad m = 1,2,\ldots, \quad n = 1,2,\ldots .$$

From the fork property it follows the next statement.

Proposition 1. *BCF* (1) *with positive elements converges if and only if*

$$\lim_{m \to \infty} |f^{(m+1)} - f^{(m)}| = 0.$$

When studying convergence and relative stability to perturbations, we will use the following statement.

Proposition 2 ([6])**.** *Let* $\mathbf{x} \in \mathbb{R}_+^n$, $\mathbb{R}_+^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : \; x_i > 0, i = \overline{1, n}\}$, *and let* ξ *be a nonnegative number. Then*

$$\sum_{j=1}^{n} \left(\xi + \sum_{i=1}^{n} \frac{x_j}{x_i} \right)^{-1} \leq \frac{n}{n+\xi}. \tag{5}$$

Let $\{\Omega_{i(k)}\}$, $\Omega_{i(k)} \subset \mathbb{R}^2$, $i(k) \in I_k$, $k = 0, 1, \ldots$, be a sequence of element sets of BCF (1), i.e. $(a_{i(k)}, b_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k = 0, 1, \ldots$. The sequence of element sets $\{\Omega_{i(k)}\}$ is called the sequence of convergence sets of BCF (1), if the conditions $(a_{i(k)}, b_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k = 0, 1, \ldots$, ensure the convergence of this fraction.

Let $\widehat{a}_{i(k)}$, $\widehat{b}_{i(k)}$, $i(k) \in I_k$, $k = 0, 1, \ldots$, be perturbations of element values $a_{i(k)}$, $b_{i(k)}$ of BCF (1). BCF

$$\widehat{a}_0 \left(\widehat{b}_0 + \mathop{D}_{k=1}^{\infty} \sum_{i_k=1}^{N} \frac{\widehat{a}_{i(k)}}{\widehat{b}_{i(k)}} \right)^{-1}$$

is called the BCF perturbation to the fraction (1).

Let $\alpha_{i(k)}$, $\beta_{i(k)}$, $\varepsilon^{(s)}_{i(p)}$, $\varepsilon^{(s)}$ be relative errors of elements $a_{i(k)}$, $b_{i(k)}$, value $Q^{(s)}_{i(p)}$, and approximant $f^{(s)}$ of BCF (1), respectively, i.e. $\widehat{a}_{i(k)} = a_{i(k)}(1 + \alpha_{i(k)})$, $\widehat{b}_{i(k)} = b_{i(k)}(1 + \beta_{i(k)})$, $i(k) \in I_k$, $k = 0, 1, \ldots$, $\widehat{Q}^{(s)}_{i(p)} = Q^{(s)}_{i(p)}(1 + \varepsilon^{(s)}_{i(p)})$, $i(p) \in I_p$, $p = \overline{0, s}$, $s = 0, 1, \ldots$, $\widehat{f}^{(s)} = f^{(s)}(1 + \varepsilon^{(s)})$, $s = 0, 1, \ldots$, under the assumption that all $a_{i(k)} \neq 0$, $b_{i(k)} \neq 0$, $Q^{(s)}_{i(p)} \neq 0$. Further, let the values $\widehat{\alpha}_{i(k)}$, $\widehat{\beta}_{i(k)}$, $\widehat{\varepsilon}^{(s)}_{i(p)}$ be given by the relations $a_{i(k)} = \widehat{a}_{i(k)}(1 + \widehat{\alpha}_{i(k)})$, $b_{i(k)} = \widehat{b}_{i(k)}(1 + \widehat{\beta}_{i(k)})$, $i(k) \in I_k$, $k = 0, 1, \ldots$, $Q^{(s)}_{i(p)} = \widehat{Q}^{(s)}_{i(p)}(1 + \widehat{\varepsilon}^{(s)}_{i(p)})$, $i(p) \in I_p$, $p = \overline{0, s}$, $s = 0, 1, \ldots$, under the assumption that all $\widehat{a}_{i(k)} \neq 0$, $\widehat{b}_{i(k)} \neq 0$, $\widehat{Q}^{(s)}_{i(p)} \neq 0$.

Definition 1. *All sets $\{\Omega_{i(k)}\}$, $i(k) \in I_k$, $k = 0, 1, \ldots$, are called a sequence of sets of relative stability to perturbations of BCF* (1)*, if for each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for all $(a_{i(k)}, b_{i(k)}) \in \Omega_{i(k)}$, $a_{i(k)} \neq 0$, $b_{i(k)} \neq 0$, $i(k) \in I_k$, $k = 0, 1, \ldots$, and all $(\widehat{a}_{i(k)}, \widehat{b}_{i(k)}) \in \Omega_{i(k)}$, $i(k) \in I_k$, $k = 0, 1, \ldots$, such that*

$$\left| \frac{\widehat{a}_{i(k)} - a_{i(k)}}{a_{i(k)}} \right| < \delta, \quad \left| \frac{\widehat{b}_{i(k)} - b_{i(k)}}{b_{i(k)}} \right| < \delta, \quad i(k) \in I_k, \quad k = 0, 1, \ldots,$$

the inequalities $|(\widehat{f}^{(s)} - f^{(s)})/f^{(s)}| < \varepsilon$, $s = 0, 1, \ldots$, are satisfied.

For the relative error of the sth approximant of BCF (1) the following equality

$$\begin{aligned} \varepsilon^{(s)} = \alpha_0 + (1 + \alpha_0) \Bigg(\beta'_0 \Bigg(1 - \sum_{i_1=1}^{N} q^{(s)}_{i(1)} \Bigg) \\ + \sum_{k=1}^{s} \sum_{i_1, \ldots, i_k = 1}^{N} \prod_{l=1}^{k} q^{(s)}_{i(l)} \Bigg(\beta'_{i(k)} \Bigg(1 - \sum_{i_{k+1}=1}^{N} q^{(s)}_{i(k+1)} \Bigg) + \gamma_{i(k)} \Bigg) \Bigg) \end{aligned} \tag{6}$$

holds (see [15]), where

$$q^{(s)}_{i(l)} = \begin{cases} g^{(s)}_{i(l)}, & l = 2n, \\ \widehat{g}^{(s)}_{i(l)}, & l = 2n + 1, \end{cases}$$

$$\beta'_{i(k)} = \begin{cases} \beta_{i(k)}, & k = 2n + 1, \\ \widehat{\beta}_{i(k)}, & k = 2n, \end{cases}$$

$$\gamma_{i(k)} = \begin{cases} \alpha_{i(k)}(1 + \varepsilon^{(s)}_{i(k)}), & k = 2n, \\ \widehat{\alpha}_{i(k)}(1 + \widehat{\varepsilon}^{(s)}_{i(k)}), & k = 2n + 1, \end{cases}$$

$$q^{(s)}_{i(s+1)} = 0, \quad s = 0, 1, \ldots.$$

1 Convergence sets

Let sets

$$\Omega_0=(0,\mu_0^{(2)}]\times[\nu_0^{(1)},+\infty),\quad \Omega_{i(k)}=[\mu_k^{(1)},\mu_k^{(2)}]\times[\nu_k^{(1)},\nu_k^{(2)}],\quad i(k)\in I_k,\quad k=1,2,\ldots, \tag{7}$$

where $\nu_0^{(1)}>0$, $0<\mu_k^{(1)}<\mu_k^{(2)}$, $0<\nu_k^{(1)}<\nu_k^{(2)}$, $k=1,2,\ldots$, be a sequence of element sets of BCF (1).

In what follows, $[\,\cdot\,]$ denotes an integer part of a number.

Theorem 1. *The sets* (7) *are sequence of convergence sets of BCF* (1)*, if*

$$\lim_{s\to+\infty}\prod_{k=1}^{2s}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}=0, \tag{8}$$

where

$$r_k^{(p(2s,k))}=\nu_k^{(1)}+\cfrac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}+\cfrac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}+\cdots+\cfrac{N\mu_{p(2s,k)}^{(2)}}{\nu_{p(2s,k)}^{(1)}}}},\quad k=\overline{1,2s-2},\quad s=2,3,\ldots,$$
$$r_{2s-1}^{(p(2s,2s-1))}=\nu_{2s-1}^{(1)},\quad r_{2s}^{(p(2s,2s))}=\nu_{2s}^{(1)},\quad s=1,2,\ldots, \tag{9}$$

$$p(n,m)=n+(-1)^{n+1}(n-m-2[(n-m)/2]),\quad m=\overline{1,n},\quad n=1,2,\ldots. \tag{10}$$

Proof. From (4) it follows the formula for the difference of two approximants of the BCF (1), which, taking into account the notations (3), (10) and setting $n=2s$, $m=2s-1$, we write as

$$f^{(2s)}-f^{(2s-1)}=\sum_{i_1,i_2,\ldots,i_{2s}=1}^{N}\prod_{k=0}^{2s}a_{i(k)}\left(\prod_{k=0}^{2s-1}Q_{i(k)}^{(2s-1)}\prod_{k=0}^{2s}Q_{i(k)}^{(2s)}\right)^{-1}$$
$$=\frac{a_0}{Q_0^{(2s)}}\sum_{i_1,i_2,\ldots,i_{2s}=1}^{N}\prod_{k=1}^{s}g_{i(2k)}^{(2s)}\prod_{k=1}^{s}g_{i(2k-1)}^{(2s-1)}=\frac{a_0}{Q_0^{(2s)}}\sum_{i_1,i_2,\ldots,i_{2s}=1}^{N}\prod_{k=1}^{2s}g_{i(k)}^{(p(2s,k))}.$$

Using the recurrence relations (2), the values $g_{i(k)}^{(p(2s,k))}$, $i(k)\in I_k$, $k=\overline{1,2s}$, we write in the form

$$g_{i(k)}^{(p(2s,k))}=\frac{a_{i(k)}}{Q_{i(k)}^{(p(2s,k))}}\left(b_{i(k-1)}+\sum_{j=1}^{N}\frac{a_{i(k-1)j}}{Q_{i(k-1)j}^{(p(2s,k))}}\right)^{-1}$$
$$=\left(\frac{b_{i(k-1)}Q_{i(k)}^{(p(2s,k))}}{a_{i(k)}}+\sum_{j=1}^{N}\frac{Q_{i(k)}^{(p(2s,k))}a_{i(k-1)j}}{a_{i(k)}Q_{i(k-1)j}^{(p(2s,k))}}\right)^{-1}.$$

Then the formula for the difference of two approximants of the BCF (1) will have the form

$$f^{(2s)}-f^{(2s-1)}=\frac{a_0}{Q_0^{(2s)}}\sum_{i_1,i_2,\ldots,i_{2s}=1}^{N}\prod_{k=1}^{2s}\left(\frac{b_{i(k-1)}Q_{i(k)}^{(p(2s,k))}}{a_{i(k)}}+\sum_{j=1}^{N}\frac{Q_{i(k)}^{(p(2s,k))}a_{i(k-1)j}}{a_{i(k)}Q_{i(k-1)j}^{(p(2s,k))}}\right)^{-1}.$$

Since $p(2s,k)-k=2[(2s-k)/2]$, then

$$Q_{i(k)}^{(p(2s,k))}=b_{i(k)}+\sum_{i_{k+1}=1}^{N}\cfrac{a_{i(k+1)}}{b_{i(k+1)}+\sum_{i_{k+2}=1}^{N}\cfrac{a_{i(k+2)}}{b_{i(k+2)}+\cdots+\sum_{i_{k+2[(2s-k)/2]}=1}^{N}\cfrac{a_{i(k+2[(2s-k)/2])}}{b_{i(k+2[(2s-k)/2])}}}}$$
$$\geq\nu_k^{(1)}+\cfrac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}+\cfrac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}+\cdots+\cfrac{N\mu_{p(2s,k)}^{(2)}}{\nu_{p(2s,k)}^{(1)}}}}=r_k^{(p(2s,k))},$$

where $k = \overline{1, 2s-2}$, $s = 2, 3, \ldots$. Moreover,

$$Q_{i(2s-1)}^{(p(2s,2s-1))} = b_{i(2s-1)} \geq \nu_{2s-1}^{(1)} = r_{2s-1}^{(p(2s,2s-1))}, \quad Q_{i(2s)}^{(p(2s,2s))} = b_{i(2s)} \geq \nu_{2s}^{(1)} = r_{2s}^{(p(2s,2s))},$$

where $s = 1, 2, \ldots$. Using the inequality (5), we have

$$\begin{aligned}\sum_{i_k=1}^{N} g_{i(k)}^{(p(2s,k))} &\leq \sum_{i_k=1}^{N} \left(\frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{\mu_k^{(2)}} + \sum_{j=1}^{N} \frac{Q_{i(k)}^{(p(2s,k))} a_{i(k-1)j}}{a_{i(k)} Q_{i(k-1)j}^{(p(2s,k))}} \right)^{-1} \\ &\leq \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N \mu_k^{(2)}} \right)^{-1},\end{aligned}$$

where $i(k-1) \in I_{k-1}$, $k = \overline{1, 2s}$, $s = 1, 2, \ldots$.

Thus, for the difference of the approximants $f^{(2s)}$ and $f^{(2s-1)}$ of (1) the estimate

$$f^{(2s)} - f^{(2s-1)} \leq \frac{\mu_0^{(2)}}{r_0^{(2s)}} \prod_{k=1}^{2s} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N \mu_k^{(2)}} \right)^{-1} \tag{11}$$

holds, where $r_0^{(2s)}$ is defined in (9) when $k = 0$.

Let us put $n = 2s+1$, $m = 2s$ in the formula (4), then

$$|f^{(2s+1)} - f^{(2s)}| = f^{(2s)} - f^{(2s+1)} < f^{(2s)} - f^{(2s-1)} \leq \frac{\mu_0^{(2)}}{r_0^{(2s)}} \prod_{k=1}^{2s} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N \mu_k^{(2)}} \right)^{-1}.$$

Since the sequence $\mu_0^{(2)} / r_0^{(2s)}$, $s = 1, 2, \ldots$, is bounded, it follows from the estimate (11) and Proposition 1 that the sets (7) form a sequence of convergence sets of the BCF (1), if the condition (8) is satisfied. □

Corollary 1. *The sets* (7) *are sequence of convergence sets of BCF* (1)*, if*

$$\lim_{s \to +\infty} \sum_{k=1}^{2s} \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{\mu_k^{(2)}} = +\infty, \tag{12}$$

where the values $r_k^{(p(2s,k))}$*,* $k = \overline{1, 2s}$*,* $s = 1, 2, \ldots$*, are defined by* (9)*.*

Proof. We consider two cases.

1. Let the sequence $\nu_{k-1}^{(1)} r_k^{(p(2s,k))} / N \mu_k^{(2)}$, $s = 1, 2, \ldots$, $k = \overline{1, 2s}$, is bounded from below by some number $c > 0$. Then

$$\left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N \mu_k^{(2)}} \right)^{-1} \leq \frac{1}{1+c}.$$

Since

$$\lim_{s \to +\infty} \frac{1}{(1+c)^{2s}} = 0,$$

the condition (8) are satisfied and, therefore, the sets (7) form a sequence of convergence sets of BCF (1).

2. Let

$$\lim_{s\to+\infty,\,k\to+\infty}\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}=0.$$

Then the sequence

$$\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}},\quad s=1,2,\ldots,\quad k=\overline{1,2s},$$

is bounded from above by some number $C>0$, and

$$\begin{aligned}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}&\le 1-\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}\\&\le\exp\left(-\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}\right)\\&\le\exp\left(-\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{N(1+C)\mu_k^{(2)}}\right).\end{aligned}$$

Thus, for the difference of two approximants of the BCF (1), the estimate

$$f^{(2s)}-f^{(2s-1)}\le\frac{\mu_0^{(2)}}{r_0^{(2s)}}\exp\left(-\frac{1}{N(1+C)}\sum_{k=1}^{2s}\frac{\nu_{k-1}^{(1)}r_k^{(p(2s,k))}}{\mu_k^{(2)}}\right)$$

is valid, from which it follows that the condition (12) ensures the convergence of branched continued fraction (1). □

Note that Corollary 1 is a generalization of the convergence criteria of branched continued fraction with positive partial denominators and partial numerators equal to one to the case of BCF of the general form [6].

Corollary 2. *The sets*

$$\Omega_{i(k)}=(0,\mu_k]\times[\nu_k,+\infty),\quad \mu_k>0,\quad \nu_k>0,\quad i(k)\in I_k,\quad k=0,1,\ldots,\tag{13}$$

are sequence of convergence sets of the BCF (1)*, if the series*

$$\sum_{k=1}^{\infty}\frac{\nu_{k-1}\nu_k}{\mu_k}$$

is divergent.

2 Sets of relative stability to perturbations

We will prove the following theorem.

Theorem 2. *Let there exist constants* $\alpha, \beta, 0 \le \alpha < 1, 0 \le \beta < 1, \alpha + \beta \ne 0$, *such that*

$$|\alpha_{i(k)}| \le \alpha, \quad i(k) \in I_k, \quad k = 0, 1, 2, \ldots, \tag{14}$$

$$|\beta_{i(k)}| \le \beta, \quad i(k) \in I_k, \quad k = 0, 1, 2, \ldots. \tag{15}$$

If the sequence

$$\sum_{n=1}^{s} \frac{\mu_n^{(2)}}{r_{n-1}^{(s)} r_n^{(s)}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1}, \quad s = 1, 2, \ldots, \tag{16}$$

is bounded, where

$$r_k^{(s)} = \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} \underset{+}{} \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} \underset{+}{} \cdots \underset{+}{} \frac{N\mu_s^{(p_2(s,k))}}{\nu_s^{(p_1(s,k))}}, \tag{17}$$

$k = \overline{0, s-1}$, $r_s^{(s)} = \nu_s^{(1)}$, $s = 1, 2, \ldots$, $p_j(s,k) = j + (-1)^{j+1}(s - k - 2[(s-k)/2])$, $k = \overline{0, s-1}$, $s = 1, 2, \ldots$, $j \in \{1, 2\}$, *then the sets* (7) *form a sequence of sets of relative stability to perturbations of the BCF* (1). *Further, for* $s = 0, 1, 2, \ldots$ *the estimate*

$$|\varepsilon^{(s)}| \le \alpha + (1+\alpha)\left(\frac{\beta}{1-\beta} + \frac{\alpha N}{1-\alpha} \sum_{n=1}^{s} \frac{\mu_n^{(2)}}{r_{n-1}^{(s)} r_n^{(s)}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1}\right) \tag{18}$$

is valid.

Proof. From the formula (6) it follows the following estimate of the relative error of the sth approximant of the BCF (1)

$$|\varepsilon^{(s)}| \le |\alpha_0| + (1 + |\alpha_0|)\max\{|\tilde{\beta}_{i(k)}| : \ i(k) \in I_k, \ k = \overline{0, s}\}$$
$$+ (1 + |\alpha_0|)\max\{|\tilde{\alpha}_{i(k)}| : \ i(k) \in I_k, \ k = \overline{1, s}\}$$
$$\times \sum_{n=1}^{s} \sum_{i_1, i_2, \ldots, i_n = 1}^{N} \gamma_{i(n)} \prod_{k=1}^{n} q_{i(k)}^{(s)}, \quad s = 0, 1, \ldots.$$

Let us estimate the values

$$\sum_{i_k=1}^{N} g_{i(k)}^{(s)}, \quad i(k-1) \in I_k, \quad k = \overline{1, s}.$$

For $s - k = 2m + 1$, $m = 0, 1, 2, \ldots$, we have

$$Q_{i(k)}^{(s)} = b_{i(k)} + \sum_{i_{k+1}=1}^{N} \frac{a_{i(k+1)}}{b_{i(k+1)}} \underset{+}{} \sum_{i_{k+2}=1}^{N} \frac{a_{i(k+2)}}{b_{i(k+2)}} \underset{+}{} \sum_{i_s=1}^{N} \frac{a_{i(k+2m+1)}}{b_{i(k+2m+1)}}$$
$$\ge \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} \underset{+}{} \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} \underset{+}{} \cdots \underset{+}{} \frac{N\mu_{k+2m+1}^{(1)}}{\nu_{k+2m+1}^{(2)}}, \quad i(k) \in I_k.$$

Let $s-k=2m$, $m=1,2,\ldots$. Then

$$Q_{i(k)}^{(s)}=b_{i(k)}+\sum_{i_{k+1}=1}^{N}\frac{a_{i(k+1)}}{b_{i(k+1)}}+\sum_{i_{k+2}=1}^{N}\frac{a_{i(k+2)}}{b_{i(k+2)}}+\cdots+\sum_{i_s=1}^{N}\frac{a_{i(k+2m)}}{b_{i(k+2m)}}$$
$$\geq \nu_k^{(1)}+\frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}}+\frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}}+\cdots+\frac{N\mu_{k+2m}^{(2)}}{\nu_{k+2m}^{(1)}},\quad i(k)\in I_k.$$

Moreover, for $k=s$ we have $Q_{i(s)}^{(s)}=b_{i(s)}\geq \nu_s^{(1)}$, $i(s)\in I_s$. Thus, $Q_{i(k)}^{(s)}\geq r_k^{(s)}$, $i(k)\in I_k$, $k=\overline{0,s}$, where the values $r_k^{(s)}$ are defined by (17). Using the inequality (5), we obtain

$$\sum_{i_k=1}^{N}g_{i(k)}^{(s)}\leq\sum_{i_k=1}^{N}\left(\frac{\nu_{k-1}^{(1)}r_k^{(s)}}{\mu_k^{(2)}}+\sum_{j=1}^{N}\frac{Q_{i(k)}^{(s)}a_{i(k-1)j}}{a_{i(k)}Q_{i(k-1)j}^{(s)}}\right)^{-1}\leq\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1},$$

for $i(k-1)\in I_{k-1}$, $k=\overline{1,s}$. Since $(\widehat{a}_{i(k)},\widehat{b}_{i(k)})\in\Omega_{i(k)}$, $i(k)\in I_k$, $k=0,1,2,\ldots$, then $\widehat{Q}_{i(k)}^{(s)}\geq r_k^{(s)}$, $i(k)\in I_k$, $k=\overline{0,s}$, and

$$\sum_{i_k=1}^{N}\widehat{g}_{i(k)}^{(s)}\leq\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1},\quad i(k-1)\in I_{k-1},\quad k=\overline{1,s}.$$

Thus,

$$\sum_{i_k=1}^{N}q_{i(k)}^{(s)}\leq\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1},\quad i(k-1)\in I_{k-1},\quad k=\overline{1,s}.\tag{19}$$

The values

$$\sum_{i_k=1}^{N}q_{i(k)}^{(s)}\gamma_{i(k)}^{(s)},\quad i(k-1)\in I_{k-1},\quad k=\overline{1,s},$$

will be estimated taking into account the parity of the number k. If $k=2m$, $m=1,2,\ldots$, we have

$$\sum_{i_{2m}=1}^{N}q_{i(2m)}^{(s)}\gamma_{i(2m)}^{(s)}=\sum_{i_k=1}^{N}\frac{a_{i(2m)}}{Q_{i(2m-1)}^{(s)}Q_{i(2m)}^{(s)}(1+\varepsilon_{i(2m)}^{(s)})}=\sum_{i_k=1}^{N}\frac{a_{i(2m)}}{Q_{i(2m-1)}^{(s)}\widehat{Q}_{i(2m)}^{(s)}}\leq\frac{N\mu_{2m}^{(2)}}{r_{2m-1}^{(s)}r_{2m}^{(s)}}.$$

When $k=2m+1$, $m=0,1,2,\ldots$, we obtain

$$\sum_{i_{2m+1}=1}^{N}q_{i(2m+1)}^{(s)}\gamma_{i(2m+1)}^{(s)}=\sum_{i_k=1}^{N}\frac{\widehat{a}_{i(2m+1)}}{\widehat{Q}_{i(2m)}^{(s)}\widehat{Q}_{i(2m+1)}^{(s)}(1+\varepsilon_{i(2m+1)}^{(s)})}$$
$$=\sum_{i_k=1}^{N}\frac{\widehat{a}_{i(2m+1)}}{\widehat{Q}_{i(2m)}^{(s)}Q_{i(2m+1)}^{(s)}}\leq\frac{N\mu_{2m+1}^{(2)}}{r_{2m}^{(s)}r_{2m+1}^{(s)}}.$$

Thus,

$$\sum_{i_k=1}^{N}q_{i(k)}^{(s)}\gamma_{i(k)}^{(s)}\leq\frac{N\mu_k^{(2)}}{r_{k-1}^{(s)}r_k^{(s)}},\quad i(k-1)\in I_{k-1},\quad k=\overline{1,s}.\tag{20}$$

Now, taking into account the inequalities (14), (15), (19), and (20), we obtain (18). Since the sequence (16) is bounded, there exists a constant $M > 0$ such that

$$\sum_{n=1}^{s} \frac{\mu_n^{(2)}}{r_{n-1}^{(s)} r_n^{(s)}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1} \leq M, \quad s = 1, 2, \ldots,$$

and

$$|\varepsilon^{(s)}| \leq \alpha + (1+\alpha)(\beta(1-\beta)^{-1} + \alpha(1-\alpha)^{-1} NM).$$

It is easy to show that for $|\alpha_{i(k)}| \leq \alpha < f(\varepsilon)$, $|\beta_{i(k)}| \leq \beta < f(\varepsilon)$, $i(k) \in I_k$, $k = 0, 1, 2, \ldots$, where

$$f(\varepsilon) = \frac{1}{2}\left(\sqrt{\left(1 + \frac{2+\varepsilon}{NM}\right)^2 + \frac{4\varepsilon}{NM}} - \frac{2+\varepsilon}{NM} - 1\right),$$

ε is an arbitrary positive constant, the inequalities $|\varepsilon^{(s)}| < \varepsilon$, $s = 0, 1, 2, \ldots$, for relative errors of approximants of the BCF (1) hold, which proves the fulfillment of the conditions for determining the sequence of sets of relative stability to perturbations of this BCF. □

The following theorem gives conditions under which the sets (7) form a sequence of sets of convergence and relative stability to perturbations of the BCF (1).

Theorem 3. *Let the relative errors of the elements of the BCF* (1) *satisfy the conditions* (14), (15). *If there exists a limit of the sequence*

$$\sum_{n=1}^{s} \frac{\mu_n^{(2)}}{r_{n-1}^{(p(s,n-1))} r_n^{(p(s,n))}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(s,k))}}{N\mu_k^{(2)}}\right)^{-1}, \quad s = 1, 2, \ldots, \tag{21}$$

where

$$\begin{aligned} r_k^{(p(s,k))} &= \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} \underset{+}{} \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} \underset{+}{} \cdots \underset{+}{} \frac{N\mu_{p(s,k)}^{(2)}}{\nu_{p(s,k)}^{(1)}}, \quad k = \overline{0, s-2}, \quad s = 2, 3, \ldots, \\ r_{s-1}^{(p(s,s-1))} &= \begin{cases} \nu_s^{(1)}, & s = 2l, \\ \nu_{s-1}^{(1)} + \frac{N\mu_s^{(1)}}{\nu_s^{(2)}} \underset{+}{} \frac{N\mu_{s+1}^{(2)}}{\nu_{s+1}^{(1)}}, & s = 2l+1, \end{cases} \\ r_s^{(p(s,s))} &= \nu_s^{(1)}, \quad s = 1, 2, \ldots, \end{aligned} \tag{22}$$

and

$$p(s,k) = s + (-1)^{s+1}(s - k - 2[(s-k)/2]), \quad k = \overline{0, s},$$

then the sets (7) *form a sequence of convergence sets and relative stability to perturbations of BCF* (1). *Further, for* $s = 0, 1, 2, \ldots$ *the estimate*

$$|\varepsilon^{(s)}| \leq \alpha + (1+\alpha)\left(\frac{\beta}{1-\beta} + \frac{\alpha N}{1-\alpha} \sum_{n=1}^{s} \frac{\mu_n^{(2)}}{r_{n-1}^{(p(s,n-1))} r_n^{(p(s,n))}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(s,k))}}{N\mu_k^{(2)}}\right)^{-1}\right)$$

is valid.

Proof. We will show that the condition (8) follows from the convergence of the sequence (21). Since $r_k^{(p(2s,k))} = r_k^{(p(2s-1,k))}$, $k = \overline{1, 2s-1}$, $s = 1, 2, \ldots$, then

$$\sum_{n=1}^{2s} \frac{\mu_n^{(2)}}{r_{n-1}^{(p(2s,n-1))} r_n^{(p(2s,n))}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}$$
$$- \sum_{n=1}^{2s-1} \frac{\mu_n^{(2)}}{r_{n-1}^{(p(2s-1,n-1))} r_n^{(p(2s-1,n))}} \prod_{k=1}^{n-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s-1,k))}}{N\mu_k^{(2)}}\right)^{-1}$$
$$= \frac{\mu_{2s}^{(2)}}{r_{2s-1}^{(p(2s,2s-1))} r_{2s}^{(p(2s,2s))}} \prod_{k=1}^{2s-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1} = \frac{\mu_{2s}^{(2)}}{\nu_{2s-1}^{(1)} \nu_{2s}^{(1)}} \prod_{k=1}^{2s-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}.$$

In addition,

$$\frac{\mu_{2s}^{(2)}}{\nu_{2s-1}^{(1)} \nu_{2s}^{(1)}} \prod_{k=1}^{2s-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1} \to 0 \quad \text{as} \quad s \to +\infty.$$

Taking into account the inequalities

$$\frac{\mu_{2s}^{(2)}}{\nu_{2s-1}^{(1)} \nu_{2s}^{(1)}} \prod_{k=1}^{2s-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1} > \frac{1}{N} \prod_{k=1}^{2s} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1} > 0, \quad s = 1, 2, \ldots,$$

and the convergence to zero of the sequence

$$\frac{\mu_{2s}^{(2)}}{\nu_{2s-1}^{(1)} \nu_{2s}^{(1)}} \prod_{k=1}^{2s-1} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(p(2s,k))}}{N\mu_k^{(2)}}\right)^{-1}, \quad s = 1, 2, \ldots,$$

we conclude that the sequence of sets (7) form a sequence of convergence sets of the BCF (1).

Let us prove that $r_k^{(s)} > r_k^{(p(s,k))}$, if $s - k = 2m + 1$, $m = 0, 1, 2, \ldots$, and $r_k^{(s)} = r_k^{(p(s,k))}$, if $s - k = 2m$, $m = 0, 1, 2, \ldots$, where the values $r_k^{(s)}$, $r_k^{(p(s,k))}$ are defined by (17), (22), respectively.

Let $s - k = 2m + 1$, $m = 0, 1, 2, \ldots$. Then for $s = 2n$, $n = 1, 2, \ldots$, we have

$$\begin{aligned} r_k^{(2n)} &= \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{k+2m+1}^{(1)}}{\nu_{k+2m+1}^{(2)}} \\ &> \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{k+2m}^{(2)}}{\nu_{k+2m}^{(1)}} \\ &= \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{2n-1}^{(2)}}{\nu_{2n-1}^{(1)}} = r_k^{(p(2n,k))}, \end{aligned}$$

and for $s = 2n - 1$, $n = 1, 2, \ldots$, we obtain

$$\begin{aligned} r_k^{(2n+1)} &= \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{k+2m+1}^{(1)}}{\nu_{k+2m+1}^{(2)}} \\ &> \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{k+2m+2}^{(2)}}{\nu_{k+2m+2}^{(1)}} \\ &= \nu_k^{(1)} + \frac{N\mu_{k+1}^{(1)}}{\nu_{k+1}^{(2)}} + \frac{N\mu_{k+2}^{(2)}}{\nu_{k+2}^{(1)}} + \cdots + \frac{N\mu_{2n}^{(2)}}{\nu_{2n}^{(1)}} = r_k^{(p(2n-1,k))}. \end{aligned}$$

Let $s-k=2m$, $m=0,1,2,\dots$. Then $p(s,k)=s$ and $r_k^{(s)}=r_k^{(p(s,k))}$.

From the obtained relations between the values $r_k^{(s)}$ and $r_k^{(p(s,k))}$ it follows that for the elements of the sequences (16), (21) the inequalities

$$0<\sum_{n=1}^{s}\frac{\mu_n^{(2)}}{r_{n-1}^{(s)}r_n^{(s)}}\prod_{k=1}^{n-1}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1}<\sum_{n=1}^{s}\frac{\mu_n^{(2)}}{r_{n-1}^{(p(s,n-1))}r_n^{(p(s,n))}}\prod_{k=1}^{n-1}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(s,k))}}{N\mu_k^{(2)}}\right)^{-1}$$

hold, where $s=1,2,\dots$. From the obtained inequalities, we conclude that the convergence of the sequence (21) implies the boundedness of the sequence (16). Thus, the sets (7) form a sequence of sets of relative stability to perturbations of BCF (1). □

Directing the values $\mu_k^{(1)}\to 0+$, $\nu_k^{(2)}\to+\infty$, $k=1,2,\dots$, we get the following result.

Corollary 3. *Let relative errors of the elements of the BCF* (1) *satisfy the conditions* (14)*,* (15)*. If series*

$$\sum_{n=1}^{\infty}\frac{\mu_n}{\nu_{n-1}\nu_n}\prod_{k=1}^{n-1}\left(1+\frac{\nu_{k-1}\nu_k}{N\mu_k}\right)^{-1}$$

converges, then the family of sets (13) *form a sequence of sets of convergence and relative stability to perturbations of the BCF* (1)*. Further, the following estimate holds*

$$|\varepsilon^{(s)}|\le\alpha+(1+\alpha)\left(\frac{\beta}{1-\beta}+\frac{\alpha N}{1-\alpha}\sum_{n=1}^{s}\frac{\mu_n}{\nu_{n-1}\nu_n}\prod_{k=1}^{n-1}\left(1+\frac{\nu_{k-1}\nu_k}{N\mu_k}\right)^{-1}\right),\quad s=0,1,2,\dots .$$

Example 1. *Let relative errors of the elements of the BCF*

$$a_0\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{N}\frac{a_{i(k)}}{1}\right)^{-1}\tag{23}$$

satisfy the conditions (14)*. Then the sets*

$$E_0=(0,+\infty),\quad E_{i(k)}=(0,\ k^{\gamma}/N],\quad i(k)\in I_k,\quad k=1,2,\dots,\quad \gamma<1,$$

form a sequence of sets of convergence and relative stability to perturbations of the BCF (23)*. Further, the following estimate holds*

$$|\varepsilon^{(s)}|\le\alpha+\frac{\alpha(1+\alpha)}{1-\alpha}\sum_{n=1}^{s}n^{\gamma}\prod_{k=1}^{n-1}\left(1+\frac{1}{k^{\gamma}}\right)^{-1},\quad s=0,1,\dots .$$

Example 2. *Let relative errors of the elements of the BCF* (23) *satisfy the conditions* (14)*. Then the sets*

$$E_0=(0,+\infty),\quad E_{i(2k-1)}=(0,\ 1/N],\quad E_{i(2k)}=(0,\ 2k/N],$$

where $i(2k)\in I_{2k}$, $i(2k-1)\in I_{2k-1}$, $k=1,2,\dots$, form a sequence of sets of convergence and relative stability to perturbations of the BCF (23)*. Further, the following estimate holds*

$$|\varepsilon^{(s)}|\le\alpha+\frac{\alpha(1+\alpha)}{1-\alpha}\sum_{n=1}^{s}\frac{[n/2]!}{(2[(n-1)/2]+1)!!},\quad s=0,1,\dots .$$

Let us consider the problem of relative stability to perturbations of the BCF (1) in the case when the partial numerators $a_{i(2k)}$, $i(2k) \in I_{2k}$, $k = 0, 1, 2, \ldots$, are perturbed by a shortage, and partial numerators $a_{i(2k+1)}$, $i(2k+1) \in I_{2k+1}$, $k = 0, 1, 2, \ldots$, by an excess, i.e. under the condition that the relative errors of the partial numerators have alternating signs.

Theorem 4. *Let relative errors of the elements of the BCF* (1) *satisfy the conditions* (14), (15), *and*

$$\alpha_{i(2k)} \leq 0, \quad i(2k) \in I_{2k}, \quad \alpha_{i(2k+1)} \geq 0, \quad i(2k+1) \in I_{2k+1}, \quad k = 0, 1, 2, \ldots. \tag{24}$$

If the sequence

$$\sum_{n=1}^{s} \prod_{k=1}^{n} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1}, \quad s = 1, 2, \ldots,$$

is bounded, then the sets (7) *form a sequence of sets of relative stability to perturbations of the BCF* (1). *Further, for* $s = 0, 1, 2, \ldots$, *the estimate*

$$|\varepsilon^{(s)}| \leq \frac{\beta}{1-\beta} + \alpha \left(1 + \sum_{n=1}^{s} \prod_{k=1}^{n} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1}\right) \tag{25}$$

is valid.

Proof. From (6) it follows

$$\begin{aligned} |\varepsilon^{(s)}| \leq& \max\left\{|\tilde{\beta}_{i(k)}| : \ i(k) \in I_k, \ k = \overline{0, s}\right\} \\ &\times \left((1+\alpha_0)\left(1 - \sum_{i_1=1}^{N} q_{i(1)}^{(s)}\right) + \sum_{n=1}^{s} \sum_{i_1, i_2, \ldots, i_n = 1}^{N} \prod_{k=1}^{n} q_{i(k)}^{(s)} \prod_{k=0}^{n} (1 + \tilde{\alpha}_{i(k)}) \left(1 - \sum_{i_{k+1}=1}^{N} q_{i(n+1)}^{(s)}\right)\right) \\ &+ \max\left\{|\tilde{\alpha}_{i(k)}| : i(k) \in I_k, k = \overline{0, s}\right\} \left(1 + \sum_{n=1}^{s} \sum_{i_1, i_2, \ldots, i_n = 1}^{N} \prod_{k=0}^{n} q_{i(k)}^{(s)} \prod_{k=1}^{n-1} (1 + \tilde{\alpha}_{i(k)})\right). \end{aligned}$$

Taking into account the conditions (24) and the inequalities (14), (15), (19), we obtain the estimate (25). Then for $|\alpha_{i(k)}| \leq \alpha < f(\varepsilon)$, $|\beta_{i(k)}| \leq \beta < f(\varepsilon)$, $i(k) \in I_k$, $k = 0, 1, 2, \ldots$, where

$$f(\varepsilon) = \frac{2 + M + \varepsilon - \sqrt{(M-\varepsilon)^2 + 4(M+1)}}{2(M+1)},$$

ε is an arbitrary positive constant, M is a positive constant such that

$$\sum_{n=1}^{s} \prod_{k=1}^{n} \left(1 + \frac{\nu_{k-1}^{(1)} r_k^{(s)}}{N\mu_k^{(2)}}\right)^{-1} \leq M, \quad s = 1, 2, \ldots,$$

the inequalities $|\varepsilon^{(s)}| < \varepsilon$, $s = 0, 1, 2, \ldots$, for the relative errors of the approximants of the BCF (1), are valid, which proves the fulfillment of the conditions for determining the sequence of sets of relative stability to perturbations of (1). □

The following result can be proved in much the same way as Theorem 3.

Theorem 5. *Let relative errors of the elements of the BCF* (1) *satisfy the conditions* (14)*,* (15)*, and* (24)*. If there exist a limit of the sequence*

$$\sum_{n=1}^{s}\prod_{k=1}^{n}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(s,k))}}{N\mu_l^{(2)}}\right)^{-1},$$

then the sets (7) *form a sequence of sets of convergence and relative stability to perturbations of the BCF* (1)*. Further, for* $s=0,1,2,\ldots$*, the estimate*

$$|\varepsilon^{(s)}|\leq\frac{\beta}{1-\beta}+\alpha\left(1+\sum_{n=1}^{s}\prod_{k=1}^{n}\left(1+\frac{\nu_{k-1}^{(1)}r_k^{(p(s,k))}}{N\mu_k^{(2)}}\right)^{-1}\right)$$

is valid.

Corollary 4. *Let relative errors of the elements of the BCF* (1) *satisfy the conditions* (14)*,* (15)*, and* (24)*. If the series*

$$\sum_{n=1}^{\infty}\prod_{k=1}^{n}\left(1+\frac{\nu_{k-1}\nu_k}{N\mu_k}\right)^{-1}$$

converges, then the sets (13) *form a sequence of sets of relative stability to perturbations of the BCF* (1)*. Further, the following estimate holds*

$$|\varepsilon^{(s)}|\leq\frac{\beta}{1-\beta}+\alpha\left(1+\sum_{n=1}^{s}\prod_{k=1}^{n}\left(1+\frac{\nu_{k-1}\nu_k}{N\mu_k}\right)^{-1}\right),\quad s=0,1,2,\ldots.$$

Example 3. *Let relative errors of the elements of the BCF* (23) *satisfy the conditions* (14)*,* (24)*. Then the sets*

$$E_0=(0,+\infty),\quad E_{i(k)}=(0,\ k/(2N)],\quad k=1,2,\ldots,$$

form a sequence of sets of relative stability to perturbations of the BCF (23)*. Further, the following estimate holds*

$$|\varepsilon^{(s)}|\leq\alpha+\sum_{k=1}^{s}\frac{\alpha}{([k/2]+1)(2k+1-2[k/2])},\quad s=0,1,2,\ldots.$$

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Received 02.12.2023
Revised 03.02.2024

Гладун В.Р., Боднар Д.І., Русин Р.С. *Множини збіжності та відносної стійкості до збурень гіллястого ланцюгового дробу з додатними елементами* // Карпатські матем. публ. — 2024. — Т.16, №1. — С. 16–31.

У роботі досліджуються питання збіжності та відносної стійкості до збурень гіллястого ланцюгового дробу з додатними елементами та фіксованою кількістю гілок розгалуження. Встановлено умови, за яких множини елементів

$$\Omega_0 = (0, \mu_0^{(2)}] \times [\nu_0^{(1)}, +\infty), \quad \Omega_{i(k)} = [\mu_k^{(1)}, \mu_k^{(2)}] \times [\nu_k^{(1)}, \nu_k^{(2)}], \quad i(k) \in I_k, \quad k = 1, 2, \dots,$$

де $\nu_0^{(1)} > 0$, $0 < \mu_k^{(1)} < \mu_k^{(2)}$, $0 < \nu_k^{(1)} < \nu_k^{(2)}$, $k = 1, 2, \dots$, є послідовністю множин збіжності та відносної стійкості до збурень гіллястого ланцюгового дробу

$$\frac{a_0}{b_0} + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}} + \sum_{i_2=1}^{N} \frac{a_{i(2)}}{b_{i(2)}} + \cdots + \sum_{i_k=1}^{N} \frac{a_{i(k)}}{b_{i(k)}} + \cdots.$$

Отримані умови вимагають обмеженості або збіжності послідовностей, члени яких залежать від величин $\mu_k^{(j)}$, $\nu_k^{(j)}$, $j = 1, 2$. У випадку, якщо множинами елементів гіллястого ланцюгового дробу є множини $\Omega_{i(k)} = (0, \mu_k] \times [\nu_k, +\infty)$, $i(k) \in I_k$, $k = 0, 1, \dots$, де $\mu_k > 0$, $\nu_k > 0$, $k = 0, 1, \dots$, то умови збіжності та стійкості до збурень формулюються через збіжність рядів, члени яких залежать від величин μ_k, ν_k. Також встановлено умови відносної стійкості до збурень гіллястого ланцюгового дробу, якщо частинні чисельники на парних поверхах дробу збурюються за недостачею, а на непарних — за надлишком, тобто за умови знакопочерговості відносних похибок частинних чисельників. В усіх випадках отримано оцінки відносних похибок підхідних дробів, які виникають в результаті збурення елементів гіллястого ланцюгового дробу.

Ключові слова і фрази: гіллястий ланцюговий дріб, збіжність, стійкість до збурень, множина збіжності, множина стійкості до збурень.