ALGEBRA IN SUPEREXTENSIONS OF SEMILATTICES

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Abstract. Given a semilattice X we study the algebraic properties of the semigroup $\upsilon(X)$ of upfamilies on X. The semigroup $\upsilon(X)$ contains the Stone-Čech extension $\beta(X)$, the superextension $\lambda(X)$, and the space of filters $\varphi(X)$ on X as closed subsemigroups. We prove that $\upsilon(X)$ is a semilattice iff $\lambda(X)$ is a semilattice iff $\varphi(X)$ is a semilattice iff the semilattice X is finite and linearly ordered. We prove that the semigroup $\beta(X)$ is a band if and only if X has no infinite antichains, and the semigroup $\lambda(X)$ is commutative if and only if X is a bush with finite branches.

Introduction

One of powerful tools in the modern Combinatorics of Numbers is the method of ultrafilters based on the fact that each (associative) binary operation $*: X\times X\to X$ defined on a discrete topological space X extends to a right-topological (associative) operation $*:\beta(X)\times\beta(X)\to\beta(X)$ on the Stone-Čech compactification $\beta(X)$ of X, see [HS], [P]. The Stone-Čech extension $\beta(X)$ is the space of ultrafilters on X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula:

$$\mathcal{U}*\mathcal{V}=\Big\langle\bigcup_{x\in U}x{*}V_x:U\in\mathcal{U},\ \ (V_x)_{x\in U}\in\mathcal{V}^U\Big\rangle, \tag{1}$$

where $\langle\mathcal{B}\rangle=\{A\subset X:\exists B\in\mathcal{B}\ \ B\subset A\}$ is the upper closure of a family $\mathcal{B}$. In this case $\mathcal{B}$ is called a *base* of $\langle\mathcal{B}\rangle$.

Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [HS], [P].

In [G2] it was observed that the binary operation $*$ extends not only to $\beta(X)$ but also to the space $\upsilon(X)$ of all upfamilies on X. By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *upfamily* if for any sets $A\subset B\subset X$ the inclusion $A\in\mathcal{F}$ implies $B\in\mathcal{F}$. The space $\upsilon(X)$ is a closed subspace of the double power-set $\mathcal{P}(\mathcal{P}(X))$ endowed with the compact Hausdorff topology of the Tychonoff power $\{0,1\}^{\mathcal{P}(X)}$. In the papers [G1], [G2], [BGN]–[BG4] the space $\upsilon(X)$ was denoted by $G(X)$ and its elements were called inclusion hyperspaces[1]. The extension of a binary operation $*$ from X to $\upsilon(X)$ can be defined in the same way as for ultrafilters, i.e., by the formula (1) applied to any two upfamilies $\mathcal{U},\mathcal{V}\in\upsilon(X)$. If X is a semigroup, then $\upsilon(X)$ is a compact Hausdorff right-topological semigroup containing $\beta(X)$ as closed subsemigroups. The algebraic properties of this semigroups were studied in details in [G2].

The space $\upsilon(X)$ of upfamilies over a discrete space X contains many interesting subspaces. First we recall some definitions. An upfamily $\mathcal{A}\in\upsilon(X)$ is defined to be

- *a filter* if $A_1\cap A_2\in\mathcal{A}$ for all sets $A_1,A_2\in\mathcal{A}$;
- *an ultrafilter* if $\mathcal{A}=\mathcal{A}'$ for any filter $\mathcal{A}'\in\upsilon(X)$ containing $\mathcal{A}$;

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[1] We decided to change the terminology and notation after discovering the paper [SS, 2.7.4] that discusses monadic properties of the up-set functor υ.

- *linked* if $A \cap B \neq \emptyset$ for any sets $A, B \in \mathcal{A}$;
- *maximal linked* if $\mathcal{A} = \mathcal{A}'$ for any linked upfamily $\mathcal{A}' \in \upsilon(X)$ containing $\mathcal{A}$.

By $\varphi(X)$, $\beta(X)$, $N_2(X)$, and $\lambda(X)$ we denote the subspaces of $\upsilon(X)$ consisting of filter, ultrafilters, linked upfamilies, and maximal linked upfamilies, respectively. The space $\lambda(X)$ is called *the superextension* of X, see [vM], [Ve]. In [G2] it was observed that for a discrete semigroup X the subspaces $\varphi(X)$, $\beta(X)$, $N_2(X)$, $\lambda(X)$ are closed subsemigroups of the semigroup $\upsilon(X)$. The following diagram describes the inclusion relations between these subspaces of $\upsilon(X)$ (an arrow $A \to B$ indicates that A is a subset of B).

$$
\begin{array}{ccccc}
\beta(X) & \longrightarrow & \lambda(X) & & \\
\downarrow & & \downarrow & & \\
\varphi(X) & \longrightarrow & N_2(X) & \longrightarrow & \upsilon(X)
\end{array}
$$

In [G2], [BGN] — [BG4] we studied the properties of the compact right-topological semigroup $\upsilon(X)$ and its subsemigroups for groups X. In this paper we shall study the algebraic structure of the semigroups $\lambda(X)$, $\varphi(X)$, $N_2(X)$, and $\upsilon(X)$ for semilattices X.

Let us recall that a *semilattice* is a commutative idempotent semigroup. Idempotent semigroups are called *bands*. So, in a band each element x is an *idempotent*, which means that $xx = x$. A semigroup S is *linear* if $xy \in \{x, y\}$ for any elements $x, y \in X$. It follows that each linear semigroup S is a band. Each (linear) semilattice is partially (linearly) ordered by the relation $\leq$ defined by $x \leq y$ iff $xy = x$.

A semigroup S is called *a regular semigroup* if $a \in aSa$ for any $a \in S$. Such a semigroup S is called *an inverse semigroup* if $ab = ba$ for any idempotents $a, b \in S$. A semigroup which is an union of groups is called *a Clifford semigroup*. Every band is a Clifford semigroup and every Clifford semigroup is a regular semigroup. An inverse semigroup with a unique idempotent is a group.

These algebraic properties relate as follows:

$$
\begin{array}{ccccccc}
\text{semilattice} & \longrightarrow & \text{band} & \longrightarrow & \text{Clifford semigroup} & \longrightarrow & \text{regular semigroup} \\
\downarrow & & & & \uparrow & & \uparrow \\
\text{commutative inverse semigroup} & & \longrightarrow & & \text{Clifford inverse semigroup} & \longrightarrow & \text{inverse semigroup} \\
\uparrow & & & & \uparrow & & \\
\text{commutative group} & & \longrightarrow & & \text{group} & &
\end{array}
$$

In this paper we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$ or $N_2(X)$ are bands, linear semigroups, commutative semigroups, or semilattices. In Section 5 we shall characterize lattices X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$ are lattices.

1. Semigroups whose extensions are bands

In this section we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are bands. Let us recall that a semigroup S is a (linear) band if $xx = x$ for all $x \in X$ (and $xy \in \{x, y\}$ for all $x, y \in X$).

Let us recall that an element a of a semigroup S is *regular* in S if $a \in aSa$. It is clear that each idempotent is a regular element.

Theorem 1.1. *For a semigroup X the following conditions are equivalent:*

1. *X is linear;*
2. *$\upsilon(X)$ is a band;*
3. *$\varphi(X)$ is a band;*

(4) $\lambda(X)$ *is a band.*

Proof. (1) ⇒ (2) Assume that the semigroup X is linear. To show that $\upsilon(X)$ is a band, we should check that $\mathcal{A} * \mathcal{A} = \mathcal{A}$ for any upfamily $\mathcal{A} \in \upsilon(X)$. Since X is linear, for any $A \in \mathcal{A}$ we get $A = A * A \in \mathcal{A} * \mathcal{A}$ and hence $\mathcal{A} \subset \mathcal{A} * \mathcal{A}$.

To show that $\mathcal{A} \supset \mathcal{A} * \mathcal{A}$, fix any basic subset $B = \bigcup_{x\in A} x*A_x \in \mathcal{A}*\mathcal{A}$ where $A \in \mathcal{A}$ and $A_x \in \mathcal{A}$ for all $x \in A$. Now we consider two cases.

(i) There is $x \in A$ such that $xa = a$ for all $a \in A_x$. In this case $\mathcal{A} \ni A_x = x*A_x \subset B$ and thus $B \in \mathcal{A}$.

(ii) For every $x \in A$ there is $a \in A_x$ such that $xa \neq a$ and hence $xa = x$ (as X is linear). In this case $\mathcal{A} \ni A \subset \bigcup_{x\in A} x * A_x = B$ and hence $B \in \mathcal{A}$.

The implications (2) ⇒ (3, 4) are trivial.

(3) ⇒ (1) Assume that $\varphi(X)$ is a band. Then X, being a subsemigroup of $\varphi(X)$, also is a band. To show that X is linear, take any two points $x, y \in X$ and consider the filter $\mathcal{F} = \langle\{x, y\}\rangle \in \varphi(X)$. Being an idempotent, the filter $\mathcal{F}$ is regular in $\upsilon(X)$. Consequently, we can find an upfamily $\mathcal{A} \in \upsilon(X)$ such that $\mathcal{F} * \mathcal{A} * \mathcal{F} = \mathcal{F}$. It follows that there are sets $A_x, A_y \in \mathcal{A}$ such that $(xA_x \cup yA_y) \cdot \{x, y\} \subset \{x, y\}$. In particular, for every $a_x \in A_x$ we get $xa_xy \in \{x, y\}$. If $xa_xy = x$, then $xy = xa_xyy = xa_xy = x$. If $xa_xy = y$, then $xy = xxa_xy = xa_xy = y$, witnessing that the band X is linear.

(4) ⇒ (1) Assume that $\lambda(X)$ is a band. Then X, being a subsemigroup of $\lambda(X)$, is a band as well. Assuming that the band X is not linear, we can find two points $x, y \in X$ such that $xy \notin \{x, y\}$. It can be shown that the maximal linked system $\mathcal{L} = \langle\{x, y\}, \{x, xy\}, \{y, xy\}\rangle \in \lambda(X)$ is not an idempotent and even is not regular in $\upsilon(X)$. □

Observe that the proof of Theorem 4.1 yields a bit more, namely:

Proposition 1.2. *For a band X the following conditions are equivalent:*

(1) *X is linear;*
(2) *each element of $\varphi(X)$ is regular in $\upsilon(X)$;*
(3) *each element of $\lambda(X)$ is regular in $\upsilon(X)$.*

Next we characterize semigroups X whose Stone-Čech extension $\beta(X)$ is a band.

Theorem 1.3. *For a semigroup X the semigroup $\beta(X)$ is a band if and only if for each sequence $(x_n)_{n\in\omega}$ in X there are numbers $n < m$ such that $x_nx_m \in \{x_n, x_m\}$.*

Proof. If there exists a sequence $(x_n)_{n\in\omega}$ such that $x_nx_m \notin \{x_n, x_m\}$ for all $n < m$, then we can take any free ultrafilter $\mathcal{A}$ that contains the set $A = \{x_n\}_{n\in\omega}$ and observe that $A \cap \bigcup_{n\in\omega} x_n * \{x_m\}_{m>n} = \emptyset$, which implies that $\mathcal{A} \neq \mathcal{A} * \mathcal{A}$ and hence the ultrafilter $\mathcal{A}$ is not an idempotent in $\beta(X)$.

Now assume that $\beta(X)$ is not a band and find an ultrafilter $\mathcal{F} \in \beta(X)$ with $\mathcal{F} * \mathcal{F} \neq \mathcal{F}$. In particular, $\mathcal{F} * \mathcal{F} \not\subset \mathcal{F}$. This implies that for some $A \in \mathcal{F}$ and $\{A_x\}_{x\in A} \subset \mathcal{F}$ the set $\bigcup_{x\in A} x*A_x \notin \mathcal{F}$.

Consider the set $X^{\uparrow}_{\mathcal{F}} = \{x \in X : {\uparrow}x \in \mathcal{F}\}$ where ${\uparrow}x = \{y \in X : xy = x\}$. We claim that $X^{\uparrow}_{\mathcal{F}} \notin \mathcal{F}$. Assuming that $X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}$, we conclude that $A \cap X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}$. This implies that ${\uparrow}a \in \mathcal{F}$ and ${\uparrow}a \cap A_a \in \mathcal{F}$ for any $a \in A \cap X^{\uparrow}_{\mathcal{F}}$. Therefore $a * ({\uparrow}a \cap A_a) = \{a\}$ and hence

$$\bigcup_{x\in A} x * A_x \supset \bigcup_{x\in A\cap X^{\uparrow}_{\mathcal{F}}} x * ({\uparrow}x \cap A_x) = \bigcup_{x\in A\cap X^{\uparrow}_{\mathcal{F}}} \{x\} = A \cap X^{\uparrow}_{\mathcal{F}} \in \mathcal{F}.$$

Thus $\bigcup_{x\in A} x*A_x \in \mathcal{F}$. This contradiction shows that $X^{\uparrow}_{\mathcal{F}} \notin \mathcal{F}$.

Next, consider the set $X^{\downarrow}_{\mathcal{F}}=\{x\in X:{\downarrow}x\in\mathcal{F}\}$ where ${\downarrow}x=\{y\in X:xy=y\}$. We claim that $X^{\downarrow}_{\mathcal{F}}\notin\mathcal{F}$. Assume that $X^{\downarrow}_{\mathcal{F}}\in\mathcal{F}$, then $A\cap X^{\downarrow}_{\mathcal{F}}\in\mathcal{F}$. This implies that ${\downarrow}a\in\mathcal{F}$ and ${\downarrow}a\cap A_a\in\mathcal{F}$ for any $a\in A\cap X^{\downarrow}_{\mathcal{F}}$. Therefore

$$ {\downarrow}a\cap A_a\subset a*({\downarrow}a\cap A_a)\subset a*A_a\subset\bigcup_{x\in A}x*A_x. $$

Thus $\bigcup_{x\in A}x{*}A_x\in\mathcal{F}$. This contradiction shows that $X^{\downarrow}_{\mathcal{F}}\notin\mathcal{F}$.

Since $\mathcal{F}$ is an ultrafilter, $X^{\uparrow}_{\mathcal{F}}\cup X^{\downarrow}_{\mathcal{F}}\notin\mathcal{F}$ and $Z_{\mathcal{F}}=X\setminus(X^{\uparrow}_{\mathcal{F}}\cup X^{\downarrow}_{\mathcal{F}})\in\mathcal{F}$. Let $x_0\in Z_{\mathcal{F}}$ be arbitrary and by induction, for every $n\in\omega$ choose a point $x_{n+1}\in Z_{\mathcal{F}}\setminus\bigcup_{i\le n}({\uparrow}x_i\cup{\downarrow}x_i)\in\mathcal{F}$. Then the sequence $(x_n)_{n\in\omega}$ has the required property: $x_nx_m\notin\{x_n,x_m\}$ for $n<m$ (which follows from $x_m\notin{\downarrow}x_n\cup{\uparrow}x_n$). □

A subset A of a semigroup X is called an *antichain* if $ab\notin\{a,b\}$ for any distinct points $a,b\in A$. Theorem implies the following characterization:

Corollary 1.4. *For a commutative semigroup X the semigroup $\beta(X)$ is a band if and only if each antichain in X is finite.*

2. Semilattices whose extensions are commutative

In this section we recognize the structure of semilattices X whose extensions $\upsilon(X)$, $N_2(X)$ or $\lambda(X)$ are commutative.

Commutative semigroups of ultrafilters were characterized in [HS, 4.27] as follows:

Theorem 2.1. *The Stone-Čech extension $\beta(X)$ of a semigroup S is not commutative if and only if there are sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ in X such that $\{x_ky_n:k<n\}\cap\{y_kx_n:k<n\}=\emptyset$.*

This characterization implies the following (well-known) fact:

Corollary 2.2. *If the Stone-Čech extension $\beta(X)$ of a semilattice X is commutative, then each linear subsemigroup in X in finite.*

Proof. Assume conversely that X contains an infinite linear subsemilattice L. Being linear, L is linearly ordered by the order $\le$ defined by $x\le y$ iff $xy=x$. Since L is infinite, we can apply Ramsey Theorem in order to find an injective sequence $(z_n)_{n\in\omega}$ in L, which is either strictly increasing or strictly decreasing. Put $x_n=z_{2n}$ and $y_n=z_{2n+1}$ for $n\in\omega$. Applying Theorem 2.1 to the sequences $(x_n)_{n\in\omega}$ and $(y_n)_{n\in\omega}$ we conclude that the semigroup $\beta(L)$ is not commutative. Then $\beta(X)$ is not commutative neither. □

In spite of Theorem 2.1 the following problem seems to be open.

Problem 2.3. *Describe the structure of semilattice X whose Stone-Čech extension $\beta(X)$ is commutative.*

A similar problem on commutativity of semigroups $\upsilon(X)$ also is open:

Problem 2.4. *Characterize semigroups X whose extension $\upsilon(X)$ is commutative.*
(It can be shown that if $\upsilon(X)$ is commutative, then X is a commutative semigroup with finite linear idempotent band $E=\{x\in X:xx=x\}$ and $x^3=x^4$ for all $x\in X$).

We shall resolve this problem for bands. First we prove a useful result on multiplication of upfamilies on linear semigroups.

For a semigroup X denote by $\upsilon^\bullet(X)$ the subsemigroup of $\upsilon(X)$ consisting of all upfamilies $\mathcal{A}\in\upsilon(X)$ such that for each set $A\in\mathcal{A}$ there is a finite subset $F\in\mathcal{A}$ with $F\subset A$.

For a semigroup X and two upfamilies $\mathcal{A}, \mathcal{B} \in \upsilon(X)$ let

$$\mathcal{A} \otimes \mathcal{B} = \langle A * B : A \in \mathcal{A},\ B \in \mathcal{B} \rangle.$$

It is clear that $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$. In the following theorem we show that for finite linear semigroups the converse inclusion also holds.

Theorem 2.5. *If X is a linear semigroup, then $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B}$ for any upfamilies $\mathcal{A} \in \upsilon^\bullet(X)$ and $\mathcal{B} \in \upsilon(X)$.*

Proof. On the semigroup X consider the relation $\le$ defined by: $x \le y$ iff $yx = x$. This relation is reflexive and transitive. For a subsets $A \subset X$ and a point $x \in X$ we write $A \le x$ if $a \le x$ for all $a \in A$. It follows from the definition of the semigroup operation $*$ on $\upsilon(X)$ that $\mathcal{A} \otimes \mathcal{B} \subset \mathcal{A} * \mathcal{B}$. To prove the reverse inclusion, fix any basic set $C = \bigcup_{a \in A} a * B_a \in \mathcal{A} * \mathcal{B}$ where $A \in \mathcal{A}$ and $B_a \in \mathcal{B}$ for all $a \in A$. Since $\mathcal{A} \in \upsilon^\bullet(X)$, we can assume that the set A is finite and hence can be enumerated as $A = \{a_1, \dots, a_n\}$ so that $a_i \le a_{i+1}$ for all $i < n$. Now let us consider two cases.

1. For some $i \le n$ we get $B_{a_i} \le a_i$, which means that $a_i b = b$ for all $b \in B_{a_i}$ and hence $a_i * B_{a_i} = B_{a_i}$. For every $j \ge i$ the inequality $B_{a_i} \le a_i \le a_j$ implies $a_j * B_{a_i} = B_{a_i}$. Consequently, $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i}$.

The minimality of i implies that $B_{a_j} \not\le a_j$ for all $j < i$. This means $b_j \not\le a_j$ for some $b_j \in B_{a_j}$ and hence $a_j b_j = a_j$ (as $a_j b_j \in \{a_j, b_j\}$ and $a_j b_j \ne b_j$). Then $a_j * B_{a_j} \ni a_j b_j = a_j$ and thus $A * B_{a_i} \subset \{a_1, \dots, a_{i-1}\} \cup B_{a_i} \subset \bigcup_{j=1}^n a_j B_{a_j}$, which implies that $C \in \mathcal{A} \otimes \mathcal{B}$.

2. $B_{a_i} \not\le a_i$ for all $i \le n$. In this case $a_i \in a_i * B_{a_i}$ for all i, and hence $A * B_{a_n} \subset \{a_1, \dots, a_n\} \cup a_n * B_{a_n} \subset \bigcup_{i=1}^n a_i * B_{a_i} = C$, so $C \in \mathcal{A} \otimes \mathcal{B}$. □

Now we are able to characterize bands X with commutative extensions $\upsilon(X)$ and $N_2(X)$.

Theorem 2.6. *For a band X the following conditions are equivalent:*

(1) *X is a finite linear semilattice;*
(2) *the semigroup $\upsilon(X)$ is commutative;*
(3) *the semigroup $N_2(X)$ is commutative.*

Proof. The implication (1) ⇒ (2) follows from Theorem 2.5 as $\mathcal{A} * \mathcal{B} = \mathcal{A} \otimes \mathcal{B} = \mathcal{B} \otimes \mathcal{A} = \mathcal{B} * \mathcal{A}$ for every $\mathcal{A}, \mathcal{B} \in \upsilon^\bullet(X) = \upsilon(X)$.

The implication (2) ⇒ (3) is trivial.

(3) ⇒ (1) Assume that the semigroup $N_2(X)$ is commutative. Then so is the semigroup X. Being a commutative band, the semigroup X is a semilattice. Assuming that X is not linear, we can find two points $x, y \in X$ with $xy \notin \{x, y\}$. It can be shown that the linked upfamilies $\mathcal{A} = \langle \{x, y\} \rangle$ and $\mathcal{B} = \langle \{x, xy\}, \{y, xy\} \rangle \in N_k(X)$ do not commute because $\{xy\} \in \mathcal{A} * \mathcal{B} \setminus \mathcal{B} * \mathcal{A}$. Therefore, X is a linear semilattice. Since $\beta(X) \subset \upsilon(X)$ is commutative, Corollary 2.2 implies that the linear semilattice X is finite. □

Now we shall characterize semilattices X with commutative superextension $\lambda(X)$. A semilattice X is called a *bush* if for any maximal linear subsemilattices $A, B \subset X$ the product $A * B$ is the singleton $\{\min X\}$ containing the smallest element $\min X$ of X. This definition implies that $A \cap B = A * B = \{\min X\}$. By a *branch* of a bush X we understand a maximal linear subsemilattice of X.

Theorem 2.7. *A semilattice X has commutative superextension $\lambda(X)$ if and only if X is a bush with finite branches.*

Proof. First assume that X is a bush with finite branches, and take any two maximal linked systems $\mathcal{A}, \mathcal{B} \in \lambda(X)$. Since the products $\mathcal{A} * \mathcal{B}$ and $\mathcal{B} * \mathcal{A}$ are maximal linked upfamilies, the equality $\mathcal{A} * \mathcal{B} = \mathcal{B} * \mathcal{A}$ will follow as soon as we check that any two basic sets $C_{AB} = \bigcup_{a\in A} a{*}B_a \in \mathcal{A} * \mathcal{B}$ and $C_{BA} = \bigcup_{b\in B} b{*}A_b \in \mathcal{B} * \mathcal{A}$ have non-empty intersection. Here $A \in \mathcal{A}$, $(B_a)_{a\in A} \in \mathcal{B}^A$, $B \in \mathcal{B}$, and $(A_b)_{b\in B} \in \mathcal{A}^B$. Assume conversely that $C_{AB} \cap C_{BA} = \emptyset$. Then either $\min X \notin C_{AB}$ or $\min X \notin C_{BA}$.

Without loss of generality, $\min X \notin C_{AB}$. Then $\min X \notin A$ and for each $a \in A$ the set $\{a\} \cup B_a$ lies in a branch of X. Since branches of X meet only at the point $\min X$, all the sets $\{a\} \cup B_a$, $a \in A$, lie in the same (finite) branch. Repeating the argument of Theorem 2.5, we can show that $C_{AB} \supset AB'$ for some set $B' \in \mathcal{B}$. Since $\mathcal{B}$ is linked, there is a point $b \in B \cap B'$. By the same reason, there is a point $a \in A \cap A_b$. Then $ab = ba \in AB' \cap bA_b \subset C_{AB} \cap C_{BA}$ and we are done.

Now assume that X is a semilattice with commutative superextension $\lambda(X)$. Corollary 2.2 implies that all branches of X are finite. We claim that for every $z \in X$ the lower set ${\downarrow}z = \{x \in X : xz = x\}$ is linear. Assuming the converse, find two points $x, y \in {\downarrow}z$ such that $xy \notin \{x, y\}$. It follows that the points x, y, z, xy are pairwise distinct. It is easy to check that the maximal linked upfamilies $\mathcal{A} = \langle\{x, y\}, \{x, z\}, \{y, z\}\rangle$ and $\mathcal{B} = \langle\{x, y\}, \{x, xy\}, \{y, xy\}\rangle$ do not commute because $\{x, y\} \in \mathcal{B}{*}\mathcal{A} \setminus \mathcal{A}{*}\mathcal{B}$. Thus ${\downarrow}z$ is linear for every $z \in X$, which means that X is a tree.

Assuming that the tree X is not a bush, we can find two points $x, y \in X$ such that $xy \notin \{x, y, z\}$ where $z = \min X$. Now consider the maximal linked systems $\mathcal{A} = \langle\{x, y\}, \{x, z\}, \{y, z\}\rangle$ and $\mathcal{B} = \langle\{x, y\}, \{x, xy\}, \{y, xy\}\rangle$ and observe that they do not commute as $\{xy\} \in \mathcal{A} * \mathcal{B}$ misses the set $\{x, y, z\} \in \mathcal{B} * \mathcal{A}$. □

3. Semigroups whose extensions are semilattices

In this section we shall characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$, $\varphi(X)$, or $N_2(X)$ are semilattices.

Theorem 3.1. *For a semigroup X the following conditions are equivalent:*

(1) *X is finite linear semilattice;*
(2) *$\upsilon(X)$ is a semilattice;*
(3) *$\lambda(X)$ is a semilattice;*
(4) *$\varphi(X)$ is a semilattice.*

Proof. (1) ⇒ (2) If X is a finite linear semilattice, then $\upsilon(X)$ is a semilattice (=commutative band) by Theorems 1.1 and 2.6.

The implications (2) ⇒ (3, 4) are trivial.

The implication (3) ⇒ (1) follows from Theorems 1.1 and 2.7.

(4) ⇒ (1) Assume that $\varphi(X)$ is a semilattice. Then X, being a subsemigroup of the commutative semigroup $\varphi(X)$ is commutative. Since $\varphi(X)$ is a band, X is a linear semigroup by Theorem 1.1. Thus X, being a commutative linear semigroup, is a linear semilattice. Since the subsemigroup $\beta(X) \subset \lambda(X)$ is commutative, the linear semilattice X is finite by Corollary 2.2. □

4. Semigroups whose extensions are linear

In this section we characterize semigroups X whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are linear semigroups.

A semigroup S is called a *semigroup of left* (*right*) *zeros* if $xy = x$ (resp. $xy = y$).

Theorem 4.1. *For a semigroup X the semigroup $\upsilon(X)$ is linear if and only if X is either a semigroup of right zeros or a semigroup of left zeros.*

Proof. If X is a semigroup of left zeros, then for any upfamilies $\mathcal{A},\mathcal{B}\in\upsilon(X)$ and any basic element $\bigcup_{x\in A}xB_x\in\mathcal{A}*\mathcal{B}$ we get $\bigcup_{x\in A}xB_x=\bigcup_{x\in A}\{x\}=A$ and thus $\mathcal{A}*\mathcal{B}\subset\mathcal{A}$. On the other hand, each $A\in\mathcal{A}$ belongs to $\mathcal{A}*\mathcal{B}$ as $A=A*B\in\mathcal{A}*\mathcal{B}$ for any $B\in\mathcal{B}$.

Assume that the semigroup $\upsilon(X)$ is linear. Then X, being a subsemigroup of $\upsilon(X)$, also is linear. Let x,y be any two distinct elements of X. First we prove that $xy\neq yx$. Assume conversely that $xy=yx$. Then $xy=yx\in\{x,y\}$ and we lose no generality assuming that $xy=x$. Now consider two upfamilies $\mathcal{A}=\langle\{x,y\}\rangle$ and $\mathcal{B}=\langle\{x\},\{y\}\rangle$ and observe that

$$\mathcal{B}*\mathcal{A}=\langle\{xx,xy\},\{yx,yy\}\rangle=\langle\{x\},\{x,y\}\rangle=\langle\{x\}\rangle\notin\{\mathcal{A},\mathcal{B}\},$$

so $\upsilon(X)$ is not linear and this is a required contradiction.

Thus $xy\neq yx$ for all distinct points $x,y\in X$. We call a pair $(x,y)\in X^2$ *left* if $xy=x$ and $yx=y$ and *right* if $xy=y$ and $yx=x$. Since X is linear, each pair $(x,y)\in X^2$ is either left or right. We claim that either all pairs $(x,y)\in X^2$ are left or else all such pairs are right. Assuming the opposite, find pairs $(x,y),(a,b)\in X^2$ such that (x,y) is not left and (a,b) is not right. Then $x\neq y$, $a\neq b$ and the pair (x,y) is right while (a,b) is left. Consider the filters $\mathcal{A}=\langle\{x,a\}\rangle$ and $\mathcal{B}=\langle\{y,b\}\rangle$ and observe that $\mathcal{A}*\mathcal{B}=\langle\{xy,xb,ay,ab\}\rangle=\langle\{y,xb,ay,a\}\rangle$. Since $\upsilon(X)$ is linear, either $\mathcal{A}*\mathcal{B}=\mathcal{A}$ or $\mathcal{A}*\mathcal{B}=\mathcal{B}$. In the first case $\{x,a\}\supset\{y,xb,ay,a\}\supset\{y,a\}$ and hence $y=a$. In the second case, $\{y,a\}\subset\{y,b\}$ and thus $a=y$. Now consider the filters $\mathcal{C}=\langle\{x,b\}\rangle$ and $\mathcal{D}=\langle\{a\}\rangle$ and observe that $\mathcal{C}*\mathcal{D}=\langle\{xa,ba\}\rangle=\langle\{xy,b\}\rangle=\langle\{y,b\}\rangle=\langle\{a,b\}\rangle\notin\{\mathcal{C},\mathcal{D}\}$, which contradicts the linearity of $\upsilon(X)$.

Therefore either each pair $(x,y)\in X^2$ is left and then X is a semigroup of left zeros or else each pair $(x,y)\in X^2$ is right and then X is a semigroup of right zeros. □

Theorem 4.2. *For a semigroup X the following conditions are equivalent:*

1. *the semigroup $\varphi(X)$ is linear;*
2. *the semigroup $N_2(X)$ is linear;*
3. *either X is a semigroup of left zeros or X is a semigroup of right zeros or else X is a semilattice of order $|X|\le 2$.*

Proof. (3) ⇒ (2) If $|X|=1$, then $N_2(X)$ is a singleton and hence is a linear semigroup. If X is a semilattice of order $|X|=2$, then $X=\{0,1\}$ for some elements $0,1$ with $0\cdot 1=1\cdot 0=0$. In this case $N_2(X)=\varphi(X)$ is a 3-element linear semilattice ordered as:

$$\langle\{0\}\rangle\le\langle\{0,1\}\rangle\le\langle\{1\}\rangle.$$

If X is a semigroup of left or right zeros, then the semigroup $\upsilon(X)$ is linear by Theorem 4.1 and so is its subsemigroup $N_2(X)$.

(2) ⇒ (1) Is the semigroup $N_2(X)$ is linear, then so is its subsemigroup $\varphi(X)$.

(1) ⇒ (3) Assume that the semigroup $\varphi(X)$ is linear. Then X, being a subsemigroup of $\varphi(X)$, is linear as well. If $|X|\le 2$, then either X is a linear semilattice or a semigroup or left or right zeros. So, we assume that $|X|\ge 3$. We claim that distinct elements $x,y\in X$ do not commute. Assume conversely that $xy=yx$ for some distinct elements $x,y\in X$. Since $xy=yx\in\{x,y\}$ we lose no generality assuming that $xy=yx=x$. Fix any element $z\in X\setminus\{x,y\}$. Now consider 3 cases:

1. $zx=z$. In this case we can consider the filters $\mathcal{A}=\langle\{z,y\}\rangle$ and $\mathcal{B}=\langle\{x,y\}\rangle$ and observe that $\mathcal{A}*\mathcal{B}=\langle\{zx,yx,zy,yy\}\rangle=\langle\{z,x,zy,y\}\rangle\notin\{\mathcal{A},\mathcal{B}\}$, which contradicts the linearity of $\varphi(X)$.

2. $zx=x$ and $zy=z$. In this case we can consider the filters $\mathcal{A}=\langle\{z,y\}\rangle$ and $\mathcal{B}=\langle\{x,y\}\rangle$ and observe that $\mathcal{A}*\mathcal{B}=\langle\{zx,yx,zy,yy\}\rangle=\langle\{x,x,z,y\}\rangle\notin\{\mathcal{A},\mathcal{B}\}$, which contradicts the linearity of $\varphi(X)$.

3. $zx = x$ and $zy = y$. In this case we can consider the filters $\mathcal{A} = \langle\{x, z\}\rangle$ and $\mathcal{B} = \langle\{y, z\}\rangle$ and observe that $\mathcal{A} * \mathcal{B} = \langle\{xy, xz, zy, zz\}\}\rangle = \langle\{x, xz, y, z\}\rangle \notin \{\mathcal{A}, \mathcal{B}\}$, which again contradicts the linearity of $\varphi(X)$.

Those contradictions show that distinct elements of X do not commute. Continuing as in the proof of Theorem 4.1, we can show that X is a semigroup of right or left zeros. □

Finally, we characterize commutative semigroups with linear superextensions.

Theorem 4.3. *For a commutative semigroup X the semigroup $\lambda(X)$ is linear if and only if X is a linear semilattice of order $|X| \le 3$.*

Proof. If X is a linear semilattice of order $|X| \le 2$, then the semigroup $\lambda(X) = X$ is linear.

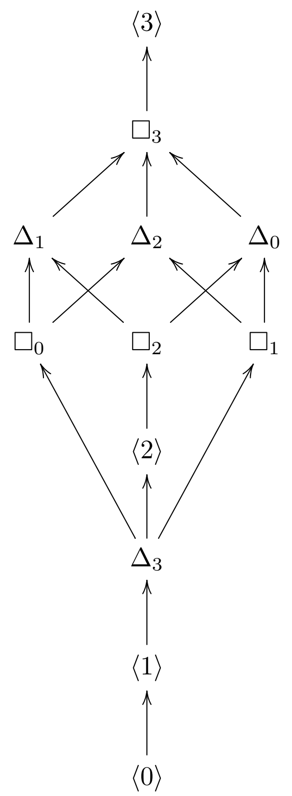
If X is a linear semilattice of order $|X| = 3$, then X can be identified with the set $3 = \{0, 1, 2\}$ endowed with the operation $xy = \min\{x, y\}$. The semigroup $\lambda(X)$ contains 4 elements: $0, 1, 2$ and $\Delta = \{A \subset 3 : |A| \ge 2\}$. One can check that $\lambda(3)$ is a linear semilattice ordered as follows:

$$0 \le \Delta \le 1 \le 2.$$

This proves the "if" part of the theorem. To prove the "only if" part we first shall analyze the structure of the superextension $\lambda(4)$ of the semilattice $4 = \{0, 1, 2, 3\}$ endowed with the operation $xy = \min\{x, y\}$. By Theorem 3.1, $\lambda(4)$ is a semilattice. It contains 12 elements:

$$\langle k\rangle,\ \ \Delta_k = \langle\{A \subset n : |A| = 2,\ k \notin A\}\ \text{ and }\ \Box_k = \langle\{n \setminus \{k\}, A : A \subset n,\ |A| = 2,\ k \in A\}\rangle\ \text{ where } k \in 4.$$

The order structure of the semilattice $\lambda(4)$ is described in the following diagram:



Looking at this diagram we see that the semilattice $\lambda(4)$ is not linear.

Now assume that X is a commutative semigroup whose superextension $\lambda(X)$ is linear. Then X is a linear semilattice. If $|X| \ge 3$, then $\lambda(X)$ is not linear as it contains a subsemigroup isomorphic to the semilattice $\lambda(3)$, which is not linear. □

5. Lattices whose extensions are lattices

In this section we characterize lattices whose extensions $\upsilon(X)$, $\lambda(X)$ or $\varphi(X)$ are lattices.

A *lattice* is a set X endowed with two semilattice operations $\wedge, \vee : X \times X \to X$ such that $(x \wedge y) \vee y = y$ and $(x \vee y) \wedge y = y$ for all $x, y \in X$.

Both operations $\wedge$ and $\vee$ of a lattice X can be extended to right-topological operations $\wedge$ and $\vee$ on the compact Hausdorff space $\upsilon(X)$. Is it natural to ask if the triple $(\upsilon(X), \wedge, \vee)$ is a lattice.

A lattice will be called *linear* if $x \wedge y, x \vee y \in \{x, y\}$ for all $x, y \in X$.

Theorem 5.1. *For a lattice X the following conditions are equivalent:*

1. *X is a linear lattice of order $|X| \le 2$.*
2. *$\upsilon(X)$ is a lattice;*
3. *$\lambda(X)$ is a lattice;*
4. *$\varphi(X)$ is a lattice.*

Proof. $(1) \Rightarrow (2)$ If X is a linear lattice of order $|X| = 1$, then $\upsilon(X) = X$ is a trivial lattice. If X is a linear lattice of order 2, then X can be identified with the lattice $2 = \{0, 1\}$ endowed with the operations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. In this case $\lambda(2) = \beta(2)$ coincides with the lattice 2, $\varphi(2) = \{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{1\}\rangle\}$ is a 3-element lattice, isomorphic to the lattice $3 = \{0, 1, 2\}$ endowed with the operations min and max, and $\upsilon(2) = \big\{\langle\{0\}\rangle, \langle\{0, 1\}\rangle, \langle\{0\}, \{1\}\rangle, \langle\{1\}\rangle\big\}$ is a 4-element lattice isomorphic to the lattice $\{0, 1\}^2$.

The implications $(2) \Rightarrow (3, 4)$ are trivial.

$(3, 4) \Rightarrow (1)$ Assume that $\lambda(X)$ or $\varphi(X)$ is a lattice. By Theorem 3.1, the lattice X is finite and linear. We claim that $|X| \le 2$. Assuming the converse, we conclude that the lattice X contains a sublattice isomorphic to the lattice $(3, \min, \max)$.

Consider the maximal linked upfamily $\Delta = \{A \subset 3 : |A| \ge 2\}$ and observe that

$$\max\{\Delta, \langle 1\rangle\} = \langle 1\rangle = \min\{\Delta, \langle 1\rangle\},$$

which implies that $\lambda(3)$ is not a lattice and then $\lambda(X)$ also is not a lattice.

Next, consider the filters $\mathcal{A} = \langle\{0, 1, 2\}\rangle$ and $\mathcal{B} = \langle\{0, 2\}\rangle$ and observe that

$$\max\{\mathcal{A}, \mathcal{B}\} = \mathcal{A} = \min\{\mathcal{A}, \mathcal{B}\}$$

implying that $\varphi(3)$ is not a lattice and then $\varphi(X)$ also cannot be a lattice. □

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