About one Euler problem associated with Viet formulas

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More than 250 years ago, in 1749, Leonard Oiler, in a letter to the St. Petersburg academician Christian Goldbach, wrote that he had spent a lot of effort to solve one problem. Its essence was to find triples of positive integers $a,b,c,$ for which $a+b+c,$ $ab+bc+ca$, abc are squares of positive integers.

But for the maximum triples of such numbers he found, the condition $(a,b,c)=1$ was not met.

In addition to this condition, we will additionally require that the sum of the squares of such numbers is also an exact square.

Theorem. The set of triples of positive integers $a,b,c,$ for which $a+b+c,$ $ab+bc+ca,$ abc and $a^2+b^2+c^2$ are squares of positive integers, and $(a,b,c)=1$, is infinite.

Proof. The conditions of the theorem are satisfied by a trio of numbers:

$$a=136=8\cdot(8+9),\quad b=153=9\cdot(8+9),\quad c=72=8\cdot 9.$$

Really

$a+b+c=19^2,\quad ab+bc+ca=204^2,\quad abc=1224^2,\quad a^2+b^2+c^2=217^2.$

Let's put $a=x(x+y)$, $b=y(x+y)$, $c=xy$, where x,y are positive integers. Then

$$abc=\left(xy(x+y)\right)^2,\quad a^2+b^2+c^2=\left(x^2+xy+y^2\right)^2,$$

$$ab+bc+ca=2xy(x+y)^2,\quad a+b+c=x^2+3xy+y^2.$$

Therefore, it is enough to find such positive integers $x,y,$ for which $2xy$ and $x^2+3xy+y^2$ are squares of positive integers and $(x,y)=1.$

Let for positive integers $x,y,$ where $(x,y)=1,$ equalities are performed

$$2xy=r^2\,2xy=r^2,\ x^2+3xy+y^2.$$

Denote $|x-y|=q,\quad x+y=s$ and put

$$X=2r^2t^2>x,\ Y=q^2s^2>y.$$

Then

$$2XY=\left(2qrst\right)^2,\quad X^2+3XY+Y^2=\left(X+Y+q^2r^2\right)^2.$$

Let's prove that $\left(X,Y\right)=1.$

From equality $2xy=r^2$ and condition $\left(x,y\right)=1$ it follows that numbers x and y different parity. Therefore, the numbers q and s are odd. Besides

$$q^2+2r^2=s^2,\ \ 2s^2+r^2=2t^2,\ \ q^2+4t^2=5s^2.$$

Assuming that $\left(X,Y\right)\neq 1,$ from here for each of the possible four cases: $\left(q,r\right)\vdots\, p,$ $\left(s,r\right)\vdots\, p,$ $\left(q,t\right)\vdots\, p,$ $\left(s,t\right)\vdots\, p,$ where p – some prime number, we obtain that $\left(q,s\right)\vdots\, p.$ Then and $\left(x,y\right)\vdots\, p,$ what contradicts the condition $\left(x,y\right)=1.$

Let now $x_1=8,$ $y_1=9,$ and, for all positive integers $n,$

$$x_{n+1}=4x_ny_n\left(x_n^2+3x_ny_n+y_n^2\right),$$

$$y_{n+1}=(x_n-y_n)^2(x_n+y_n)^2.$$

From the proven it follows that triples of numbers

$$a_n=x_n\left(x_n+y_n\right),\quad b_n=y_n\left(x_n+y_n\right),\quad c_n=x_ny_n$$

satisfy the conditions of the theorem for each positive integer $n,$ and $(a_n,b_n,c_n)=1.$

The theorem is proven.

We also found all pairs of positive integers x,y of different parity (x – even, y – odd), for which the value of the expression $x^2+3xy+y^2$ is the square of a positive integer, and $\left(x,y\right)=1$:

1) $x=8k,\ y=20k^2-12k+1,\ \ x^2+3xy+y^2=\left(20k^2-1\right)^2;$

2) $x=8k,\ y=4k^2-12k+5,\ k\geq 3,\ k\neq 5l,\ \ x^2+3xy+y^2=\left(4k^2-5\right)^2;$

3) $x=4\left(5k^2+k\right), y=4k+1,\ \ x^2+3xy+y^2=\left(20k^2+10k+1\right)^2;$

4) $x=4\left(k^2-k\right), y=4k+1,\ \ k\geq 2,\ \ x^2+3xy+y^2=\left(4k^2+2k-1\right)^2.$

In particular, from 1) for $k=1$ or from 4) for $k=2$ we get $x_1=8,$ $y_1=9,$ and from 3) for $k=72$ we will find $x_2=103968,$ $y_2=289.$ Such pairs of numbers are the first two elements of the sequence of pairs that we built when proving the theorem.

But it was not possible to identify at least one pair of numbers $\left(x,y\right),$ different from the elements of the sequence $\left(x_n,y_n\right),$ of pairs built above, for which $2xy$ is also the square of the positive integer.

The question of the existence of triples $a,b,c,$ that satisfy the conditions of the theorem, but are not represented as $a=x(x+y), b=y(x+y), c=xy.$