ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2023, **15** (2), 449–467
doi:10.15330/cmp.15.2.449-467

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2023, Т.15, №2, С.449–467

New U-Bernoulli, U-Euler and U-Genocchi polynomials and their matrices

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In this paper, we introduce the U-Bernoulli, U-Euler, and U-Genocchi polynomials, their numbers, and their relationship with the Riemann zeta function. We also derive the Apostol-type generalizations to obtain some of their algebraic and differential properties. We introduce generalized U-Bernoulli, U-Euler and U-Genocchi polynomial Pascal-type matrix. We deduce some product formulas related to this matrix. Furthermore, we establish some explicit expressions for the U-Bernoulli, U-Euler, and U-Genocchi polynomial matrices, which involves the generalized Pascal matrix.

Key words and phrases: U-Bernoulli polynomial, U-Euler polynomial, generalized U-Bernoulli polynomial, generalized U-Euler polynomial, U-Bernoulli polynomials matrix, U-Euler polynomials matrix, Pascal matrix.

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Introduction

The classical Bernoulli, Euler, and Genocchi polynomials, their respective numbers, and their various extensions, play a role in number theory, special functions, combinatorics, and classical analysis, see, for example, [5, 7, 9–11, 21, 25, 26]. We will review some known results that will suggest various generalizations. For arbitrary real or complex parameters α, λ and $1^\alpha := 1$, the generalized Apostol-Bernoulli polynomials $\mathcal{B}_n^{(\alpha)}(x;\lambda)$, Apostol-Euler polynomials $\mathcal{E}_n^{(\alpha)}(x;\lambda)$ and Apostol-Genocchi polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda)$ are defined using the following generating functions (see [26]):

$$\left(\frac{z}{\lambda e^z-1}\right)^\alpha e^{zx}=\sum_{n=0}^{\infty}\mathcal{B}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{1}$$

$|z|<2\pi$, when $\lambda=1$; $|z|<|\log\lambda|$, when $\lambda\neq 1$;

УДК 517.589
2020 *Mathematics Subject Classification:* 33E20, 11B83, 11B68, 30H50.

This research was funded by the International Telematic University Uninettuno (Italia) and the Universidad de la Costa (Colombia) for all the support provided supported by the project with order code SAP E11P1070121C

$$\left(\frac{2}{\lambda e^z + 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} \mathcal{E}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{2}$$

$|z| < \pi$, when $\lambda = 1$; $|z| < |\log(-\lambda)|$, when $\lambda \neq 1$; and

$$\left(\frac{2z}{\lambda e^z + 1}\right)^{\alpha} e^{zx} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{3}$$

$|z| < \pi$, when $\lambda = 1$; $|z| < |\log(-\lambda)|$, when $\lambda \neq 1$.

Clearly if $\alpha = 1$, the Apostol-Bernoulli numbers $\mathcal{B}_n(\lambda)$, the Apostol-Euler numbers $\mathcal{E}_n(\lambda)$ and Apostol-Genocchi numbers $\mathcal{G}_n(\lambda)$ are given by

$$\mathcal{B}_n(\lambda) := \mathcal{B}_n(0;\lambda), \quad \mathcal{E}_n(\lambda) := \mathcal{E}_n(0;\lambda), \quad \text{and} \quad \mathcal{G}_n(\lambda) := \mathcal{G}_n(0;\lambda), \quad n \in \mathbb{N}_0.$$

These classes of polynomials have been studied in [1, 14, 16, 20, 21, 23–25]. They provide a generalization of the polynomials of Bernoulli, Euler, and Genocchi, and hence, they also generalize the classical Bernoulli, Euler and Genocchi numbers.

The Pascal matrix has been used in different areas of pure and applied mathematics, for instance, in probability problems, combinatorics, and others. Particularly interesting are those contexts in which such a matrix representation is related to classes of polynomials, namely, Bernoulli polynomials, Euler polynomials, Bell polynomials, Jacobi polynomials, Laguerre polynomials, their generalizations, and q-analogs, and so on [19–21, 24, 30]. Understanding these facts, we introduce and study the U-Bernoulli, U-Euler, U-Genocchi polynomials, their numbers, and their Apostol-type generalizations. The focus of this paper is to obtain some properties explicit for them. Also, we introduce the matrices of the three families of polynomials (U-Bernoulli, U-Euler, and U-Genocchi) and study some of their properties, which connect them with Pascal matrices.

Throughout this paper, we use the following standard notations: $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The letters $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of integers, real numbers and complex numbers, respectively. For the complex logarithm, we consider the principal branch, and by $w = z^{\alpha}$ we denote the single branch of the a multiplevalued function $w = z^{\alpha}$ such that $1^{\alpha} = 1$. All matrices are in the set $M_{n+1}(\mathbb{R})$ of all $(n+1) \times (n+1)$ matrices over the field $\mathbb{R}$.

For $k, n \geq 1$ the binomial coefficients satisfy the following relation (see [26]):

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} = \frac{n-k+1}{k}\binom{n}{k-1}. \tag{4}$$

Also, for i, j any nonnegative integers we adopt the following convention

$$\binom{i}{j} = 0, \quad \text{whenever} \quad j > i.$$

Let $n \in \mathbb{N}_0$ and α, β, λ are suitable (real or complex) parameters. By using (1) and (2), we have (see [26, p. 94]):

$$\mathcal{B}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{B}_k^{(\alpha)}(x;\lambda)\mathcal{B}_{n-k}^{(\beta)}(y;\lambda),$$

$$\mathcal{E}_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k^{(\alpha)}(x;\lambda)\mathcal{E}_{n-k}^{(\beta)}(y;\lambda).$$

In [28, p. 1324], it is shown the following relationship between $\mathcal{B}_n^{(\alpha)}(x;\lambda)$ and $\mathcal{E}_n^{(\alpha)}(x;\lambda)$:

$$\mathcal{B}_n^{(\alpha)}(x+y;\lambda)=\frac{1}{2^\beta}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{m=0}^{\infty}\binom{\beta}{m}\lambda^m\mathcal{B}_{n-k}^{(\alpha)}(y+m;\lambda)\right)\mathcal{E}_k^{(\beta)}(x;\lambda).$$

From (3), it also follows that (see [17, p. 5706]):

$$\mathcal{G}_n^{(\alpha+\beta)}(x+y;\lambda)=\sum_{k=0}^{n}\binom{n}{k}\mathcal{G}_k^{(\alpha)}(x;\lambda)\mathcal{G}_{n-k}^{(\beta)}(y;\lambda),$$

$$(n-\alpha)\mathcal{G}_n^{(\alpha)}(x;\lambda)=nx\mathcal{G}_{n-1}^{(\alpha)}(x;\lambda)-\frac{\alpha\lambda}{2}\mathcal{G}_n^{(\alpha+1)}(x+1;\lambda).$$

Let x be any nonzero real number. The generalized Pascal matrix $P[x]\in M_{n+1}(\mathbb{R})$ of first kind is a matrix whose entries are given by (see [8,29]):

$$p_{i,j}(x)=\begin{cases}\binom{i}{j}x^{i-j}, & i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

Also, $P[x]$ is an invertible matrix, and its inverse is given by $P^{-1}[x]:=(P[x])^{-1}=P[-x]$.

Let $P[x]\in M_{n+1}(\mathbb{R})$ be the generalized Pascal matrix of first kind and if we adopt the convention $0^0=1$ it is possible to define $P[0]:=I_{n+1}=\operatorname{diag}(1,1,\ldots,1)$, where I_{n+1} denotes the identity matrix of order $n+1$. The following statements hold:

1) addition theorem of the argument (see [8, Theorem 2]): $P[x+y]=P[x]P[y]$;

2) the matrix $P[x]$ can be factorized as follows (see [29, Theorem 1]):

$$P[x]=G_n[x]G_{n-1}[x]\ldots G_1[x],$$

where $G_k[x]$ is the $(n+1)\times(n+1)$ summation matrix given by

$$G_k[x]=\begin{cases}\begin{bmatrix}I_{n-k} & 0\\ 0 & S_k[x]\end{bmatrix}, & k=1,\ldots,n-1,\\ S_n[x], & k=n,\end{cases}$$

being $S_k[x]$ the $(k+1)\times(k+1)$ matrix whose entries $S_k(x;i,j)$ are given by

$$S_k(x;i,j)=\begin{cases}x^{i-j}, & j\leq i,\\ 0, & j>i,\end{cases}\quad 0\leq i,j\leq k.$$

Let $s=\rho+i\sigma\in\mathbb{C}$ be such that as $\rho>1$. We remind the definition of the Riemann zeta function $\zeta(s)$ (see [3]):

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}. \tag{5}$$

For $n\in\mathbb{N}_0$ and $\kappa,\beta>-1$, the nth Jacobi polynomial $P_n^{(\kappa,\beta)}(x)$ may be defined by means of Rodrigues' formula (see [4,22,27]):

$$P_n^{(\kappa,\beta)}(x)=(1-x)^{-\kappa}(1+x)^{-\beta}\frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left\{(1-x)^{n+\kappa}(1+x)^{n+\kappa}\right\},\quad x\in\mathbb{C}\setminus\{-1,1\}.$$

The connection between the nth monomial x^n and the nth Jacobi polynomial $P_n^{(\kappa,\beta)}(x)$ may be written as follows (see [22, equation (2), p. 262]):

$$x^n = n!\sum_{k=0}^{n}\binom{n+\kappa}{n-k}(-1)^k\frac{(1+\kappa+\beta+2k)}{(1+\kappa+\beta+k)_{n+1}}P_k^{(\kappa,\beta)}(1-2x). \tag{6}$$

For $n \in \mathbb{N}_0$ and $x \in \mathbb{R}$, the Stirling numbers of second kind $S(n,k)$ are defined by means of the following expansion (see [12, Theorem B, p. 207]):

$$x^n = \sum_{k=0}^{n}\binom{x}{k}k!\,S(n,k), \tag{7}$$

so that $S(n,0) = \delta_{n,0}$, $S(n,1) = S(n,n) = 1$ and $S(n,n-1) = \binom{n}{2}$.

Proposition 1. *For $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$ and $\{G_n(x)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level m and Genocchi polynomials, respectively. Then the following identities are satisfied:*

$$x^n = \sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(k+m)!}B_{n-k}^{[m-1]}(x) \quad \text{(see [18, equation (2.6)])}, \tag{8}$$

$$x^n = \frac{1}{2(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(x) + G_{n+1}(x)\right] \quad \text{(see [15, Remark 7])}. \tag{9}$$

1 New U-Bernoulli, U-Euler and U-Genocchi polynomials and their matrices

In this section, we introduce new families of U-Bernoulli, U-Euler, and U-Genocchi polynomials. We also explore some analytic properties concerned with these polynomials and other associated results.

Definition 1. *For $n \in \mathbb{N}_0$, we define the new family of U-Bernoulli polynomials $M_n(x)$ of degree n in the variable x by the power series expansion at 0 of the following generating function*

$$f(x;z) = \left(\frac{z}{e^{-z}-1}\right)e^{-xz} = \sum_{n=0}^{\infty}M_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{10}$$

The first U-Bernoulli polynomials are

$$M_0(x) = -1, \quad M_1(x) = x - \frac{1}{2}, \quad M_2(x) = -x^2 + x - \frac{1}{6}, \quad M_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x,$$

$$M_4(x) = -x^4 + 2x^3 - x^2 + \frac{1}{30}, \quad M_5(x) = x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x.$$

Note that, if $x = 0$ in (10), the U-Bernoulli numbers are defined by the generating function

$$f(z) = \frac{z}{e^{-z}-1} = \sum_{n=0}^{\infty}\frac{M_n z^n}{n!}, \quad |z| < 2\pi. \tag{11}$$

Some of these numbers are

$$M_0 = -1, \quad M_1 = -\frac{1}{2}, \quad M_2 = -\frac{1}{6}, \quad M_3 = 0, \quad M_4 = \frac{1}{30}, \quad M_5 = 0.$$

Proposition 2. *If n is an odd integer greater than or equal to 3, then $M_n = 0$. As a consequence, $(-1)^n M_n = M_n$ for all positive integers n except for $n = 1$.*

Proof. Since

$$\frac{z}{e^{-z}-1} = -1 - \frac{z}{2} + \sum_{k=2}^{\infty} M_k \frac{z^k}{k!},$$

we have

$$\sum_{k=2}^{\infty} M_k \frac{z^k}{k!} - 1 = \frac{z}{e^{-z}-1} + \frac{z}{2} = \frac{z}{2}\Big[\frac{1+e^z}{1-e^z}\Big]. \tag{12}$$

Note that (12) is an even function, therefore

$$\sum_{k=2}^{\infty} M_k \frac{z^k}{k!} = \sum_{k=2}^{\infty} (-1)^k M_k \frac{z^k}{k!}.$$

So, $M_k = (-1)^k M_k$, if $k \geq 3$ is odd, we obtain $M_k = -M_k$ and $M_k = 0$. □

Proposition 3. *Let M_n be the U-Bernoulli numbers defined in* (11)*. Then, we have*

$$z \cot z = 1 - \sum_{n=1}^{\infty} (-1)^n 2^{2n} M_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < \pi, \tag{13}$$

$$z \tan z = \sum_{n=1}^{\infty} (-1)^n 2^{2n} M_{2n} (1 + 2^{2n}) \frac{z^{2n}}{(2n)!}, \quad |z| < \frac{\pi}{2}. \tag{14}$$

Proof. Let us prove (13). In Proposition 2, we noticed that

$$-\frac{z}{2}\Big[\frac{e^{-z/2}+e^{z/2}}{e^{z/2}-e^{-z/2}}\Big] = -1 + \sum_{k=0}^{\infty} M_k \frac{z^k}{k!}. \tag{15}$$

Now,

$$z \cot z = -iz\Big[\frac{e^{-iz}+e^{iz}}{e^{-iz}-e^{iz}}\Big]. \tag{16}$$

Thus, from (15) and (16) we obtain (13). For the proof of (14), it is sufficient to consider (13) and the identity $z \tan z = z \cot z - 2z \cot 2z$. □

Theorem 1. *For all $k \geq 1$, we have*

$$\zeta(2k) = (-1)^k 2^{2k-1} \pi^{2k} \frac{M_{2k}}{(2k)!}.$$

Proof. Consider the following function (cf. [2, p. 205]):

$$\cot z = \sum_{n \in \mathbb{Z}} \frac{1}{z - \pi n}. \tag{17}$$

By (5) and (17), we get

$$z \cot z = 1 - 2\Big(\sum_{k \geq 1} \frac{z^{2k}}{\pi^{2k}}\Big)\Big(\sum_{n \geq 1} \frac{1}{n^{2k}}\Big) = 1 - 2\sum_{k \geq 1} \frac{\zeta(2k)}{\pi^{2k}} z^{2k}. \tag{18}$$

In view of (13) and (18), we have the result. □

In light of the Definition 1 and the assertion (11), by using (4), the following proposition can be obtained.

Proposition 4. *Let* $\{M_n(x)\}_{n\geq 0}$ *be a U-Bernoulli sequence of polynomials defined as in* (10). *Then, we obtain:*

$$M_n(x) = (-1)^{n+1}x^n + \sum_{k=1}^{n} \frac{n-k+1}{k}\binom{n}{k-1}(-1)^{n-k}M_k x^{n-k}, \quad M_0 = -1;$$
$$M_n(x+1) - M_n(x) = n(-1)^{n-1}x^{n-1}, \quad n \geq 1;$$
$$M'_{n+1}(x) = -(n+1)M_n(x);$$
$$\int_x^{x+1} M_n(t)\,dt = (-1)^{n+1}x^n;$$
$$\int_x^y M_n(t)\,dt = -\frac{M_{n+1}(y) - M_{n+1}(x)}{n+1}.$$

Theorem 2. *For* $n, k \geq 1$, *the U-Bernoulli numbers satisfy the following recurrence relationship*

$$(n+1) = 1 - \sum_{k=1}^{n} \frac{n+2-k}{k}\binom{n+1}{k-1}M_k.$$

Proof. By using (11) and (4), we have

$$-\sum_{n=1}^{\infty}\frac{z^{n-1}}{n!}\sum_{n=10}^{\infty}M_n\frac{z^n}{n!} = e^z, \quad -\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}\sum_{n=0}^{\infty}M_n\frac{z^n}{n!} = e^z, \quad -\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{M_k z^n}{(n-k+1)!k!} = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Thus,

$$-\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+1}{k}\frac{1}{(n+1)!}M_k z^n = \sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

By equating coefficients of z^n we complete the proof. □

Now we define the new family of U-Euler polynomials and prove some of their properties.

Definition 2. *For* $n \in \mathbb{N}_0$, *we defined the new family of U-Euler polynomials* $A_n(x)$ *of degree* n *in the variable* x *by the power series expansion at* 0 *of the following generating function*

$$\left(\frac{2}{e^{-z/2}+1}\right)e^{-xz/2} = \sum_{n=0}^{\infty} A_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{19}$$

The first U-Euler polynomials $A_n(x)$ are

$$A_0(x) = 1, \quad A_1(x) = \frac{1}{4} - \frac{x}{2}, \quad A_2(x) = \frac{1}{4}x^2 - \frac{1}{4}x, \quad A_3(x) = -\frac{1}{8}x^3 + \frac{3}{16}x^2 - \frac{1}{32},$$
$$A_4(x) = \frac{1}{16}x^4 - \frac{1}{8}x^3 + \frac{1}{16}x, \quad A_5(x) = -\frac{1}{32}x^5 + \frac{5}{64}x^4 - \frac{5}{64}x^2 + \frac{1}{64}.$$

For $x = 0$ in (19) the U-Euler numbers are defined by the generating function

$$\frac{2}{e^{-\frac{z}{2}}+1} = \sum_{n=0}^{\infty}\frac{A_n z^n}{n!}, \quad |z| < 2\pi. \tag{20}$$

Some of these numbers are

$$A_0 = 1, \quad A_1 = \frac{1}{4}, \quad A_2 = 0, \quad A_3 = -\frac{1}{32}, \quad A_4 = 0, \quad A_5 = -\frac{1}{64}.$$

From (19), (20) by using (4), we get the following result.

Proposition 5. *Let $\{A_n(x)\}_{n\geq 0}$ be a U-Euler sequence of polynomials defined as in* (19). *Then the following statements hold:*

$$A_n(x) = \Big(-\frac{1}{2}\Big)^n x^n + \sum_{k=1}^{n} \frac{n-k+1}{k}\binom{n}{k-1}\Big(-\frac{1}{2}\Big)^{n-k} A_k x^{n-k}, \quad A_0 = 1;$$

$$\frac{A_n(x+1)+A_n(x)}{2} = \Big(-\frac{1}{2}\Big)^n x^n;$$

$$A'_{n+1}(x) = -\frac{1}{2}(n+1)A_n(x);$$

$$-\frac{1}{4}\int_x^{x+1} A_n(t)\,dt = \frac{1}{n+1}\Big(A_{n+1}(x+1) - \Big(-\frac{1}{2}\Big)^{n+1} x^{n+1}\Big);$$

$$\int_x^y A_n(t)\,dt = -\frac{A_{n+1}(y)-A_{n+1}(x)}{(n+1)}.$$

Theorem 3. *For every $n, k \in \mathbb{N}$, we have:*

$$\sum_{k=1}^{n} \frac{n-k+1}{k}\binom{n}{k-1}\frac{A_k}{2^{n-k}} = \frac{1}{2^n} - A_n.$$

Proof. By virtue of (20) and (4), we get

$$2e^{z/2} = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{2^n}\frac{z^n}{n!}\sum_{n=0}^{\infty} A_n \frac{z^n}{n!},$$

$$2\sum_{n=0}^{\infty} 2^{1-n}\frac{z^n}{n!} = \sum_{n=0}^{\infty} A_n \frac{z^n}{n!} + \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{2^{n-k}} A_k \frac{z^n}{(n-k)!k!}.$$

Therefore,

$$\sum_{n=0}^{\infty} 2^{1-n}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\Big[A_n + \sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^{n-k}}A_k\Big]\frac{z^n}{n!}.$$

Comparing the coefficients, we get the result. □

Taking into account the ideas of Proposition 2, we obtain the following result for U-Euler polynomials.

Proposition 6. *If n is an even integer greater than or equal to* 2, *the $A_n = 0$. Consequently, $A_n = -A_n$ for all positive integers n except for $n = 0$.*

Next, we define the new U-Genocchi polynomials and give some of their properties.

Definition 3. *For $n \in \mathbb{N}_0$, we define the new family of U-Genocchi polynomials $V_n(x)$ of degree n in the variable x by the power series expansion at* 0 *of the following generating function*

$$\Big(\frac{2z}{e^{-z/2}+1}\Big)e^{-xz/2} = \sum_{n=0}^{\infty} V_n(x)\frac{z^n}{n!}, \quad |z| < 2\pi. \tag{21}$$

Let us note that the first U-Genocchi polynomials $V_n(x)$ are

$$V_0(x) = 0, \quad V_1(x) = 1, \quad V_2(x) = -x + \frac{1}{2}, \quad V_3(x) = \frac{3}{4}x^2 - \frac{3}{4}x,$$

$$V_4(x) = -\frac{1}{2}x^3 + \frac{3}{4}x^2 - \frac{1}{8}, \quad V_5(x) = \frac{5}{16}x^4 - \frac{5}{8}x^3 + \frac{5}{16}x.$$

For $x = 0$ in (21) the U-Genocchi numbers are defined by the generating function

$$\frac{2z}{e^{-\frac{z}{2}} + 1} = \sum_{n=0}^{\infty} \frac{V_n z^n}{n!}, \quad |z| < 2\pi. \tag{22}$$

Some of these numbers are

$$V_0 = 0, \quad V_1 = 1, \quad V_2 = \frac{1}{2}, \quad V_3 = -\frac{3}{4}, \quad V_4 = -\frac{1}{8}, \quad V_5 = 0.$$

Now, by using (21), (22) and (4), we can obtain the following result.

Proposition 7. *Let* $\{V_n(x)\}_{n\geq 0}$ *be a U-Genocchi sequence of polynomials defined as in* (21). *Then we have:*

$$V_n(x) = \sum_{k=1}^{n} \frac{n-k+1}{k} \binom{n}{k-1} \Big(-\frac{1}{2}\Big)^{n-k} V_k x^{n-k};$$

$$\frac{V_n(x+1) + V_n(x)}{2} = n\Big(-\frac{1}{2}\Big)^{n-1} x^{n-1};$$

$$V'_{n+1}(x) = -\frac{1}{2}(n+1)V_n(x);$$

$$\int_x^{x+1} V_n(t)\,dt = -\frac{4}{(n+1)}\Big(V_{n+1}(x) - (n+1)\Big(-\frac{1}{2}\Big)^n x^n\Big);$$

$$\int_x^y V_n(t)\,dt = -\frac{V_{n+1}(y) - V_{n+1}(x)}{(n+1)}.$$

Theorem 4. *For every* $n \geq 2$ *and* $k \geq 1$, *we have*

$$\sum_{k=1}^{n} \frac{n-k+1}{k} \binom{n}{k-1} \frac{V_k}{2^{n-k}} = n2^{2-n} - V_n.$$

Proof. By using generating functions (22) and by (4), we obtain

$$2ze^{z/2} = \sum_{n=0}^{\infty} V_n \frac{z^n}{n!} + \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{z^n}{n!} \sum_{n=0}^{\infty} V_n \frac{z^n}{n!},$$

$$2z + \sum_{n=1}^{\infty} 2^{1-n} \frac{z^{n+1}}{n!} = \sum_{n=0}^{\infty} V_n \frac{z^n}{n!} + \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n-k}} V_n \frac{z^n}{n!}.$$

Therefore,

$$\sum_{n=2}^{\infty} 2^{2-n} \frac{z^n}{(n-1)!} = \sum_{n=2}^{\infty} \left[V_n + \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n-k}} V_k \right] \frac{z^n}{n!}.$$

Comparing the coefficients of z^n, the result follows. □

Proposition 8. *For every $n \in \mathbb{N}_0$, the following relations hold true:*

$$x^n = -\frac{1}{(n+1)} \sum_{k=0}^{n} \binom{n+1}{k} M_k(1-x), \tag{23}$$

$$x^n = 2^{n-1}\left[A_n(1-x) + \sum_{k=0}^{n} \binom{n}{k} \frac{1}{2^{n-k}} A_k(1-x)\right], \tag{24}$$

$$x^n = \frac{2^{n-1}}{n+1}\left[V_{n+1}(1-x) + \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{2^{n+1-k}} V_k(1-x)\right]. \tag{25}$$

Proof. Let us prove (24). By virtue of (19), we get

$$e^{-xz/2} = \left(\frac{e^{-z/2}+1}{2}\right) \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!},$$

$$e^{(1-x)z/2} = \frac{1}{2} \sum_{n=0}^{\infty} A_n(x) \frac{z^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{2^{n-k}(n-k)!} A_k(x) \frac{z^k}{k!},$$

$$\sum_{n=0}^{\infty} \frac{x^n}{2^n} \frac{z^n}{n!} = \frac{1}{2} \sum_{n=0}^{\infty} A_n(1-x) \frac{z^n}{n!} + \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^{n-k}}{2^{n-k}(n-k)!} A_k(1-x) \frac{z^k}{k!}.$$

It follows the proof of (24). The proofs of (23) and (25) are similar. □

By using generating functions (20) and (22), we derive a relation between the U-Euler numbers and the U-Genocchi numbers as follows.

Proposition 9. *Let $n \in \mathbb{N}_0$. Then we have*

$$V_n = nA_{n-1}. \tag{26}$$

Let us finish this section by the following result.

Theorem 5. *For all $n \geq 1$, we have the relationships:*

$$\zeta(2n) = \frac{(-1)^n 4^{n-1} (2\pi)^{2n}}{2(2n-1)!(1-4^n)} A_{2n-1}, \tag{27}$$

$$\zeta(2n) = \frac{(-1)^n 4^{n-2} (2\pi)^{2n}}{n(2n-1)!(1-4^n)} V_{2n}. \tag{28}$$

Proof. By using Euler's formula $e^{ix} = \cos x + i \sin x$, we have

$$i\tan(x) = 1 - \frac{e^{-ix}}{\cos(x)}. \tag{29}$$

Now

$$\cos(x) = \frac{e^{ix}+e^{-ix}}{2}, \qquad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}. \tag{30}$$

Then, if we apply (30) to (29), we get

$$i\tan(x) = 1 - \frac{2}{e^{2ix}+1}. \tag{31}$$

Also, using (30), we obtain

$$i\tan(x) = \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = 1 - \frac{2}{e^{2ix} - 1} + \frac{4}{e^{4ix} - 1}.$$

So,

$$x\tan(-x) = -ix + \frac{2ix}{e^{-2ix} - 1} - \frac{4ix}{e^{-4ix} - 1}. \tag{32}$$

Then, if we apply (11) to (32) with $z = ix$, $z = 4ix$, by using the Proposition 2, we get

$$x\tan(-x) = -ix + \sum_{n=0}^{\infty} M_n \frac{(2ix)^n}{n!} - \sum_{n=0}^{\infty} M_n \frac{(4ix)^n}{n!} = \sum_{n=1}^{\infty} \frac{M_{2n}}{(2n)!} 4^n [i - 4^n] x^{2n}.$$

Therefore,

$$\tan(-x) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{M_{2n+2}}{(2n+2)!} 4^{n+1} [1 - 4^{n+1}] x^{2n+1}. \tag{33}$$

Now, from (20) and (31) with $z = 4ix$, by using the Proposition 6, we get

$$\begin{aligned} i\tan(-x) &= 1 - \frac{2}{e^{-2ix} + 1} = 1 - \frac{2}{e^{-2ix} + 1} = 1 - \sum_{n=0}^{\infty} A_n \frac{(4ix)^n}{n!} = -\sum_{n=1}^{\infty} A_n (4i)^n \frac{x^n}{n!} \\ &= -i \sum_{n=0}^{\infty} A_{n+1} 4^{n+1} i^n \frac{x^{n+1}}{(n+1)!} = i \sum_{n=0}^{\infty} (-1)^{n+1} A_{2n+1} 4^{2n+1} \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Thus,

$$\tan(-x) = \sum_{n=0}^{\infty} (-1)^{n+1} A_{2n+1} 4^{2n+1} \frac{x^{2n+1}}{(2n+1)!}. \tag{34}$$

From (33) and (34), we obtain

$$(-1)^{n+1} \frac{M_{2n+2}}{(2n+2)!} 4^{n+1} [1 - 4^{n+1}] = (-1)^{n+1} \frac{A_{2n+1}}{(2n+1)!} 4^{2n+1}.$$

Therefore,

$$M_{2n+2} = \frac{A_{2n+1} 4^n (2n+2)}{1 - 4^{n+1}}.$$

So,

$$M_{2n} = \frac{A_{2n-1} 4^{n-1} (2n)}{1 - 4^n}. \tag{35}$$

Taking into account the Theorem 1 and (35), we have

$$\frac{A_{2n-1} 4^{n-1} (2n)}{1 - 4^n} = \frac{2(2n)!(-1)^n}{(2\pi)^{2n}} \zeta(2n).$$

From the above equation, the result (27) follows. Now combining (26) with (27) we get (28). This completes the proof. □

2 Generalized U-Apostol Bernoulli, U-Apostol Euler and U-Apostol Genocchi polynomials

Definition 4. *For arbitrary $\alpha, \lambda \in \mathbb{C}$ and $1^\alpha := 1$, the generalized U-Apostol Bernoulli, U-Apostol Euler and U-Apostol Genocchi polynomials of degree n in the variable x and order α are respectively defined by the power series expansion at 0 of the meromorphic generating function*

$$\Big(\frac{z}{\lambda e^{-z}-1}\Big)^{\alpha} e^{-xz} = \sum_{n=0}^{\infty} M_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{36}$$

$|z| < 2\pi$, *when* $\lambda = 1$; $|z| < |\log \lambda|$, *when* $\lambda \neq 1$;

$$\Big(\frac{2}{\lambda e^{-z/2}+1}\Big)^{\alpha} e^{-xz/2} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{37}$$

$|z| < 2\pi$, *when* $\lambda = 1$; $|z| < |-2\log(-1/\lambda)|$, *when* $\lambda \neq 1$;

$$\Big(\frac{2z}{\lambda e^{-z/2}+1}\Big)^{\alpha} e^{-xz/2} = \sum_{n=0}^{\infty} V_n^{(\alpha)}(x;\lambda)\frac{z^n}{n!}, \tag{38}$$

$|z| < 2\pi$, *when* $\lambda = 1$; $|z| < |-2\log(-1/\lambda)|$, *when* $\lambda \neq 1$.

Furthermore, by $M_n^{(\alpha)}(0;\lambda) := M_n^{(\alpha)}(\lambda)$, $A_n^{(\alpha)}(0;\lambda) := A_n^{(\alpha)}(\lambda)$ and $V_n^{(\alpha)}(0;\lambda) := V_n^{(\alpha)}(\lambda)$ we denote the corresponding generalized of U-Apostol Bernoulli numbers, U-Apostol Euler numbers and U-Apostol Genocchi numbers, respectively.

From (36)–(38) it is fairly straightforward to deduce the addition formulas:

$$M_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} M_k^{(\alpha)}(x;\lambda) M_{n-k}^{(\beta)}(y;\lambda), \tag{39}$$

$$A_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(\alpha)}(x;\lambda) A_{n-k}^{(\beta)}(y;\lambda), \tag{40}$$

$$V_n^{(\alpha+\beta)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} V_k^{(\alpha)}(x;\lambda) V_{n-k}^{(\beta)}(y;\lambda). \tag{41}$$

Making an adequate substitution in (39)–(41), we get

$$M_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} M_k^{(\alpha)}(y;\lambda)(-1)^{n-k}x^{n-k}, \tag{42}$$

$$A_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} A_k^{(\alpha)}(y;\lambda)\Big(-\frac{1}{2}\Big)^{n-k}x^{n-k}, \tag{43}$$

$$V_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n} \binom{n}{k} V_k^{(\alpha)}(y;\lambda)\Big(-\frac{1}{2}\Big)^{n-k}x^{n-k}. \tag{44}$$

Proposition 10. *For every $n \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$, the following relations hold true*

$$x^n = (\lambda-1)M_{n+1}(1-x;\lambda) - \frac{1}{(n+1)}\sum_{k=0}^{n}\binom{n+1}{k}M_k(1-x;\lambda), \tag{45}$$

$$x^n = 2^{n-1}\Big[\lambda A_n(1-x;\lambda) + \sum_{k=0}^{n}\binom{n}{k}\frac{1}{2^{n-k}}A_k(1-x;\lambda)\Big], \tag{46}$$

$$x^n = \frac{2^{n-1}}{n+1}\Big[\lambda V_{n+1}(1-x;\lambda) + \sum_{k=0}^{n+1}\binom{n+1}{k}\frac{1}{2^{n+1-k}}V_k(1-x;\lambda)\Big]. \tag{47}$$

Note that to prove (45)–(47), it is sufficient to use (36)–(38), respectively.

Theorem 6. *The generalized U-Apostol Bernoulli polynomials $M_n^{(\alpha)}(x;\lambda)$ of order α satisfy the following relation*

$$M_n^{(\alpha)}(x+y;\lambda) = \frac{1}{2}\sum_{k=0}^{n} 2^{n-k}\binom{n}{k}[(-1)^{n-k}M_k^{(\alpha)}(y;\lambda) + M^{(\alpha)}(1+y;\lambda)]A_{n-k}(1-x).$$

Proof. If we replace (24) in (42) we have,

$$\begin{aligned} M_n^{(\alpha)}(x+y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda)A_{n-k}(1-x) \\ &+ \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda)\frac{1}{2^{n-k}}\sum_{j=0}^{n-k}\binom{n-k}{j}2^{j}A_j(1-x). \end{aligned}$$

Reversing the order of summation, we have

$$\begin{aligned} M_n^{(\alpha)}(x+y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda)A_{n-k}(1-x) \\ &+ \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}2^{j}A_j(1-x)\sum_{k=0}^{n-j}\binom{n-j}{k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda). \end{aligned} \tag{48}$$

On the other hand if in (42) $x=1$, we get

$$M_n^{(\alpha)}(1+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda), \tag{49}$$

replacing (49) in (48) and $n \longmapsto n-j$, $0 \leqslant j \leqslant n$, $n, j \in \mathbb{N}_0$, we obtain

$$\begin{aligned} M_n^{(\alpha)}(x+y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda)A_{n-k}(1-x) \\ &+ \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}2^{j}A_j(1-x)M_{n-j}^{(\alpha)}(1+y;\lambda). \end{aligned}$$

The following expression is equivalent to the previous one

$$\begin{aligned} M_n^{(\alpha)}(x+y;\lambda) &= \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}(-1)^{n-k}M_k^{(\alpha)}(y;\lambda)A_{n-k}(1-x) \\ &+ \frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}A_{n-k}(1-x)M_k^{(\alpha)}(1+y;\lambda). \end{aligned}$$

Factoring, we get the result. □

3 Some relationships for the polynomials $M_n^{(\alpha)}(x;\lambda)$, $A_n^{(\alpha)}(x;\lambda)$ and $V_n^{(\alpha)}(x;\lambda)$

In this section, we deduce some relations by connecting the polynomials given in Section 2 and other families of polynomials such as Jacobi polynomials and generalized Bernoulli polynomials of level m, Genocchi polynomials and Stirling numbers of the second kind.

Theorem 7. *For $n \in \mathbb{N}_0$, the generalized U-Apostol Bernoulli polynomials $M_n^{(\alpha)}(x;\lambda)$ are related with the Jacobi polynomials $P_n^{(\kappa,\beta)}(x)$ by means of the following identity*

$$M_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^{n+k}j!\binom{j+\kappa}{j-k}\binom{n}{j}\frac{1+\kappa+\beta+2k}{(1+\kappa+\beta+k)_{j+1}}M_{n-j}^{(\alpha)}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x). \quad (50)$$

Proof. By substituting (6) into the right-hand side of (42), we have the following result.

$$\begin{aligned}
M_n^{(\alpha)}(x+y;\lambda) &= \sum_{j=0}^{n}\binom{n}{j}M_j^{(\alpha)}(y;\lambda)\frac{(n-j)!}{(-1)^{j-n}}\sum_{k=0}^{n-j}(-1)^k\binom{n-j+\kappa}{n-j-k}\frac{(1+\kappa+\beta+2k)P_k^{(\kappa,\beta)}(1-2x)}{(1+\kappa+\beta+k)_{n-j+1}}\\
&= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}M_j^{(\alpha)}(y;\lambda)\frac{(n-j)!}{(-1)^{j-n-k}}\binom{n-j+\kappa}{n-j-k}\frac{(1+\kappa+\beta+2k)P_k^{(\kappa,\beta)}(1-2x)}{(1+\kappa+\beta+k)_{n-j+1}}\\
&= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{j}\binom{n-j+\kappa}{n-j-k}M_j^{(\alpha)}(y;\lambda)\frac{(n-j)!}{(-1)^{j-n-k}}\frac{(1+\kappa+\beta+2k)P_k^{(\kappa,\beta)}(1-2x)}{(1+\kappa+\beta+k)_{n-j+1}}\\
&= \sum_{k=0}^{n}\sum_{j=k}^{n}j!(-1)^{n-j+k}\binom{j+\kappa}{j-k}\binom{n}{j}\frac{1+\kappa+\beta+2k}{(1+\kappa+\beta+k)_{j+1}}M_{n-j}^{(\alpha)}(y;\lambda)P_k^{(\kappa,\beta)}(1-2x).
\end{aligned}$$

Therefore, the identity (50) holds. ☐

Theorem 8. *For $n \in \mathbb{N}_0$, the generalized U-Apostol Euler polynomials $A_n^{(\alpha)}(x;\lambda)$ are related with the generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ of level m by means of the following identity*

$$A_n^{(\alpha)}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{j=k}^{n}\Big(-\frac{1}{2}\Big)^j\frac{k!}{(k+m)!}\binom{n}{j}\binom{j}{k}A_n^{(\alpha)}(y;\lambda)B_{j-k}^{[m-1]}(x).$$

Proof. By substituting (8) into the right-hand side of (43), one gets the result. ☐

Theorem 9. *For $n \in \mathbb{N}_0$, the generalized U-Apostol Genocchi polynomials $V_n^{(\alpha)}(x;\lambda)$ are related with the Genocchi polynomials $G_n(x)$ by means of the following identity*

$$V_n^{(\alpha)}(x+y;\lambda) = \frac{1}{2}\sum_{k=0}^{n}\Big(-\frac{1}{2}\Big)^k\frac{1}{k+1}\Big[\binom{n}{k}V_{n-k}^{(\alpha)}(y;\lambda)+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}V_{n-j}^{(\alpha)}(y;\lambda)\Big(-\frac{1}{2}\Big)^{j-k}\Big]G_{k+1}(x). \quad (51)$$

Proof. By substituting (9) into the right-hand side of (44), we see that

$$\begin{aligned}
&V_n^{(\alpha)}(x+y;\lambda)\\
&=\sum_{j=0}^{n}\binom{n}{j}V_j^{(\alpha)}(y;\lambda)\left(\frac{-1}{2}\right)^{n-j}\frac{1}{2(n-j+1)}\left[\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}G_{k+1}(x)+G_{n-j+1}(x)\right]\\
&=\sum_{j=0}^{n}\binom{n}{j}V_j^{(\alpha)}(y;\lambda)\left(\frac{-1}{2}\right)^{n-j}\frac{1}{2(n-j+1)}\sum_{k=0}^{n-j}\binom{n-j+1}{k+1}G_{k+1}(x)\\
&\quad+\sum_{j=0}^{n}\binom{n}{j}V_j^{(\alpha)}(y;\lambda)\left(\frac{-1}{2}\right)^{n-j}\frac{1}{2(n-j+1)}G_{n-j+1}(x).
\end{aligned}$$

Then,

$$\begin{aligned}
&V_n^{(\alpha)}(x+y;\lambda)\\
&=\frac{1}{2}\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^{k}\frac{1}{k+1}\left[\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}V_{n-j}^{(\alpha)}(y;\lambda)\left(\frac{-1}{2}\right)^{j-k}+\binom{n}{k}V_{n-k}^{(\alpha)}(y;\lambda)\right]G_{k+1}(x).
\end{aligned}$$

Therefore, the identity (51) holds. □

Theorem 10. *For $n\in\mathbb{N}$, $\alpha,\lambda\in\mathbb{C}$, we have*

$$M_n^{(\alpha)}(x+y;\lambda)=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=k}^{n}\binom{n}{n-j}M_{n-j}^{(\alpha)}(y;\lambda)(-1)^{n-j}S(j,k).$$

Proof. By substituting (7) into the right-hand side of (42) and using appropriate binomial coefficient identities, we obtain the result. □

4 Generalized U-Bernoulli, U-Euler and U-Genocchi polynomials matrices

In this section, we define the generalized U-Bernoulli, U-Euler and U-Genocchi polynomials matrices and show some of their properties (see [19,20,24]).

Definition 5. *The generalized $(n+1)\times(n+1)$ U-Bernoulli polynomial matrix $\mathcal{M}^{(\alpha)}(x)=m_{i,j}^{(\alpha)}[x]$ is defined by*

$$m_{i,j}^{(\alpha)}[x]=\begin{cases}\binom{i}{j}M_{i-j}^{(\alpha)}(x), & i\geq j,\\ 0, & \text{otherwise},\end{cases}$$

where $\mathcal{M}^{(1)}(x):=\mathcal{M}(x)$ and $\mathcal{M}(0):=\mathcal{M}$ are called the U-Bernoulli polynomial matrix and U-Bernoulli matrix, respectively.

Let us consider $n=3$. It follows from the Definition 5 that

$$\mathcal{M}=\begin{bmatrix}-1 & 0 & 0 & 0\\ -\frac{1}{2} & -1 & 0 & 0\\ -\frac{1}{6} & -1 & -1 & 0\\ 0 & -\frac{1}{2} & -\frac{3}{2} & -1\end{bmatrix}$$

and

$$\mathcal{M}(x)=\begin{bmatrix} -1 & 0 & 0 & 0\\ x-\frac{1}{2} & -1 & 0 & 0\\ -x^2-x-\frac{1}{6}+\frac{1}{4} & 2x-1 & -1 & 0\\ x^3-3x^2+\frac{1}{2}x & -3x^2-3x-\frac{1}{2} & 3x-\frac{3}{2} & -1 \end{bmatrix}.$$

Theorem 11. *The generalized U-Bernoulli polynomial matrix $\mathcal{M}^{(\alpha)}(x)$ satisfies the following product formula*

$$\mathcal{M}^{(\alpha+\beta)}(x+y)=\mathcal{M}^{(\alpha)}(x)\mathcal{M}^{(\beta)}(y)=\mathcal{M}^{(\beta)}(x)\mathcal{M}^{(\alpha)}(y)=\mathcal{M}^{(\alpha)}(y)\mathcal{M}^{(\beta)}(x). \tag{52}$$

Proof. Let $D_{i,j}^{(\alpha,\beta)}(x,y)$ be the (i,j)th entry of the matrix product $\mathcal{M}^{(\alpha)}(x)\mathcal{M}^{(\beta)}(y)$. By the addition formula (39) and $\lambda=1$, we have

$$\begin{aligned} D_{i,j}^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n}\binom{i}{k}\mathcal{M}_{i-k}^{(\alpha)}(x)\binom{k}{j}\mathcal{M}_{k-j}^{(\beta)}(y)=\sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathcal{M}_{i-k}^{(\alpha)}(x)\mathcal{M}_{k-j}^{(m-1,\beta)}(y)\\ &=\binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathcal{M}_{i-j-k}^{(\alpha)}(x)\mathcal{M}_{k}^{(\beta)}(y)=\binom{i}{j}\mathcal{M}_{i-j}^{(\alpha+\beta)}(x+y), \end{aligned}$$

which implies (52). □

Let $\mathcal{F}=f_{i,j}$ be the $(n+1)\times(n+1)$ matrix whose entries are defined by

$$f_{i,j}=\begin{cases} -\dfrac{(-1)^{i-j}}{i-j+1}\dbinom{i}{j}, & i\geq j,\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 12. *The inverse U-Bernoulli matrix $\mathcal{M}$ is given by $\mathcal{M}^{-1}=\mathcal{F}$.*

Proof. Given

$$\sum_{k=0}^{n}\frac{(-1)^k}{k+1}\binom{n}{k}M_{n-k}=\delta_{n,0},$$

where $\delta_{n,0}$ is the Kronecker delta, we have

$$\begin{aligned} \sum_{k=j}^{i}-\binom{i}{k}M_{i-k}\frac{(-1)^{k-j}}{k-j+1}\binom{k}{j} &= -\binom{i}{j}\sum_{k=j}^{i}\frac{(-1)^{k-j}}{k-j+1}\binom{i-j}{k-j}M_{i-k}\\ &= -\binom{i}{j}\sum_{k=0}^{i-j}\frac{(-1)^{k}}{k+1}\binom{i-j}{k}M_{i-j-k}=-\binom{i}{j}\delta_{i-j,0}. \end{aligned}$$

The proof is finished. □

Definition 6. *Let x be any nonzero real number. We define the generalized type U-Pascal matrix $\mathcal{G}[x]=g_{i,j}(x)$ of first kind as an $(n+1)\times(n+1)$ matrix whose entries are given by*

$$g_{i,j}(x)=\begin{cases} (-1)^{i-j}\dbinom{i}{j}x^{i-j}, & i\geq j,\\ 0, & \textit{otherwise.}\end{cases}$$

Theorem 13. *The U-Bernoulli polynomial matrix $\mathcal{M}(x)$ satisfies the following relations*

$$\begin{aligned}\mathcal{M}(x+y) &= \mathcal{G}[x]\mathcal{M}(y) = \mathcal{G}[y]\mathcal{M}(x), \\ \mathcal{M}(x) &= \mathcal{G}[x]\mathcal{M}.\end{aligned} \tag{53}$$

Proof. The substitution $\beta = 0$ into (52) yields

$$\mathcal{M}^{(\alpha)}(x+y) = \mathcal{M}^{(\alpha)}(x)\mathcal{M}^{(0)}(y) = \mathcal{M}^{(0)}(x)\mathcal{M}^{(\alpha)}(y) = \mathcal{M}^{(\alpha)}(y)\mathcal{M}^{(0)}(x).$$

Since for $\alpha = 0$ and $\lambda = 1$ in (36), we have $\mathcal{M}^{(0)}(x) = \mathcal{G}[x]$, then

$$\mathcal{M}^{(\alpha)}(x+y) = \mathcal{G}[x]\mathcal{M}^{(\alpha)}(y). \tag{54}$$

Next, the substitution $\alpha = 1$ into (54) yields (53). □

Definition 7. *For $\alpha \in \mathbb{C}$, the generalized $(n+1) \times (n+1)$ U-Euler polynomial matrix $\mathcal{A}(x) = a_{i,j}^{(\alpha)}[x]$ is defined by*

$$a_{i,j}^{(\alpha)}(x) = \begin{cases} \binom{i}{j} A_{i-j}^{(\alpha)}(x), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathcal{A}^{(1)}(x) := \mathcal{A}(x)$ and $\mathcal{A}(0) := \mathcal{A}$ are called the U-Euler polynomial matrix and the U-Euler matrix, respectively.

Let us consider $n = 3$. It follows from the Definition 7 that

$$\mathcal{A} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ -\frac{1}{32} & 0 & \frac{3}{4} & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{A}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} - \frac{1}{4}x & 1 & 0 & 0 \\ \frac{1}{4}x^2 - \frac{1}{4}x & \frac{1}{2} - x & 1 & 0 \\ -\frac{1}{8}x^3 + \frac{3}{16}x^2 - \frac{1}{32} & \frac{3}{4}x^2 - \frac{3}{4}x & \frac{3}{4} - \frac{3}{2}x & 1 \end{bmatrix}.$$

Theorem 14. *The generalized U-Euler polynomial matrix $\mathcal{A}^{(\alpha)}(x)$ satisfies the following product formula*

$$\mathcal{A}^{(\alpha+\beta)}(x+y) = \mathcal{A}^{(\alpha)}(x)\mathcal{A}^{(\beta)}(y) = \mathcal{A}^{(\beta)}(x)\mathcal{A}^{(\alpha)}(y) = \mathcal{A}^{(\alpha)}(y)\mathcal{A}^{(\beta)}(x). \tag{55}$$

Proof. Let $W_{i,j}^{(\alpha,\beta)}(x,y)$ be the (i,j)th entry of the matrix product $\mathcal{A}^{(\alpha)}(x)\mathcal{A}^{(\beta)}(y)$. By the addition formula (40) and $\lambda = 1$, we have

$$\begin{aligned} W_{i,j}^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n} \binom{i}{k} \mathcal{A}_{i-k}^{(\alpha)}(x) \binom{k}{j} \mathcal{A}_{k-j}^{(\beta)}(y) = \sum_{k=j}^{i} \binom{i}{j} \binom{i-j}{i-k} \mathcal{A}_{i-k}^{(\alpha)}(x) \mathcal{A}_{k-j}^{(m-1,\beta)}(y) \\ &= \binom{i}{j} \sum_{k=0}^{i-j} \binom{i-j}{k} \mathcal{A}_{i-j-k}^{(\alpha)}(x) \mathcal{A}_{k}^{(\beta)}(y) = \binom{i}{j} \mathcal{A}_{i-j}^{(\alpha+\beta)}(x+y), \end{aligned}$$

which implies (55). □

Definition 8. *Let x be any nonzero real number. We define the generalized U-Pascal-type matrix $\mathcal{P}[x] = p_{i,j}(x)$ of first kind as an $(n+1)\times(n+1)$ matrix whose entries are given by*

$$p_{i,j}(x) = \begin{cases} \dfrac{(-1)^{i-j}}{2^{i-j}}\dbinom{i}{j}x^{i-j}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 15. *The U-Euler polynomial matrix $\mathcal{A}(x)$ satisfies the following relations*

$$\mathcal{A}(x+y) = \mathcal{P}[x]\mathcal{A}(y) = \mathcal{P}[y]\mathcal{A}(x), \qquad (56)$$
$$\mathcal{A}(x) = \mathcal{P}[x]\mathcal{A}.$$

Proof. The substitution $\beta = 0$ into (55) yields

$$\mathcal{A}^{(\alpha)}(x+y) = \mathcal{A}^{(\alpha)}(x)\mathcal{A}^{(0)}(y) = \mathcal{A}^{(0)}(x)\mathcal{A}^{(\alpha)}(y) = \mathcal{A}^{(\alpha)}(y)\mathcal{A}^{(0)}(x).$$

Since for $\alpha = 0$ and $\lambda = 1$ in (37), we have $\mathcal{A}^{(0)}(x) = \mathcal{P}[x]$, then

$$\mathcal{A}^{(\alpha)}(x+y) = \mathcal{P}[x]\mathcal{A}^{(\alpha)}(y). \qquad (57)$$

Next, the substitution $\alpha = 1$ into (57) yields (56). □

Definition 9. *For $\alpha \in \mathbb{C}$, the generalized $(n+1)\times(n+1)$ U-Genocchi polynomial matrix $\mathcal{V}^{(\alpha)}(x) = v_{i,j}^{(\alpha)}[x]$ is defined by*

$$v_{i,j}^{(\alpha)}[x] = \begin{cases} \dbinom{i+1}{j+1}V_{i-j+1}^{(\alpha)}(x), & i \geq j, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{V}^{(1)}(x) := \mathcal{V}(x)$ and $\mathcal{V}(0) := \mathcal{V}$ are called U-Genocchi polynomial matrix and U-Genocchi matrix, respectively.

Theorem 16. *The generalized U-Genocchi polynomial matrix $\mathcal{V}^{(\alpha)}(x)$ satisfies the following product formula*

$$\mathcal{V}^{(\alpha+\beta)}(x+y) = \mathcal{V}^{(\alpha)}(x)\mathcal{V}^{(\beta)}(y) = \mathcal{V}^{(\beta)}(x)\mathcal{V}^{(\alpha)}(y) = \mathcal{V}^{(\alpha)}(y)\mathcal{V}^{(\beta)}(x). \qquad (58)$$

Proof. Let $K_{i,j}^{(\alpha,\beta)}(x,y)$ be the (i,j)th entry of the matrix product $\mathcal{V}^{(\alpha)}(x)\mathcal{V}^{(\beta)}(y)$. By the addition formula (41) and $\lambda = 1$, we have

$$\begin{aligned} K_{i,j}^{(\alpha,\beta)}(x,y) &= \sum_{k=0}^{n}\binom{i}{k}\mathcal{V}_{i-k}^{(\alpha)}(x)\binom{k}{j}\mathcal{V}_{k-j}^{(\beta)}(y) = \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathcal{V}_{i-k}^{(\alpha)}(x)\mathcal{V}_{k-j}^{(m-1,\beta)}(y) \\ &= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathcal{V}_{i-j-k}^{(\alpha)}(x)\mathcal{V}_{k}^{(\beta)}(y) = \binom{i}{j}\mathcal{V}_{i-j}^{(\alpha+\beta)}(x+y), \end{aligned}$$

which implies (58). □

Theorem 17. *The U-Genocchi polynomial matrix $\mathcal{V}(x)$ satisfies the following relations*

$$\mathcal{V}(x+y) = \mathcal{P}[x]\mathcal{V}(y) = \mathcal{P}[y]\mathcal{V}(x), \qquad (59)$$
$$\mathcal{V}(x) = \mathcal{P}[x]\mathcal{V}.$$

Proof. The substitution $\beta = 0$ into (58) yields

$$\mathcal{V}^{(\alpha)}(x+y) = \mathcal{V}^{(\alpha)}(x)\mathcal{V}^{(0)}(y) = \mathcal{V}^{(0)}(x)\mathcal{V}^{(\alpha)}(y) = \mathcal{V}^{(\alpha)}(y)\mathcal{V}^{(0)}(x).$$

Since for $\alpha = 0$ and $\lambda = 1$ in (38), we have $\mathcal{V}^{(0)}(x) = \mathcal{P}[x]$, then

$$\mathcal{V}^{(\alpha)}(x+y) = \mathcal{P}[x]\mathcal{V}^{(\alpha)}(y). \qquad (60)$$

Next, the substitution $\alpha = 1$ into (60) yields (59). □

5 Conclusion

Our research aimed to introduce novel families of U-Bernoulli, U-Euler, and U-Genocchi polynomials, coupled with their unique properties. While it is important to note that the utilization of the Cauchy product of power series underpins some of our formulations, it is not a new method. Nonetheless, it has proven invaluable in generating fresh sets of special polynomials, whether or not they adhere to Appell-type conditions. Even in recent times, this approach has led to significant discoveries. For an in-depth exploration of the latest developments in this expansive field, we refer the interested reader to [6, 11, 13] and the references therein. Finally, we unveiled novel properties of the generalized matrices associated with the U-Bernoulli, U-Euler, and U-Genochhi polynomials. These discoveries shed light on their inherent properties and offer insightful factorizations.

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Received 30.06.2023
Revised 11.11.2023

Рамірез В., Бедоя Д., Урієлес А., Чезарано К., Ортега М. *Нові поліноми U-Бернуллі, U-Ойлера і U-Дженоккі та їх матриці* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 449–467.

У цій статті означено поліноми U-Бернуллі, U-Ойлера та U-Дженоккі, їх числа та показано зв'язок цих поліномів із дзета-функцією Рімана. Для встановлення деяких їхніх алгебраїчних і диференціальних властивостей уведено узагальнення типу Апостола. Визначено узагальнені U-Бернуллі, U-Ойлера та U-Дженоккі поліноміальні матриці типу Паскаля та доведено деякі формули добутку, пов'язані з цими матрицями. Крім того, встановлено деякі явні вирази для поліноміальних матриць U-Бернуллі, U-Ойлера та U-Дженоккі, які включають узагальнену матрицю Паскаля.

Ключові слова і фрази: поліном U-Бернуллі, поліном U-Ойлера, узагальнений поліном U-Бернуллі, узагальнений поліном U-Ойлера, матриця поліномів U-Бернуллі, матриця поліномів U-Ойлера, матриця Паскаля.