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Research Article

Combinatorial Determinant Formulas for Boubaker Polynomials

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In this paper, we evaluate several families of Toeplitz–Hessenberg determinants whose entries are the Boubaker polynomials. Equivalently, these determinant formulas may be also rewritten as combinatorial identities involving sum of products of Boubaker polynomials and multinomial coefficients. We also present new formulas for Boubaker polynomials via recurrent three-diagonal determinants.

1. Introduction

The Boubaker polynomials of order n, denoted by $B_n(x)$, constitute a nonorthogonal polynomial sequence defined by

$$B_0(x) = 1,$$
$$B_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n-4k}{n-k} \binom{n-k}{k} x^{n-2k}, \quad (1)$$

where $n \geq 1$, $\lfloor s \rfloor$ is the floor of s, and $\binom{n-k}{k}$ the binomial coefficient.

The Boubaker polynomials can be expressed also by the recurrence:

$$B_n(x) = xB_{n-1}(x) - B_{n-2}(x), \quad n \geq 3, \quad (2)$$

with $B_0(x) = 1$, $B_1(x) = x$, and $B_2(x) = x^2 + 2$.

The first few terms of this polynomial sequence starting from $B_3(x)$ are

$$\begin{aligned} B_3(x) &= x^3 + x, \\ B_4(x) &= x^4 - 2, \\ B_5(x) &= x^5 - x^3 - 3x, \\ B_6(x) &= x^6 - 2x^4 - 3x^2 + 2, \\ B_7(x) &= x^7 - 3x^5 - 2x^3 + 5x, \\ B_8(x) &= x^8 - 4x^6 + 8x^2 - 2, \\ B_9(x) &= x^9 - 5x^7 + 3x^5 + 10x^3 - 7x, \\ B_{10}(x) &= x^{10} - 6x^8 + 7x^6 + 10x^4 - 15x^2 + 2. \end{aligned} \quad (3)$$

The ordinary generating function of the Boubaker polynomials is

$$\sum_{n=0}^{\infty} B_n(x) t^n = \frac{1 + 3t^2}{1 - tx + t^2}. \quad (4)$$

Polynomials (2) can be expressed in terms of Chebyshev polynomials of the first and second kind, $T_n(x)$ and $U_n(x)$, respectively, as follows:

$$B_n(x) = 2T_n\left(\frac{x}{2}\right) + 4U_{n-2}\left(\frac{x}{2}\right),$$
$$B_n(x) = U_n\left(\frac{x}{2}\right) + 3U_{n-2}\left(\frac{x}{2}\right), \tag{5}$$

where $U_{-1}(x) := 0$; see [1].

The Boubaker polynomials play an important role in different scientific and engineering fields, such as thermodynamics, mechanics, cryptography, biology and biophysics, heat transfer, nonlinear dynamics, approximation theory, hydrology, electrical engineering, and nuclear engineering physics (see, among others, [2–11] and related references therein).

Solutions of many applied problems are based on the Boubaker Polynomials Expansion Scheme (BPES), using the subsequence $\{B_{4k}(x)\}_{k\geq 0}$. Such polynomials satisfy the recurrence

$$B_{4k}(x) = \left(x^2 - 4x^2 + 2\right)B_{4(k-1)}(x) - \beta_k B_{4(k-2)}(x), \quad k \geq 0, \tag{6}$$

with $B_0(x) = 1$ and $B_4(x) = x^4 - 2$, where $\beta_0 = 0$, $\beta_1 = -2$, and $\beta_2 = \beta_3 = \cdots = 1$.

For example, few boundary value problems of ordinary differential equations and many physical models involving ordinary differential equations systems were solved more efficiently by the BPES compared to other methods [3]. Physical models in terms of partial differential equations were reliably addressed through the BPES [6]. Davaeifar and Rashidinia [4] and Milovanović and Joksimović [1] used the BPES to solve certain integral equations. In [12], Barry and Hennesy outlined the role of the Boubaker polynomials and their associated integer sequences in array analysis and approximation theory (see also related work [13]). Dubey et al. [5] provided analytical solution to the Lotka–Volterra Predator-Prey equations in the case of quickly satiable predators. In [9], Vazquez-Leal et al. presented the BPES to construct semianalytical solutions for the transient of a nonlinear circuit. In [14, 15], Rabiei et al. focused on Boubaker polynomials in fractional calculus.

The purpose of the present paper is to investigate some families of Toeplitz–Hessenberg determinants whose entries are Boubaker polynomials with successive, odd, or even subscripts. As a consequence, we obtain for these polynomials new identities involving multinomial coefficients.

Some of the results of this paper were announced without proof in [16].

2. Toeplitz–Hessenberg Determinants and Related Formulas

A *Toeplitz–Hessenberg determinant* takes the form

$$T_n(a_0; a_1, \ldots, a_n) = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{vmatrix}, \tag{7}$$

where $a_0 \neq 0$ and $a_k \neq 0$ for at least one $k > 0$.

This class of determinants has been encountered in various scientific and engineering applications (see, e.g., [17, 18] and related references contained therein).

Expanding the determinant T_n along the last row repeatedly, we obtain the recurrence:

$$T_n = \sum_{k=1}^{n} (-a_0)^{k-1} a_k T_{n-k}, \quad n \geq 1, \tag{8}$$

where, by definition, $T_0 = 1$.

The following result, which provides a multinomial expansion of T_n, is known as *Trudi's formula*, the $a_0 = 1$ case of which is called *Brioschi's formula* [19]:

$$T_n = (-a_0)^n \cdot \sum_{s_1+2s_2+\cdots+ns_n=n} (-1)^{|s|} m_n(s) \left(\frac{a_1}{a_0}\right)^{s_1} \left(\frac{a_2}{a_0}\right)^{s_2} \cdots \left(\frac{a_n}{a_0}\right)^{s_n}, \tag{9}$$

where the summation is over all n-tuples $s = (s_1, \ldots, s_n)$ of integers $s_i \geq 0$ satisfying Diophantine equation $s_1 + 2s_2 + \cdots + ns_n = n$, $|s| = s_1 + \cdots + s_n$, and $m_n(s) = |s|!/s_1! \cdots s_n!$ denotes the multinomial coefficient.

Note that $n = s_1 + 2s_2 + \cdots + ns_n$ is partition of the positive integer n, where each positive integer i appears s_i times. Many combinatorial identities for different polynomials involving sums over integer partitions can be generated in this way. Some of these identities are presented in [20, 21] and in Section 3 of this paper.

3. Determinant Formulas with Boubaker Polynomials Entries

In this section, we find relations involving the Boubaker polynomial, which arise as certain families of Toeplitz–Hessenberg determinants.

In the interest of brevity and convenience, we omit the argument in the functional notation, when there is no ambiguity; so B_k will mean $B_k(x)$.

Theorem 1. *Let $n \geq 2$, except when noted otherwise. The following formulas hold:*

$$T_n(1; B_1, B_2, \ldots, B_n) = (-1)^{\lfloor n/2 \rfloor} 2^{((-1)^n - 1)/2} 3^{\lfloor (n-1)/2 \rfloor} (x + 1 + (-1)^n (1 - x)), \quad n \ge 1,$$
$$T_n(1; B_2, B_4, \ldots, B_{2n}) = \frac{3^{n-1}(3x^2 + 4) - x^2}{2}, \quad n \ge 1, \tag{10}$$

$$T_n(B_1; B_2, B_3, \ldots, B_{n+1}) = 2^{n-2}(3x^2 + 4), \tag{11}$$

$$\begin{aligned} T_n(B_1; B_3, B_5, \ldots, B_{2n+1}) &= 3^{n-2}(3x^2 + 4)x^n, \\ T_n(B_2; B_3, B_4, \ldots, B_{n+2}) &= (-x)^{n-2}(3x^2 + 4), \\ T_n(B_2; B_4, B_6, \ldots, B_{2n+2}) &= 2^{n-2}x^2(3x^2 + 4), \\ T_n(B_3; B_4, B_5, \ldots, B_{n+3}) &= (-1)^n(3x^2 + 4)(x^2 + 2)^{n-2}, \\ T_n(B_3; B_5, B_7, \ldots, B_{2n+3}) &= (-x)^n(3x^2 + 4), \\ T_n(-2; B_1, B_2, \ldots, B_n) &= 3^{n-2}(3x^2 + 4)x^{n-2}, \end{aligned} \tag{12}$$

$$T_n(-2; B_2, B_4, \ldots, B_{2n}) = x^2(3x^2 + 4)(3x^2 - 2)^{n-2}. \tag{13}$$

Proof. We will prove formulas (11) and (13) by induction on n; the other proofs, which we omit, are similar.

Proof of Identity (11). To make the notation simpler, we will write D_n instead of $T_n(B_1; B_2, B_3, \ldots, B_{n+1})$. When $n = 2$ and $n = 3$ the formula is seen to hold. Suppose it is true for all $k \le n - 1$, where $n \ge 4$. Using recurrences (8) and (2), we then have

$$\begin{aligned} D_n &= \sum_{i=1}^{n} (-x)^{i-1} B_{i+1} D_{n-i} \\ &= B_2 D_{n-1} - x B_3 D_{n-2} + \sum_{i=3}^{n} (-x)^{i-1} (x B_i - B_{i-1}) D_{n-i} \\ &= (x^2 + 2) D_{n-1} - x(x^3 + x) D_{n-2} + x \sum_{i=3}^{n} (-x)^{i-1} B_i D_{n-i} - \sum_{i=3}^{n} (-x)^{i-1} B_{i-1} D_{n-i} \\ &= (x^2 + 2) D_{n-1} - x^2(x^2 + 1) D_{n-2} + x \sum_{i=2}^{n-1} (-x)^{i} B_{i+1} D_{n-i-1} - \sum_{i=1}^{n-2} (-x)^{i+1} B_{i+1} D_{n-i-2} \\ &= (x^2 + 2) D_{n-1} - x^2(x^2 + 1) D_{n-2} + x \left(-x \sum_{i=1}^{n-1} (-x)^{i-1} B_{i+1} D_{n-i-1} + x B_2 D_{n-2} \right) - x^2 \sum_{i=1}^{n-2} (-x)^{i-1} B_{i+1} D_{n-i-2} \\ &= (x^2 + 2) D_{n-1} - x^2(x^2 + 1) D_{n-2} + x(-x D_{n-1} + x(x^2 + 2) D_{n-2}) - x^2 D_{n-2} = 2 D_{n-1}. \end{aligned} \tag{14}$$

Now, using the induction hypothesis, we obtain

$$D_n = 2 \cdot 2^{n-3}(3x^2 + 4) = 2^{n-2}(3x^2 + 4). \tag{15}$$

Consequently, formula (11) is true in the case n and thus, by induction, it holds for all positive integers.

Proof of Identity (13). Let $D_n = T_n(-2; B_2, B_4, \ldots, B_{2n})$. One may verify that formula (13) holds when $n = 2$ and $n = 3$. Suppose it is true for all $k \le n - 1$, where $n \ge 4$. Using (8), (2), and formula

$$B_{2i-1} = x \sum_{s=1}^{i-1} (-1)^s B_{2s} - (-1)^i x, \quad i \ge 2, \tag{16}$$

we then have

$$\begin{aligned}
D_n &= \sum_{i=1}^{n} 2^{i-1} B_{2i} D_{n-i} \\
&= B_2 D_{n-1} + \sum_{i=2}^{n} 2^{i-1}\left(xB_{2i-1} - B_{2i-2}\right) D_{n-i} \\
&= \left(x^2+2\right) D_{n-1} + x\sum_{i=2}^{n} 2^{i-1} B_{2i-1} D_{n-i} - \sum_{i=2}^{n} 2^{i-1} B_{2i-2} D_{n-i} \\
&= \left(x^2+2\right) D_{n-1} + x\sum_{i=2}^{n} 2^{i-1}\left(x\sum_{s=1}^{i-1}(-1)^s B_{2s} - (-1)^i x\right) D_{n-i} - 2\sum_{i=1}^{n-1} 2^{i-1} B_{2i} D_{n-i-1} \\
&= \left(x^2+2\right) D_{n-1} + x^2 \sum_{i=2}^{n}\sum_{s=1}^{i-1}(-1)^s 2^{i-1} B_{2s} D_{n-i} + x^2\sum_{i=2}^{n}(-2)^{i-1} D_{n-i} - 2D_{n-1} \\
&= x^2 D_{n-1} - x^2\sum_{s=1}^{n-1}(-2)^s \sum_{i=1}^{n-s} 2^{i-1} B_{2i} D_{n-s-i} + x^2\sum_{i=2}^{n-2}(-2)^{i-1} D_{n-i} + (-2)^{n-2} D_1 + (-2)^{n-1} D_0 \\
&= x^2 D_{n-1} - x^2\sum_{s=1}^{n-1}(-2)^s D_{n-s} + x^2\sum_{i=2}^{n-2}(-2)^{i-1} D_{n-i} + (-2)^{n-2}\left(x^2+1\right) + (-2)^{n-1} \\
&= 3x^2 D_{n-1} + x^2(-2)^{n-2}\left(3x^2+4\right) - \frac{3x^2}{2}\sum_{i=2}^{n-2}(-2)^i D_{n-i} \\
&= x^2(-2)^{n-2}\left(3x^2+4\right) - \frac{3x^2}{2}\sum_{i=1}^{n-2}(-2)^i D_{n-i}.
\end{aligned} \tag{17}$$

By the induction hypothesis, we obtain

$$\begin{aligned}
D_n &= x^2(-2)^{n-2}\left(3x^2+4\right) - \frac{3x^2}{2}\sum_{i=1}^{n-2}(-2)^i x^2 \left(3x^2+4\right)\left(3x^2-2\right)^{n-i-2} \\
&= x^2(-2)^{n-2}\left(3x^2+4\right) - \frac{3x^4\left(3x^2+4\right)\left(3x^2-2\right)^{n-2}}{2} \sum_{i=1}^{n-2}\left(\frac{2}{2-3x^2}\right)^i.
\end{aligned} \tag{18}$$

Using the geometric series, it can be seen that the following sums hold:

$$\sum_{i=1}^{n-2}\left(\frac{2}{2-3x^2}\right)^i = \frac{2}{3x^2}\left(\left(\frac{2}{2-3x^2}\right)^{n-2} - 1\right). \tag{19}$$

In view of the formula above, from (18) we find

$$\begin{aligned}
D_n &= x^2(-2)^{n-2}\left(3x^2+4\right) - x^2\left(3x^2+4\right)\left(3x^2-2\right)^{n-2} \left(\left(\frac{2}{2-3x^2}\right)^{n-2} - 1\right) \\
&= x^2\left(3x^2+4\right)\left(3x^2-2\right)^{n-2},
\end{aligned} \tag{20}$$

as desired. Since formula (13) holds for n, it follows by induction that it is true for all positive integers. The proof is complete.

4. Multinomial Extension of Toeplitz–Hessenberg Determinants

In this section, we focus on multinomial extensions of Theorem 1. The determinant formulas above may be rewritten in terms of Trudi's formula (9).

Theorem 2. *Let $n \geq 2$, except when noted otherwise. Then,*

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)B_1^{s_1}B_2^{s_2}\dots B_n^{s_n}=(-1)^{\lfloor 3n/2\rfloor}2^{((-1)^n-1)/2}3^{\lfloor (n-1)/2\rfloor}\left(x+1+(-1)^n(1-x)\right),\quad n\ge 1,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)B_2^{s_1}B_4^{s_2}\dots B_{2n}^{s_n}=-\frac{(-3)^{n-1}(3x^2+4)+(-1)^nx^2}{2},\quad n\ge 1,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_2}{B_1}\right)^{s_1}\left(\frac{B_3}{B_1}\right)^{s_2}\dots\left(\frac{B_{n+1}}{B_1}\right)^{s_n}=\frac{(-2)^{n-2}(3x^2+4)}{x^n},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_3}{B_1}\right)^{s_1}\left(\frac{B_5}{B_1}\right)^{s_2}\dots\left(\frac{B_{2n+1}}{B_1}\right)^{s_n}=3^{n-2}\left(3x^2+4\right)x^n,$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_3}{B_2}\right)^{s_1}\left(\frac{B_4}{B_2}\right)^{s_2}\dots\left(\frac{B_{n+2}}{B_2}\right)^{s_n}=\frac{(3x^2+4)x^{n-2}}{(x^2+2)^n},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_4}{B_2}\right)^{s_1}\left(\frac{B_6}{B_2}\right)^{s_2}\dots\left(\frac{B_{2n+2}}{B_2}\right)^{s_n}=\frac{(-2)^{n-2}x^2(3x^2+4)}{(x^2+2)^n},\tag{21}$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_4}{B_3}\right)^{s_1}\left(\frac{B_5}{B_3}\right)^{s_2}\dots\left(\frac{B_{n+3}}{B_3}\right)^{s_n}=\frac{(x^2+2)^{n-2}(3x^2+4)}{(x^3+x)^n},$$

$$\sum_{\sigma_n=n}(-1)^{|s|}m_n(s)\left(\frac{B_5}{B_3}\right)^{s_1}\left(\frac{B_7}{B_3}\right)^{s_2}\dots\left(\frac{B_{2n+3}}{B_3}\right)^{s_n}=\frac{3x^2+4}{(x^2+1)^n},$$

$$\sum_{\sigma_n=n}m_n(s)\left(\frac{B_1}{2}\right)^{s_1}\left(\frac{B_2}{2}\right)^{s_2}\dots\left(\frac{B_n}{2}\right)^{s_n}=\frac{(3x)^{n-2}(3x^2+4)}{2^n},$$

$$\sum_{\sigma_n=n}m_n(s)\left(\frac{B_2}{2}\right)^{s_1}\left(\frac{B_4}{2}\right)^{s_2}\dots\left(\frac{B_{2n}}{2}\right)^{s_n}=\frac{x^2(3x^2+4)(3x^2-2)^{n-2}}{2^n},$$

where $|s|=s_1+\cdots+s_n$, $\sigma_n=s_1+2s_2+\cdots+ns_n$, $m_n(s)=|s|!/s_1!\cdots s_n!$, *and the summations are over all n-tuples* $s=(s_1,\dots,s_n)$ *of integers* $s_i\ge 0$, *satisfying* $\sigma_n=n$.

Example 1. From (21), we have

$$\sum_{s_1+2s_2+3s_3+4s_4=4}(-1)^{s_1+s_2+s_3+s_4}\frac{(s_1+s_2+s_3+s_4)!}{s_1!s_2!s_3!s_4!}B_1^{s_1}B_2^{s_2}B_3^{s_3}B_4^{s_4}=B_1^4-3B_1^2B_2+2B_1B_3+B_2^2-B_4=6.\tag{22}$$

5. Recurrent Three-Diagonal Determinants with Boubaker Polynomials

In this section, we prove two formulas expressing the Boubaker polynomials $B_n(x)$ with even (odd) subscripts via recurrent determinants of the three-diagonal matrix of order n.

Let $P_n(x)$ and $Q_n(x)$ denote the $n\times n$ three-diagonal determinants having the form

$$P_n(x) = \begin{vmatrix} x & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ xB_2 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & xB_4 & B_1 & -B_1 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 0 & \cdots & xB_{2n-4} & B_{2n-7} & -B_{2n-7} \\ 0 & 0 & 0 & 0 & \cdots & 0 & xB_{2n-2} & B_{2n-5} \end{vmatrix},$$
$$Q_n(x) = \begin{vmatrix} x^2+2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ xB_3 & B_0 & -B_0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & xB_5 & B_2 & -B_2 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & xB_{2n-3} & B_{2n-6} & -B_{2n-6} \\ 0 & 0 & 0 & 0 & \cdots & 0 & xB_{2n-1} & B_{2n-4} \end{vmatrix}. \quad (23)$$

Theorem 3. *For $n \geq 1$, the following formulas hold:*

$$B_{2n-1}(x) = \frac{(-1)^{n-1}}{\prod_{i=1}^{n-2} B_{2i-1}(x)} P_n(x), \quad (24)$$

$$B_{2n}(x) = \frac{(-1)^{n-1}}{\prod_{i=1}^{n-2} B_{2i}(x)} Q_n(x). \quad (25)$$

Proof. We only prove formula (24), and formula (25) can be proved similarly. We use induction on n. Since $P_1(x) = x = B_1(x)$ and $P_2(x) = -(x^3 + x) = -B_3(x)$, the result is true when $n = 1$ and $n = 2$. Assume it true for every positive integer $k < n$. Expanding determinant $P_n(x)$ by the last row, from (24), we find

$$\begin{aligned} B_{2n-1} &= \frac{(-1)^{n-1}}{\prod_{i=1}^{n-2} B_{2i-1}} \left(B_{2n-5}P_{n-1}(x) + B_{2n-7}xB_{2n-2}P_{n-2}(x)\right) \\ &= \frac{(-1)^{n-1}}{\prod_{i=1}^{n-2} B_{2i-1}} \left(B_{2n-5}(-1)^{n-2}B_{2n-3}\prod_{i=1}^{n-3} B_{2i-1} + xB_{2n-7}B_{2n-2}(-1)^{n-3}B_{2n-5}\prod_{i=1}^{n-4} B_{2i-1}\right) \\ &= \frac{-1}{\prod_{i=1}^{n-2} B_{2i-1}} \left(B_{2n-5}B_{2n-3}\prod_{i=1}^{n-3} B_{2i-1} - xB_{2n-7}B_{2n-2}B_{2n-5}\prod_{i=1}^{n-4} B_{2i-1}\right) \\ &= xB_{2n-2} - B_{2n-3}, \end{aligned} \quad (26)$$

i.e., we have the recurrent relation (2). Therefore, the result is true for every $n \geq 1$. The proof is complete.

Example 2. Formulas (24) and (25), respectively, yield

$$B_7 = \frac{-1}{B_1B_3} \begin{vmatrix} x & 1 & 0 & 0 \\ xB_2 & 1 & -1 & 0 \\ 0 & xB_4 & B_1 & -B_1 \\ 0 & 0 & xB_6 & B_3 \end{vmatrix},$$
$$B_{10} = \frac{1}{B_2B_4B_6} \begin{vmatrix} x^2+2 & 1 & 0 & 0 & 0 \\ xB_3 & B_0 & -B_0 & 0 & 0 \\ 0 & xB_5 & B_2 & -B_2 & 0 \\ 0 & 0 & xB_7 & B_4 & -B_4 \\ 0 & 0 & 0 & xB_9 & B_6 \end{vmatrix}. \quad (27)$$

6. Conclusions

In this paper, we have found determinant formulas for several families of Toeplitz–Hessenberg determinants having various translates of the Boubaker polynomials for the nonzero entries. In Theorem 1, we found determinant formulas, where the entries were translates of the Boubaker polynomial sequence or of just the even or odd subsequence. The determinant formulas in all of these results may also be expressed (see Theorem 2) equivalently as multisum identities involving multinomial coefficients and a product of the terms of the Boubaker polynomial sequence. In Theorem 3, we present recurrent formulas for Boubaker polynomials with even or odd subscripts via three-diagonal determinants.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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