УДК 517.589, 517.926.4

Т. П. Гой, к. ф.-м. наук,[1]
Р. А. Заторський, д. ф.-м. наук[2]

Про одну неелементарну функцію типу інтеграла Доусона

Досліджена нова неелементарна функція типу інтеграла Доусона, побудована у вигляді степеневого ряду при допомозі зростаючих факторіальних степенів. Встановлений її зв'язок з функцією помилок (функцією ймовірностей). Показано, що нова функція є розв'язком рівняння Ріккаті.

Ключові слова: зростаючий факторіальний степінь, інтеграл Доусона, функція помилок, рівняння Ріккаті.

Прикарпатський національний університет імені Василя Стефаника, 76018, Івано-Франківськ, вул. Шевченка, 57.

T. P. Goy, Ph. D.,
R. A. Zatorsky, D. Sc.

On a nonelementary function of the Dawson's integral type

A new nonelementary function of the Dawson's integral type is studied. It is constructed as a power series with the help of rising factorial powers. Its connection with the error function (probability function) is determined. It is proved that the new function is a solution of the Riccati equation.

Key Words: rising factorial power, Dawson's integral, error function, Riccati equation.

Precarpathian Vasyl Stefanyk National University, 76018, Ivano-Frankivsk, 57 Shevchenko Str.

E-mail: [1]tarasgoy@yahoo.com [2]romazz@rambler.ru
Communicated by Prof. Kirichenko V.V.

Introduction

Duality of rising and falling factorial powers is a common feature in the combinatorial analysis. In other words, if a problem leads to some combinatorial identity constructed with the help of falling factorial powers, then often there is a dual combinatorial problem, which leads to a dual combinatorial identity involving rising factorial powers. One can find some examples of these dual combinatorial identities in [1], [2].

The classic exponential e^x is given by the corresponding power series with factorials, which can be written as the falling factorial power $n^{\underline{n}}$. Replacing the falling factorial powers by the corresponding rising factorial powers $n^{\overline{n}}$, we get the function $\mathrm{Exp}(x)$ [3].

Now if in the Dawson's integral [4]

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt$$

we replace the exponentials by $\mathrm{Exp}(x)$, then we get a new nonelementary function

$$D(x) = \big(\mathrm{Exp}\,(x^2)\big)^{-1}\int_0^x \mathrm{Exp}(t^2)dt,$$

the basic properties of which are to be studied in this article.

The Dawson's integral and its generalization are widely applied in astrophysics, spectroscopy, theory of electric oscillation, processes of heat conduction, viscosity mechanics, finance, applied mathematics [5]–[12].

1 Preliminaries and Notations

Definition 1. [13] *For an arbitrary $x \in \mathbb{R}$ and $m \in \mathbb{N}$, **the factorial power** m with the step of $k \in \mathbb{R}$ is the expression*

$$x^{m\{k\}} = x(x+k)(x+2k)\cdot\ldots\cdot\big(x+(m-1)k\big).$$

By definition $x^{0\{k\}} := 1$.

If $k = 0$, then we have a simple power, i.e. $x^{m\{0\}} = x^m$.

Most often, there are rising factorial powers with the step of 1 and falling factorial powers with the step of -1, which we will denote by

$$x^{\overline{m}} = x^{m\{1\}} = x(x+1)\cdot\ldots\cdot(x+m-1),$$
$$x^{\underline{m}} = x^{m\{-1\}} = x(x-1)\cdot\ldots\cdot(x-m+1),$$

respectively.

Let us assume $x^{\overline{0}} = x^{\underline{0}} = 1$. It is obvious that

$$1^{\overline{m}} = m^{\underline{m}} = m!$$

In analogy to the known power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which can be treated as the series constructed with the help of falling factorial powers ($n! = n^{\underline{n}}$). The "dual" function $\mathrm{Exp}(x)$, constructed with the help of rising factorial powers, it is studied in [3].

Definition 2. *By* $\mathrm{Exp}(x)$ *we will denote the function defined with the help of the power series*

$$\mathrm{Exp}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n^{\overline{n}}} =$$

$$= 1 + \frac{x}{1} + \frac{x^2}{2\cdot 3} + \ldots + \frac{x^n}{n\cdot(n+1)\cdot\ldots\cdot(2n-1)} + \ldots$$

It is obvious that

$$\mathrm{Exp}(x) = 1 + \sum_{n=1}^{\infty} \frac{(n-1)!}{(2n-1)!} x^n \qquad (1)$$

and the series in (1) converges on the real axis.

The graph of the function $y = \mathrm{Exp}(x)$ is shown in Figure 1.

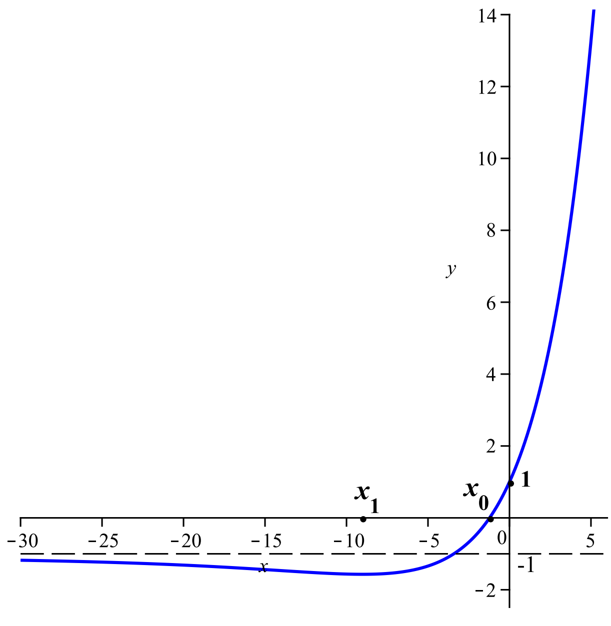

Fig. 1. *The graph of the function* $y = \mathrm{Exp}(x)$

The only zero of the function $\mathrm{Exp}(x)$ is

$$x_0 \approx -1,22041009,$$

and at $x_1 \approx -9,02371883$ it reaches its minimum.

In [3] it is proved that

$$\mathrm{Exp}(x) = 1 + \sum_{n=0}^{\infty} \frac{x^n}{4^n\, n!} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{n+1}}{(2n+1)\, 4^n\, n!} =$$

$$= 1 + \sqrt{\pi\, x}\, e^{\frac{x}{4}}\, \Phi\left(\frac{\sqrt{x}}{2}\right), \qquad (2)$$

where

$$\Phi(x) = \operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the probability function (error function). [4]

2 On operations with formal power series

Denote by

$$\left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant j}$$

the paradeterminant of the triangular matrix of order j, assuming that $\left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant 0} = 1$. [2]

Let us formulate two theorems from [2], [14] to be used hereafter.

Theorem 1. *Let* $A(z)$, $B(z)$, $C(z)$ *be the notations of the following formal power series*

$$\sum_{n=0}^{\infty} a_n z^n, \quad \sum_{n=0}^{\infty} b_n z^n, \quad \sum_{n=0}^{\infty} c_n z^n,$$

respective, where $a_0 = b_0 = c_0 = 1$ *and*

$$C(z) = \frac{A(z)}{B(z)}.$$

Then

$$c_n = \sum_{j=0}^{n-1} (-1)^j (a_{n-j} - b_{n-j}) \cdot \left\langle \frac{b_{s-r+1}}{b_{s-r}} \right\rangle_{1 \leqslant r \leqslant s \leqslant j},$$

for all $n = 1, 2, \ldots$

Theorem 2. *If*

$$C(z) = \frac{1}{A(z)},$$

then

$$c_n = (-1)^n \left\langle \begin{matrix} a_1 & & & \\ \frac{a_2}{a_1} & a_1 & & \\ \ldots & \ldots & \ddots & \\ \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \ldots & a_1 \end{matrix} \right\rangle_n,$$

for all $n = 1, 2, \ldots$

3 Function of the Dawson's integral type

Replacing in the Dawson's integral

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt$$

the exponential with the function $\mathrm{Exp}(x)$, we obtain a nonelementary function, which we will denote by $D(x)$, i.e. let

$$D(x) := \left(\mathrm{Exp}(x^2)\right)^{-1}\int_0^x \mathrm{Exp}(t^2)\,dt. \qquad (3)$$

It is easy to verify that

$$\int_0^x \mathrm{Exp}(t^2)\,dt = -x + 2\sqrt{\pi}\,e^{\frac{x^2}{4}}\,\Phi\Big(\frac{x}{2}\Big).$$

Thus from (2), (3) we obtain the formula

$$D(x) = \frac{2\sqrt{\pi}\,e^{\frac{x^2}{4}}\,\Phi\big(\frac{x}{2}\big) - x}{1+\sqrt{\pi}\,x\,e^{\frac{x^2}{4}}\Phi\big(\frac{x}{2}\big)}.$$

The graphs of the function $y = D(x)$ (solid graph) and the Dawson's integral $y = F(x)$ (dotted graph) are shown in Figure 2.

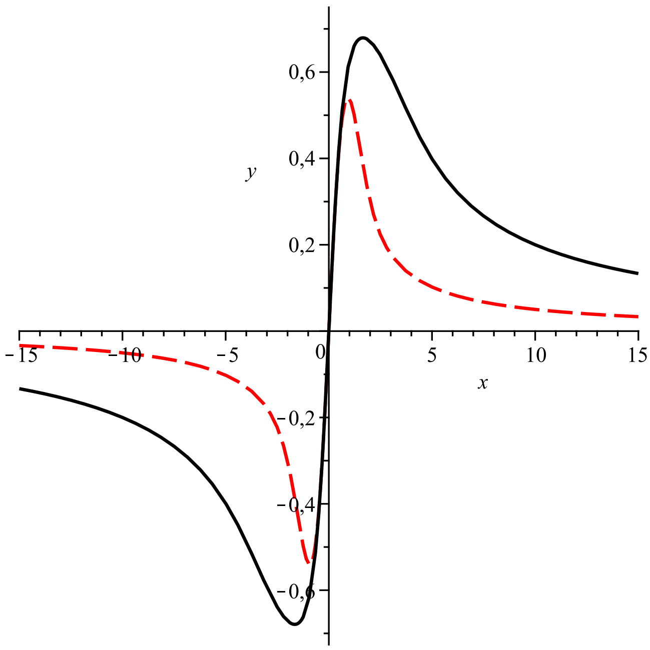

Fig. 2. *The graphs of the functions* $y = D(x)$ *and* $y = F(x)$

The function $D(x)$ can also be represented as the quotient of two power series

$$\int_0^x \mathrm{Exp}(t^2)dt =$$

$$= x + \sum_{j=1}^{\infty}\frac{(j-1)!}{(2j-1)!(2j+1)}\,x^{2j+1}$$

and

$$\mathrm{Exp}(x^2) = 1 + \sum_{j=1}^{\infty}\frac{(j-1)!}{(2j-1)!}\,x^{2j}.$$

To prove this we apply Theorems 1 and 2. If we denote

$$A(x) = 1 + x + \sum_{j=1}^{\infty}\frac{(j-1)!}{(2j-1)!(2j+1)}\,x^{2j+1},$$

$$B(x) = 1 + \sum_{j=1}^{\infty}\frac{(j-1)!}{(2j-1)!}\,x^{2j},$$

then $D(x) = R(x) - S(x)$, where

$$R(x) = \frac{A(x)}{B(x)}, \quad S(x) = \frac{1}{B(x)}.$$

We denote the coefficients of the above series by small Latin letters.

Using Theorems 1 and 2, it is easy to verify that $s_{2k-1} = 0$ and

$$r_{2k} = s_{2k} = (-1)^k B_k,$$

where

$$B_k = \left\langle \begin{matrix} b_2 & & & & & \\ \frac{b_4}{b_2} & b_2 & & & & \\ \frac{b_6}{b_4} & \frac{b_4}{b_2} & b_2 & & & \\ \vdots & \cdots & \cdots & \ddots & & \\ \frac{b_{2k-2}}{b_{2k-4}} & \frac{b_{2k-4}}{b_{2k-6}} & \frac{b_{2k-6}}{b_{2k-8}} & \cdots & b_2 & \\ \frac{b_{2k}}{b_{2k-2}} & \frac{b_{2k-2}}{b_{2k-4}} & \frac{b_{2k-4}}{b_{2k-6}} & \cdots & \frac{b_4}{b_2} & b_2 \end{matrix} \right\rangle_k,$$

for all $k = 1, 2, \ldots$

Thus $d_{2k} = 0$ and for all $k = 1, 2, \ldots$

$$d_{2k-1} = r_{2k-1} =$$

$$= a_{2k-1}r_0 + a_{2k-3}r_2 + \ldots + a_1 r_{2k-2}.$$

Notice, that the parapermanent B_k can be calculated with the help of the linear recurrent equation

$$B_0 = 1, \quad B_k = b_2B_{k-1} - b_4B_{k-2} + \ldots +$$

$$+ (-1)^{k-2}b_{2k-2}B_1 + (-1)^{k-1}b_{2k}B_0.$$

Hence, performing necessary calculations, the function $D(x)$ is represented as the power series

$$D(x) = x - \frac{2}{3}x^3 + \frac{8}{15}x^5 - \frac{55}{126}x^7 + \frac{338}{945}x^9 - \\ - \frac{121861}{415800}x^{11} + \frac{486781}{2027025}x^{13} - \ldots \quad (4)$$

Let us note that the coefficients of the power series (4) can also be found using the recurrence formulas

$$d_1 = 1, \quad d_{2n-1} = \frac{(n-2)!}{(2n-3)!(2n-1)} - \\ - \sum_{n=2}^{\infty}\sum_{j=1}^{n} \frac{(n-1-j)!}{(2n-2j-1)!} d_{2j-1}, \quad n = 2, 3, \ldots$$

4 Differential equation of the function $D(x)$

The Dawson's integral is a solution of the linear nonhomogeneous equation $y' + 2xy = 1$ [4]. Let us prove that the function $D(x)$ is a solution of the Riccati equation.

Theorem 3. *The function $y = D(x)$ is a solution of the Cauchy problem*

$$\left.\begin{array}{l} y' = \frac{x^2-2}{2(x^2+2)}\, y^2 - \frac{x(x^2+6)}{2(x^2+2)} y + 1, \\ y(0) = 0. \end{array}\right\} \quad (5)$$

Proof. It follows from the formula (3) that the integral curve $y = D(x)$ passes through the origin.

Let us prove that the function $y = D(x)$ is a solution of the Riccati equation (5). Indeed, since

$$y' = \frac{1}{2}\left(1 + \sqrt{\pi} x e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right)\right)^{-2} \times \\ \times \left(2\sqrt{\pi}\, x e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right) + e^{\frac{x^2}{4}} x^3 \Phi\left(\frac{x}{2}\right) + 2x^2 - \right. \\ \left. - 4\pi e^{\frac{x^2}{2}} \Phi^2\left(\frac{x}{2}\right) + 2\right),$$

excluding from this formula and from (3) the expression $\sqrt{\pi}\, e^{\frac{x^2}{4}} \Phi\left(\frac{x}{2}\right)$, we obtain the relation

$$y' = \frac{x^2-2}{2(x^2+2)}\, y^2 - \frac{x(x^2+6)}{2(x^2+2)}\, y + 1.$$

The Theorem 3 is proved.

References

1. *Graham R., Knuth D., Patashnik O.* Concrete Mathematics: a Foundation for Computer Science. Addison-Wesley, 1990.
2. *Zatorsky R.A.* Calculus of Triangular Matrices and Its Applications. Simyk, Ivano-Frankivsk, 2010. (in Ukrainian)
3. *Goy T.P., Zatorsky R.A.* New functions generated by rising factorials and their properties // Bukovyn. Math. J., **1**, № 1-2 (2013), 28-33. (in Ukrainian)
4. *Abramowitz M., Stegun I.A. (eds).* Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York, 1972.
5. *Cody W.J., Paciorek K.A., Thacher H.C.* Chebyshev approximations for Dawson's integral // Math. Comp., **24** (1970), 171-178.
6. *Dijkstra D.* A continued fraction expansion for a generalization of Dawson's integral // Math. Comp., **31** (1977), 503-510.
7. *Iscoe I., Kreinin A.* Recursive valuation of basket default swaps // J. Comp. Finance, **9**, 3 (2006), 95-116.
8. *Kaiser A., Rethfeld B., Vicanek M., Simon G.* Microscopic processes in dielectrics under irradiation by subpicosecond laser pulses // Phys. Rev. B, **61** (2000), 11437-11450.
9. *Lehle H., Kriegl J.M., Nienhaus K., Deng P., Fengler S., Ulrich N.G.* Probing electric fields in protein cavities by using the vibrational Stark effect of carbon monoxide // Biophys. J., **88**(2005), 1978-1990.
10. *McCabe J.H.* A continued fraction expansion, with a truncation error estimate, for Dawson's integral // Math. Comp., **28**, 127 (1974), 811-816.
11. *Messina R., Jivulescu M.A., Messina A., Napoli A.* Riccati equation-based generalization Dawson's integral function // Math. Meth. Appl. Sci., **30**, 16 (2007), 2055-2064.
12. *Sampoornaa M., Nagendraa K.N., Frischb H.* Generalized Voigt functions and their derivatives // J. Quant. Spectr. Rad. Transf., **104**, 1 (2007), 71–85.
13. *Gelfond A.O.* Calculus of Finite Differences. GIFML, Moscow, 1959. (in Russian)
14. *Zatorsky R.A. Operations with formal power series wich have nontrivial free numbers //* Bulletin of Kyiv University. Series: Physical and Mathematical Sciences. — 2008. — № 1. — P. 36–39. (in Ukrainian)

Received: 01.10.2013