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On the approximation of fixed points for the class of mappings satisfying (CSC)-condition in Hadamard spaces

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In this paper, we consider the class of mappings satisfying (CSC)-condition. Further, we prove the strong and $\triangle$-convergence theorems of the JF-iteration process for this class of mappings in Hadamard spaces. In the end, we provide a numerical example to show that the JF-iteration process is faster than some well known iterative processes. Our results improve and extend the corresponding recent results of the current literature.

Key words and phrases: $\triangle$-convergence, strong convergence, fixed point, CAT(0) space, JF-iteration process, (CSC)-condition.

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1 Introduction

Let (X,d) be a metric space, Y be a non-empty subset of X, and $T : Y \to Y$ be a mapping. A point $p \in Y$ is called a *fixed point* of T if $Tp = p$. We denote by $F(T)$ the set of all fixed points of T. The mapping T is called *nonexpansive* if $d(Tu,Tv) \le d(u,v)$ for all $u,v \in Y$, and *quasi-nonexpansive* if $d(Tu,p) \le d(u,p)$ for all $u \in Y$ and for each $p \in F(T)$.

In 1973, G.E. Hardy and T.D. Rogers [1] introduced the concept of generalized nonexpansive mappings which is defined as follows.

Definition 1. *Let T be a self mapping on a non-empty subset Y of a metric space (X,d). Then T is called generalized nonexpansive mapping if for all $u,v \in Y$ we have*

$$d(Tu,Tv) \le ad(u,v) + b\big[d(u,Tu) + d(v,Tv)\big] + c\big[d(u,Tv) + d(v,Tu)\big], \tag{1}$$

where a,b,c are non-negative real numbers such that $a + 2b + 2c \le 1$.

In 2008, T. Suzuki [2] introduced a new condition on the mappings, called (C)-condition. Such mappings are also known Suzuki generalized nonexpansive mappings.

Definition 2. *A self mapping T on a non-empty subset Y of a metric space (X,d) is said to satisfy (C)-condition if*

$$\frac{1}{2}d(u,Tu) \le d(u,v) \quad \textit{implies} \quad d(Tu,Tv) \le d(u,v)$$

for all $u,v \in Y$.

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Now we list some properties of generalized nonexpansive mappings due to G.E. Hardy, T.D. Rogers and T. Suzuki which can be found in [3].

Proposition 1. *(i) The classes of generalized nonexpansive mappings satisfying* (1) *and Suzuki generalized nonexpansive mappings are independent.*

(ii) If T is a generalized nonexpansive mapping satisfying (1) and it has a fixed point, then T is quasi-nonexpansive.

(iii) If T is a generalized nonexpansive mapping satisfying (1), then

$$d(u,Tv) \leq d(u,v) + \frac{1+b+c}{1-b-c} d(u,Tu)$$

holds for all $u, v \in Y$.

In 2011, E. Karapınar and K. Taş [4] suggested (CSC)-condition which is a modification of Suzuki's (C)-condition.

Definition 3. *Let (X,d) be a metric space and Y be a non-empty subset of X. Then a mapping $T: Y \to Y$ is said to satisfy (CSC)-condition if*

$$\frac{1}{2} d(u,Tu) \leq d(u,v) \quad \textit{implies} \quad d(Tu,Tv) \leq \frac{1}{2}\big[d(Tu,v) + d(u,Tv)\big]$$

for all $u, v \in Y$.

Moreover, E. Karapınar and K. Taş [4] gave some basic properties for a mapping satisfying (CSC)-condition as follows.

Proposition 2. *(i) If a mapping T satisfies (CSC)-condition and has a fixed point, then it is a quasi-nonexpansive mapping.*

(ii) If T is a mapping satisfying (CSC)-condition, then

$$d(u,Tv) \leq 5d(u,Tu) + d(u,v)$$

holds for all $u, v \in Y$.

(iii) If T is a mapping satisfying (CSC)-condition, then the set $F(T)$ is closed.

Recently, F. Ali et al. [3] introduced a new iteration process, called JF-iteration process, in Banach spaces, defined as follows

$$\begin{cases} w_n = T\big((1-s_n)p_n + s_n T p_n\big), \\ q_n = Tw_n, \\ p_{n+1} = T\big((1-r_n)q_n + r_n T q_n\big), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{2}$$

where $\{r_n\}$ and $\{s_n\}$ are real sequences in $[0,1]$. They showed numerically that this iteration process is faster than the Mann, Ishikawa, Noor, S, Picard-S and Thakur-New iteration processes (see [5–10]) for generalized nonexpansive mappings due to G.E. Hardy and T.D. Rogers, and proved some convergence results of JF-iteration process (2) for this class of mappings in uniformly convex Banach spaces. Very recently, M. Jubair et al. [11] proved some convergence results for Suzuki generalized nonexpansive mappings via JF-iteration process (2) in uniformly convex Banach spaces.

Motivated by the above results, we modify the JF-iteration process into CAT(0) spaces as follows

$$\begin{cases} w_n = T\big((1-s_n)p_n \oplus s_n T p_n\big), \\ q_n = Tw_n, \\ p_{n+1} = T\big((1-r_n)q_n \oplus r_n T q_n\big), \quad \forall\, n \in \mathbb{N}, \end{cases} \tag{3}$$

where Y is a non-empty convex subset of a CAT(0) space, $p_1 \in Y$, $\{r_n\}$ and $\{s_n\}$ are real sequences in $[0,1]$.

In this paper, we study the convergence of the JF-iteration process (3) to a fixed point for the class of mappings satisfying (CSC)-condition in a CAT(0) space. Moreover, we provide a numerical example to support our main results. This example also shows that the JF-iteration process is faster than the Mann, Ishikawa, Noor, S, Picard-S, Thakur-New iteration processes for the mappings satisfying (CSC)-condition. Our results can be viewed as a refinement and generalization of some results in F. Ali et al. [3] and M. Jubair et al. [11].

2 Preliminaries and lemmas

Let (X,d) be a metric space and $u,v \in X$ with $d(u,v) = l$. A *geodesic path* from u to v is an isometry $c : [0,l] \to X$ such that $c(0) = u$ and $c(l) = v$. The image of c is called a *geodesic segment* joining u and v, which is denoted by $[u,v]$ whenever it is unique. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic path. Furthermore, X is said to be a *uniquely geodesic space* if there is exactly one geodesic segment joining u and v for each $u,v \in X$. A subset Y of X is called *convex* if Y includes every geodesic segment joining any two of its points. Let $u,v \in X$ and $t \in [0,1]$, we write $(1-t)u \oplus tv$ for the unique point w in $[u,v]$ such that $d(w,u) = td(u,v)$ and $d(w,v) = (1-t)d(u,v)$.

A *geodesic triangle* $\triangle(u_1,u_2,u_3)$ in a geodesic metric space (X,d) consists of three points u_1,u_2,u_3 in X (called the vertices of $\triangle$) and a geodesic segment between each pair of vertices (called the edges of $\triangle$). For any geodesic triangle, there is a *comparison triangle* $\overline{\triangle}$ in the Euclidean plane $\mathbb{R}^2$ such that $d(u_i,u_j) = d_{\mathbb{R}^2}(\overline{u}_i,\overline{u}_j)$ for $i,j \in \{1,2,3\}$.

Let $\triangle$ be a geodesic triangle in X and $\overline{\triangle}$ be a comparison triangle for $\triangle$, then $\triangle$ is said to satisfy the $CAT(0)$ *inequality* if

$$d(u,v) \le d_{\mathbb{R}^2}(\overline{u},\overline{v})$$

for all $u,v \in \triangle$ and $\overline{u},\overline{v} \in \overline{\triangle}$.

If u,v_1,v_2 are points in X and v_0 is the midpoint of the segment $[v_1,v_2]$, then the CAT(0) inequality implies

$$d^2(u,v_0) \le \frac{1}{2}d^2(u,v_1) + \frac{1}{2}d^2(u,v_2) - \frac{1}{4}d^2(v_1,v_2),$$

which is known as the *(CN) inequality* of F. Bruhat and J. Tits [12].

Definition 4 ([13]). *A geodesic space X is called a $CAT(0)$ space if all geodesic triangles satisfy the $CAT(0)$ inequality. Equivalently, X is called a $CAT(0)$ space if and only if it satisfies the (CN) inequality.*

A complete CAT(0) space is called a Hadamard space. The class of Hadamard spaces comprises Hilbert spaces, complete simply connected Riemannian manifolds of non-positive sectional curvature (for instance classic hyperbolic spaces and the manifold of positive definite matrices), Euclidean buildings, CAT(0) complexes, non-linear Lebesgue spaces, the Hilbert ball and many other spaces.

We now collect some elementary facts about CAT(0) spaces which will be used in the sequel.

Lemma 1 ([14, Lemma 2.4])**.** *Let X be a CAT(0) space. For $u, v, w \in X$ and $t \in [0,1]$, one has*

$$d\big((1-t)u \oplus tv, w\big) \leq (1-t)d(u,w) + td(v,w).$$

Lemma 2 ([15, Lemma 3.2])**.** *Let X be a CAT(0) space, $u \in X$ be a given point and $\{t_n\}$ be a sequence in $[a,b]$ with $a, b \in (0,1)$ and $0 < a(1-b) \leq \frac{1}{2}$. Let $\{p_n\}$ and $\{q_n\}$ be any sequences in X such that*

$$\limsup_{n\to\infty} d\left(p_n, u\right) \leq c, \quad \limsup_{n\to\infty} d\left(q_n, u\right) \leq c, \quad \lim_{n\to\infty} d\big((1-t_n)p_n \oplus t_n q_n, u\big) = c$$

for some $c \geq 0$. Then $\lim_{n\to\infty} d\left(p_n, q_n\right) = 0$.

We now give the concept of $\triangle$-convergence. Let $\{p_n\}$ be a bounded sequence in a CAT(0) space X. Then the *asymptotic center* $A\big(\{p_n\}\big)$ of $\{p_n\}$ is the set

$$A\big(\{p_n\}\big) = \Big\{u \in X: \ \limsup_{n\to\infty} d(u, p_n) = \inf_{\overline{u}\in X} \ \limsup_{n\to\infty} d(\overline{u}, p_n)\Big\}.$$

It is known (see, e.g., [16, Proposition 7]) that in a Hadamard space, $A\big(\{p_n\}\big)$ consists of exactly one point.

Definition 5 ([17, 18])**.** *A sequence $\{p_n\}$ in a CAT(0) space X is said to be $\triangle$-convergent to $u \in X$ if u is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{x_n\}$ of $\{p_n\}$. In this case, we write $\triangle$-$\lim_{n\to\infty} p_n = u$ and call u the $\triangle$-limit of $\{p_n\}$.*

The concept of $\triangle$-convergence in metric spaces was first introduced and studied by T.C. Lim [17]. Later, W.A. Kirk and B.A. Panyanak [18] introduced and studied this concept in CAT(0) spaces and proved that it is very similar to the weak convergence in Banach space setting.

The following lemma is very useful for proving our $\triangle$-convergence theorem.

Lemma 3. *Let X be a Hadamard space.*

(i) Every bounded sequence in X has a $\triangle$-convergent subsequence (see [18, p. 3690]).

(ii) If Y is a closed convex subset of X and $\{p_n\}$ is a bounded sequence in Y, then the asymptotic center of $\{p_n\}$ is in Y (see [19, Proposition 2.1]).

(iii) If $\{p_n\}$ is a bounded sequence in X with $A\big(\{p_n\}\big) = \{u\}$ and $\{x_n\}$ is a subsequence of $\{p_n\}$ with $A\big(\{x_n\}\big) = \{v\}$ and the sequence $\big\{d(p_n, v)\big\}$ converges, then $u = v$ (see [14, Lemma 2.8]).

3 Main results

Now we prove two key lemmas which will play very fruitful roles throughout in the sequel.

Lemma 4. *Let Y be a non-empty convex subset of a CAT(0) space X and $T : Y \to Y$ be a mapping satisfying (CSC)-condition such that $F(T) \neq \varnothing$. Let $\{p_n\}$ be an iterative sequence generated by (3) with real sequences $\{r_n\}$ and $\{s_n\}$ in $[0,1]$. Then $\lim_{n\to\infty} d(p_n, p)$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. By the item *(i)* of Proposition 2 and Lemma 1, we have

$$\begin{aligned} d(p_{n+1}, p) = d\Big(T\big((1-r_n)q_n \oplus r_n Tq_n\big), p\Big) &\leq d\big((1-r_n)q_n \oplus r_n Tq_n, p\big) \\ &\leq (1-r_n)d(q_n, p) + r_n d(Tq_n, p) \\ &\leq (1-r_n)d(q_n, p) + r_n d(q_n, p) = d(q_n, p), \end{aligned} \tag{4}$$

$$d(q_n, p) = d(Tw_n, p) \leq d(w_n, p) \tag{5}$$

and

$$\begin{aligned} d(w_n, p) = d\Big(T\big((1-s_n)p_n \oplus s_n Tp_n\big), p\Big) &\leq d\big((1-s_n)p_n \oplus s_n Tp_n, p\big) \\ &\leq (1-s_n)d(p_n, p) + s_n d(Tp_n, p) \\ &\leq (1-s_n)d(p_n, p) + s_n d(p_n, p) = d(p_n, p). \end{aligned} \tag{6}$$

Using (4), (5) and (6), we obtain

$$d(p_{n+1}, p) \leq d(p_n, p).$$

This implies that the sequence $\big\{d(x_n, p)\big\}$ is non-increasing and bounded below. Hence the limit $\lim_{n\to\infty} d(p_n, p)$ exists for all $p \in F(T)$. □

Lemma 5. *Let Y be a non-empty closed convex subset of a Hadamard space X and $T : Y \to Y$ be a mapping satisfying (CSC)-condition. Let $\{p_n\}$ be the iterative sequence (3) such that $\{r_n\}$ is a real sequence in $[0,1]$ and $\{s_n\}$ is a real sequence in $[a,b]$ for some $a, b \in (0,1)$ with $0 < a(1-b) \leq \frac{1}{2}$. Then $F(T) \neq \varnothing$ if and only if $\{p_n\}$ is bounded and $\lim_{n\to\infty} d(p_n, Tp_n) = 0$.*

Proof. First, we assume that $F(T) \neq \varnothing$. Let $p \in F(T)$. Then, by Lemma 4, $\lim_{n\to\infty} d(p_n, p)$ exists and $\{p_n\}$ is bounded. Let

$$\lim_{n\to\infty} d(p_n, p) = c \geq 0. \tag{7}$$

By the item *(i)* of Proposition 2, we have

$$\limsup_{n\to\infty} d(Tp_n, p) \leq \limsup_{n\to\infty} d(p_n, p) = c. \tag{8}$$

On the other hand, it follows from (6) that

$$\limsup_{n\to\infty} d(w_n, p) \leq c. \tag{9}$$

By using (4) and (5), we get $d(p_{n+1}, p) \leq d(w_n, p)$, which yields that

$$c \leq \liminf_{n\to\infty} d(w_n, p). \tag{10}$$

Combining (9) and (10), we obtain $\lim\limits_{n\to\infty} d(w_n, p) = c$. Now using the latter, we have

$$\begin{aligned} c = \lim_{n\to\infty} d(w_n, p) &= \lim_{n\to\infty} d\Big(T\big((1-s_n)p_n \oplus s_n T p_n\big), p\Big) \leq \lim_{n\to\infty} d\big((1-s_n)p_n \oplus s_n T p_n, p\big) \\ &\leq \lim_{n\to\infty} \big[(1-s_n)d(p_n, p) + s_n d(Tp_n, p)\big] \\ &\leq \lim_{n\to\infty} \big[(1-s_n)d(p_n, p) + s_n d(p_n, p)\big] = \lim_{n\to\infty} d(p_n, p) = c. \end{aligned}$$

This implies that

$$\lim_{n\to\infty} d\big((1-s_n)p_n \oplus s_n T p_n, p\big) = c. \tag{11}$$

From (7), (8), (11) and Lemma 2, we get $\lim\limits_{n\to\infty} d(p_n, Tp_n) = 0$.

Conversely, we assume that $\{p_n\}$ is bounded and $\lim\limits_{n\to\infty} d(p_n, Tp_n) = 0$. Let $u \in A\big(\{p_n\}\big)$. By the item *(ii)* of Proposition 2, we have

$$\begin{aligned} r\big(Tu, \{p_n\}\big) = \limsup_{n\to\infty} d(Tu, p_n) &\leq \limsup_{n\to\infty} \big[5d(p_n, Tp_n) + d(p_n, u)\big] \\ &\leq \limsup_{n\to\infty} d(u, p_n) = r\big(u, \{p_n\}\big). \end{aligned}$$

It follows that $Tu \in A\big(\{p_n\}\big)$. Since $A\big(\{p_n\}\big)$ is singleton set, we get $Tu = u$. Thus, we obtain $F(T) \neq \varnothing$. ☐

We prove the $\triangle$-convergence theorem of the JF-iteration process for a mapping satisfying (CSC)-condition in a Hadamard space.

Theorem 1. *Let Y be a non-empty closed convex subset of a Hadamard space X and $T : Y \to Y$ be a mapping satisfying (CSC)-condition such that $F(T) \neq \varnothing$. Let $\{p_n\}$ be the iterative sequence (3) such that $\{r_n\}$ is a real sequence in $[0,1]$ and $\{s_n\}$ is a real sequence in $[a,b]$ for some $a, b \in (0,1)$ with $0 < a(1-b) \leq \frac{1}{2}$. Then the sequence $\{p_n\}$ is $\triangle$-convergent to a fixed point of T.*

Proof. In order to show that the sequence $\{p_n\}$ is $\triangle$-convergent to a fixed point of T, we prove that

$$W_\triangle(p_n) = \bigcup_{\{x_n\}\subset\{p_n\}} A\big(\{x_n\}\big) \subseteq F(T)$$

and $W_\triangle(p_n)$ consists of exactly one point. Let $u \in W_\triangle(p_n)$. Then there exists a subsequence $\{x_n\}$ of $\{p_n\}$ such that $A\big(\{x_n\}\big) = \{u\}$. By the items *(i)* and *(ii)* of Lemma 3, there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $\triangle\text{-}\lim\limits_{n\to\infty} y_n = v \in Y$. By Lemma 5, we have

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0.$$

It follows similarly from the proof of Lemma 5 that v is a fixed point of T. By Lemma 4, $\lim\limits_{n\to\infty} d(p_n, v)$ exists. Hence, by the item *(iii)* of Lemma 3, we have $u = v$. This implies that $W_\triangle(p_n) \subseteq F(T)$.

Now, we prove that $W_\triangle(p_n)$ consists of exactly one point. Let $\{x_n\}$ be a subsequence of $\{p_n\}$ with $A\big(\{x_n\}\big) = \{u\}$ and let $A\big(\{p_n\}\big) = \{p\}$. We have already seen that $u = v$ and $v \in F(T)$. From Lemma 4, we know that $\big\{d(p_n, u)\big\}$ is convergent. In view of the item *(iii)* of Lemma 3, we have $p = u \in F(T)$. This shows that $W_\triangle(p_n) = \{p\}$. ☐

Next we prove the following strong convergence theorems.

Theorem 2. *Under the same assumptions of Theorem 1, if Y is a compact subset of X, then $\{p_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 5, we have $\lim_{n\to\infty} d(p_n, Tp_n) = 0$. By the compactness of Y, we can find a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ such that $\{p_{n_k}\}$ converges strongly to p for some $p \in Y$. By the item *(ii)* of Proposition 2, we have

$$d(p_{n_k}, Tp) \leq 5d(p_{n_k}, Tp_{n_k}) + d(p_{n_k}, p). \tag{12}$$

Letting $k \to \infty$, we get $p_{n_k} \to Tp$, which implies $Tp = p$. By Lemma 4, $\lim_{n\to\infty} d(p_n, p)$ exists. Hence the sequence $\{p_n\}$ converges strongly to p which is the element of $F(T)$. □

Theorem 3. *Let X, Y, T and $\{p_n\}$ be the same as in Theorem 1. Then the sequence $\{p_n\}$ converges strongly to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d\big(p_n, F(T)\big) = 0 \quad \textit{or} \quad \limsup_{n\to\infty} d\big(p_n, F(T)\big) = 0,$$

where $d\big(p_n, F(T)\big) = \inf\big\{d(p_n, p) : p \in F(T)\big\}$.

Proof. First part is trivial. So, we prove the converse part. Suppose that $\liminf_{n\to\infty} d\big(p_n, F(T)\big) = 0$. It follows from Lemma 4 that $\lim_{n\to\infty} d\big(p_n, F(T)\big)$ exists and hence $\lim_{n\to\infty} d\big(p_n, F(T)\big) = 0$. Therefore, for a given $\lambda > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$d\big(p_n, F(T)\big) = \inf\big\{d(p_n, p) : p \in F(T)\big\} < \frac{\lambda}{2}.$$

In particular, $\inf\big\{d(p_{n_0}, p) : p \in F(T)\big\} < \frac{\lambda}{2}$. Hence, there exists $p \in F(T)$ such that $d(p_{n_0}, p) < \frac{\lambda}{2}$. Now, for $m, n \geq n_0$ we have

$$d(p_{n+m}, p_n) \leq d(p_{n+m}, p) + d(p, p_n) \leq d(p_{n_0}, p) + d(p_{n_0}, p) = 2d(p_{n_0}, p) < \lambda.$$

Thus $\{p_n\}$ is a Cauchy sequence in Y. Since Y is a closed subset of complete space X, then there exists a point $q \in Y$ such that $\lim_{n\to\infty} p_n = q$. Now $\lim_{n\to\infty} d\big(p_n, F(T)\big) = 0$ implies $d\big(q, F(T)\big) = 0$, hence we get $q \in F(T)$. □

Now we prove a strong convergence theorem of the JF-iteration process under the condition (I). Before this, we give the complete definition of condition (I).

Definition 6 ([20, p. 375])**.** *A mapping $T : Y \to Y$ is said to satisfy condition (I) if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that*

$$f\big(d(u, F(T))\big) \leq d(u, Tu) \tag{13}$$

for all $u \in Y$.

Theorem 4. *Under the same assumptions of Theorem 1, if T satisfies the condition (I), then $\{p_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 5, we have $\lim_{n\to\infty} d(p_n, Tp_n) = 0$. Thus, from (13) we get

$$0 \leq \lim_{n\to\infty} f\big(d(p_n, F(T))\big) \leq \lim_{n\to\infty} d(p_n, Tp_n) = 0.$$

This implies that $\lim_{n\to\infty} f\big(d(p_n, F(T))\big) = 0$. Since the function f is non-decreasing and satisfies $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, then we have $\lim_{n\to\infty} d\big(p_n, F(T)\big) = 0$. The conclusion now follows from the proof of Theorem 3. □

Since the calculations in the following result are similar those in the above theorems with the help of the items *(ii)* and *(iii)* of Proposition 1, we omit its proof.

Theorem 5. *Let Y be a non-empty closed convex subset of a Hadamard space X and $T : Y \to Y$ be a generalized nonexpansive mapping satisfying (1) such that $F(T) \neq \varnothing$. Let $\{p_n\}$ be the iterative sequence (3) such that $\{r_n\}$ is a real sequence in $[0,1]$ and $\{s_n\}$ is a real sequence in $[a,b]$ for some $a, b \in (0,1)$ with $0 < a(1-b) \leq \frac{1}{2}$. Then the followings hold.*

(i) The sequence $\{p_n\}$ is $\triangle$-convergent to a fixed point of T.

(ii) If Y is a compact subset of X or T satisfies the condition (I), then $\{p_n\}$ converges strongly to a fixed point of T.

(iii) The sequence $\{p_n\}$ converges strongly to a fixed point of T if and only if

$$\liminf_{n\to\infty} d\big(p_n, F(T)\big) = 0 \quad \text{or} \quad \limsup_{n\to\infty} d\big(p_n, F(T)\big) = 0.$$

4 An illuminate numerical example

In this section, we present a numerical example to compare the rate of convergence for a mapping satisfying (CSC)-condition.

Example 1. *Let $X = \big(\mathbb{R}^2, \|\cdot\|_2\big)$ and $Y = [0,2] \times [0,2] \subset X$. A mapping $T : Y \to Y$ is defined by*

$$T(x,y) = \begin{cases} \left(\dfrac{x}{4}, \dfrac{y}{4}\right) & (x,y) \in [0,2) \times [0,2), \\ \left(\dfrac{1}{4}, \dfrac{1}{2}\right) & (x,y) \in \{2\} \times \{2\}. \end{cases}$$

Now, for all $x = (x_1, y_1), y = (x_2, y_2)$ in Y, we consider the following cases.

Case 1. *If $0 \leq x_1, x_2, y_1, y_2 < 2$, then*

$$\begin{aligned} \frac{1}{2}\|x - Tx\|_2 &= \frac{1}{2}\left\|(x_1, y_1) - \left(\frac{x_1}{4}, \frac{y_1}{4}\right)\right\|_2 = \frac{1}{2}\sqrt{\frac{9}{16}x_1^2 + \frac{9}{16}y_1^2} \\ &= \frac{3}{8}\sqrt{x_1^2 + y_1^2} \leq \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \|x - y\|_2 \end{aligned}$$

holds when $x_1 < \frac{16}{11}$ *and* $y_1 < \frac{16}{11}$*. This implies that*

$$\begin{aligned}
\frac{1}{2}\Big(\|Tx-y\|_2+\|x-Ty\|_2\Big) &= \frac{1}{2}\left(\left\|\left(\frac{x_1}{4},\frac{y_1}{4}\right)-(x_2,y_2)\right\|_2 + \left\|(x_1,y_1)-\left(\frac{x_2}{4},\frac{y_2}{4}\right)\right\|_2 \right)\\
&\geq \frac{1}{2}\left\|\left(\frac{5x_1}{4},\frac{5y_1}{4}\right)-\left(\frac{5x_2}{4},\frac{5y_2}{4}\right)\right\|\\
&\geq \frac{1}{2}\sqrt{\frac{25}{16}(x_1-x_2)^2+\frac{25}{16}(y_1-y_2)^2}\\
&= \frac{5}{2}\cdot\frac{1}{4}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \geq \frac{1}{4}\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\\
&= \left\|\left(\frac{x_1}{4},\frac{y_1}{4}\right)-\left(\frac{x_2}{4},\frac{y_2}{4}\right)\right\|_2 = \|Tx-Ty\|_2.
\end{aligned}$$

Case 2. *If* $0 \leq x_1, y_1 < 2$ *and* $x_2 = y_2 = 2$*, then*

$$\begin{aligned}
\frac{1}{2}\|x-Tx\|_2 &= \frac{1}{2}\left\|(x_1,y_1)-\left(\frac{x_1}{4},\frac{y_1}{4}\right)\right\|_2 = \frac{1}{2}\sqrt{\frac{9}{16}x_1^2+\frac{9}{16}y_1^2}\\
&= \frac{3}{8}\sqrt{x_1^2+y_1^2} \leq \sqrt{(x_1-2)^2+(y_1-2)^2} = \|x-y\|_2
\end{aligned}$$

holds when $x_1 \leq \frac{16}{11}$ *and* $y_1 \leq \frac{16}{11}$*. This implies that*

$$\begin{aligned}
\frac{1}{2}\Big(\|Tx-y\|_2+\|x-Ty\|_2\Big) &= \frac{1}{2}\left(\left\|\left(\frac{x_1}{4},\frac{y_1}{4}\right)-(2,2)\right\|_2 + \left\|(x_1,y_1)-\left(\frac{1}{4},\frac{1}{2}\right)\right\|_2 \right)\\
&\geq \frac{1}{2}\left\|\left(\frac{5x_1}{4},\frac{5y_1}{4}\right)-\left(\frac{9}{4},\frac{5}{2}\right)\right\|_2\\
&= \frac{5}{2}\sqrt{\left(\frac{x_1}{4}-\frac{9}{20}\right)^2+\left(\frac{y_1}{4}-\frac{1}{2}\right)^2}\\
&\geq \sqrt{\left(\frac{x_1}{4}-\frac{1}{4}\right)^2+\left(\frac{y_1}{4}-\frac{1}{2}\right)^2} = \|Tx-Ty\|_2.
\end{aligned}$$

Case 3. *If* $x_1 = y_1 = x_2 = y_2 = 2$*, then we have*

$$\frac{1}{2}\|x-Tx\|_2 = \frac{1}{2}\left\|(2,2)-\left(\frac{1}{4},\frac{1}{2}\right)\right\|_2 = \frac{1}{2}\sqrt{\left(\frac{7}{4}\right)^2+\left(\frac{3}{2}\right)^2} \geq \|x-y\|_2 = 0.$$

Case 4. *If* $x_1 = y_1 = 2$ *and* $0 \leq x_2, y_2 < 2$*, then*

$$\frac{1}{2}\|x-Tx\|_2 = \frac{1}{2}\sqrt{\left(\frac{7}{4}\right)^2+\left(\frac{3}{2}\right)^2} \leq \sqrt{(2-x_2)^2+(2-y_2)^2} = \|x-y\|_2$$

holds for $x_1 \leq \frac{9}{8}$ *and* $y_2 \leq \frac{5}{4}$*. This implies that*

$$\begin{aligned}
\frac{1}{2}\Big(\|Tx-y\|_2+\|x-Ty\|_2\Big) &= \frac{1}{2}\left(\left\|\left(\frac{1}{4},\frac{1}{2}\right)-(x_2,y_2)\right\|_2 + \left\|(2,2)-\left(\frac{x_2}{4},\frac{y_2}{4}\right)\right\|_2 \right)\\
&\geq \frac{1}{2}\left\|\left(\frac{9}{4},\frac{5}{2}\right)-\left(\frac{5x_2}{4},\frac{5y_2}{4}\right)\right\|_2 = \frac{1}{2}\sqrt{\left(\frac{9}{4}-\frac{5x_2}{4}\right)^2+\left(\frac{5}{2}-\frac{5y_2}{4}\right)^2}\\
&= \frac{5}{2}\sqrt{\left(\frac{9}{20}-\frac{x_2}{4}\right)^2+\left(\frac{1}{2}-\frac{y_2}{4}\right)^2} \geq \sqrt{\left(\frac{1}{4}-\frac{x_2}{4}\right)^2+\left(\frac{1}{2}-\frac{y_2}{4}\right)^2}\\
&= \|Tx-Ty\|_2.
\end{aligned}$$

Therefore, the mapping T satisfies (CSC)-condition. The point $(0,0)$ is the unique fixed point of T. In the below table and graphics, it can be easily seen that the JF-iteration process converges faster than the leading iteration processes. We choose the control sequences as $r_n = 0.25$, $\beta_n = 0.45$ and $s_n = 0.25$, $n \in \mathbb{N}$.

Now, we examine the step numbers at which the iteration processes converge to the fixed point for some different initial points.

Initial Points	Step number	Mann	Ishikawa	Noor	S	Thakur-New	Picard-S	JF
(1.5,1.6)	n	148	131	130	22	12	9	8
(2.0,2.0)	n	149	132	131	23	12	9	8
(0.5,0.7)	n	142	126	126	21	12	8	7
(0.1,1.9)	n	159	132	131	22	12	9	8

Table 1. The step numbers of iterations converge to fixed point with different initial points

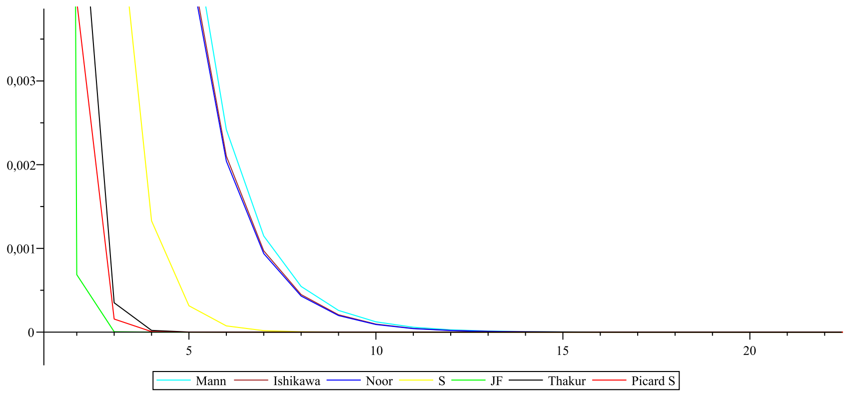

Figure 1. Rate of convergences according to the first coordinates of JF-iteration and other known iterations with the initial point $(2.0, 2.0)$

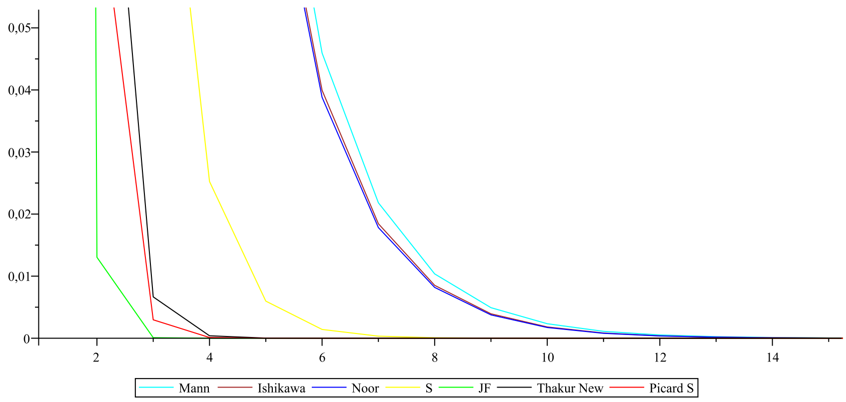

Figure 2. Rate of convergences according to the second coordinates of JF-iteration and other known iterations with the initial point $(2.0, 2.0)$

5 Conclusions

In this paper, we prove some strong and $\triangle$-convergence results of the JF-iteration process introduced by F. Ali et al. [3] in Hadamard spaces.

Theorems 1, 2, 3, 4 extend some results of M. Jubair et al. [11] in two ways:

(1) from the class of Suzuki generalized nonexpansive mappings to the class of mappings satisfying (CSC)-condition,

(2) from a uniformly convex Banach space to a Hadamard space.

Theorem 5 generalizes the corresponding results of F. Ali et al. [3] from a uniformly convex Banach space to a Hadamard space.

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Шагін А., Алагьоз О. *Про апроксимацію нерухомих точок для класу відображень, які задовольняють (CSC)-умову в просторах Адамара* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 495–506.

У цій статті ми розглядаємо відображення, що задовольняють (CSC)-умову. Ми доводимо теореми про сильну та $\triangle$-збіжність JF-ітераційного процесу для цього класу відображень у просторах Адамара. Наприкінці нами подано числовий приклад, який демонструє, що JF-ітераційний процес є швидшим за деякі добре відомі ітераційні процеси. Наші результати покращують і розширюють відповідні недавні результати, що є у сучасній літературі.

Ключові слова і фрази: $\triangle$-збіжність, сильна збіжність, нерухома точка, CAT(0) простір, JF-ітераційний процес, (CSC)-умова.