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Metric and topology on the poset of compact pseudoultrametrics

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In two ways we introduce metrics on the set of all pseudoultrametrics, not exceeding a given compact pseudoultrametric on a fixed set, and prove that the obtained metrics are compact and topologically equivalent. To achieve this, we give a characterization of the sets being the hypographs of the mentioned pseudoultrametrics, and apply Hausdorff metric to their family. It is proved that the uniform convergence metric is a limit case of metrics defined via hypographs. It is shown that the set of all pseudoultrametrics, not exceeding a given compact pseudoultrametric, with the induced topology is a Lawson compact Hausdorff upper semilattice.

Key words and phrases: pseudoultrametric, metrization, compactum, hypograph.

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Introduction

The goal of this paper is to introduce metrics (and hence topologies) on the set of all compact pseudoultrametrics on a fixed set, to compare them and to study their properties. Ultrametrics (or non-Archimedean metrics [2]) are studied since the beginning of XX century, cf. a review in [3]. They found numerous applications, e.g., in computer science.

The set of all pseudoultrametrics (i.e., of the pseudometrics that satisfy the stronger triangle inequality required for ultrametrics) on a fixed set carries a natural (i.e., pointwise) partial order, and is a lattice. Lattice operations on it were extensively studied in [5], and it was shown that their properties are unsatisfactory if all pseudoultrametrics on a set (or even all locally compact pseudoultrametrics only) are considered. To obtain reasonably good (i.e., continuous in the sense of order theory [1]) posets, one should restrict to *compact* pseudoultrametrics.

Two main approaches are proposed via uniform convergence metrics and via subgraphs (hypographs) of functions. The first approach is classical, and the second one proved to be useful in the study of non-additive measures [7]. For the both it appears to be necessary to consider not all compact pseudoultrametrics on a set but pseudoultrametrics not exceeding a fixed one only. In this case both approaches are successful and induce the same topology.

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Moreover, the uniform convergence distance between compact pseudoultrametrics is, under reasonable assumptions, the limit of the "subgraph-style" distances between these pseudoultrametrics determined with a non-decreasing net of compact pseudoultrametrics. Thus, the two methods are in fact equivalent and interchangeable. Recall that families of (pseudo-)ultrametrics were studied in [4], so a link to a previous research naturally arise.

This paper relies heavily on [5] and [6], and is, in particular, a preparatory work for a continuation of the mentioned studies.

1 Preliminaries

Recall that a poset $(D, \leq)$ is directed (resp. filtered) [1] if for all $d_1, d_2 \in D$ there is $d \in D$ such that $d_1, d_2 \leq d$ (resp. $d_1, d_2 \geq d$).

For an element d of a poset $(D, \leq)$ we denote $d\downarrow = \{d' \in D \mid d' \leq d\}$.

The following notion is a natural mixture of ones of ultrametric and pseudometric. Let X be a nonempty set.

Definition 1. *A mapping $d : X \times X \to \mathbb{R}$ that satisfies the conditions:*

- $d(x, y) \geq 0$ *for all* $x, y \in X$ *(nonnegativeness),*
- $d(x, x) = 0$ *for all* $x \in X$ *(identity),*
- $d(x, y) = d(y, x)$ *for all* $x, y \in X$ *(symmetry),*
- $d(x, y) \leq \max\{d(y, z), d(z, x)\}$ *for all* $x, y, z \in X$ *(triangle inequality),*

is called a pseudoultrametric on the set X.

It is just a pseudometric such that the usual triangle inequality $d(x, y) \leq d(y, z) + d(z, x)$ holds in a stronger form. Let $CPsU(X)$ be set of all compact pseudoultrametrics on X. The partial order on $CPsU(X)$ is defined pointwise: a pseudoultrametric d_1 precedes a pseudoultrametric d_2 (written $d_1 \leq d_2$) if $d_1(x, y) \leq d_2(x, y)$ holds for all points $x, y \in X$. The trivial pseudometric $d \equiv 0$ is the least element of $CPsU(X)$.

The least upper bound of pseudoultrametrics d_1, d_2 is the pointwise maximum $d^*(x, y) = \max\{d_1(x, y), d_2(x, y)\}$ for all $x, y \in X$.

2 Subgraphs of compact pseudoultrametrics

Definition 2. *The subgraph (or the hypograph) of a compact pseudoutrametric d on a set X is the set*

$$\operatorname{sub} d = \{(x, y, a) \colon x, y \in X, a \in [0; d(x, y)]\}.$$

Then $\operatorname{sub} d \subset X \times X \times [0, +\infty)$, and, taking into account that any compact pseudoultrametric is bounded from above and attains its least upper bound, we can state $\operatorname{sub} d \subset X \times X \times [0, M]$ for all $M \geq \max d$. Obviously, for all $d_1, d_2 \in CPsU(X)$, the inequality $d_1 \leq d_2$ is equivalent to $\operatorname{sub} d_1 \subset \operatorname{sub} d_2$.

Assume that a compact pseudoultrametric $\hat{d}$ is fixed on X, and let all compact pseudoultrametrics d from now on in this section be in $\hat{d}\downarrow$, i.e., $d \leq \hat{d}$ (hence $\operatorname{sub} d \subset \operatorname{sub} \hat{d}$).

Define a pseudometric ρ on the set $X \times X \times [0, M]$ by the formula

$$\rho\big((x_1, y_1, a_1), (x_2, y_2, a_2)\big) = \max\{\hat{d}(x_1, x_2), \hat{d}(y_1, y_2), |a_1 - a_2|\}.$$

Lemma 1. *The set* $\operatorname{sub} d$ *is closed in* $X \times X \times [0, M]$ *for all* $M \geq \max \hat{d}$*.*

Proof. Consider the complement of $\operatorname{sub} d$ in $X \times X \times [0, M]$. Show that it is an open set. Let $(x_0, y_0, a_0) \in X \times X \times [0, M] \setminus \operatorname{sub}\, d$, i.e., $a_0 > d(x_0, y_0)$. Then $\varepsilon = \frac{a_0 - d(x_0, y_0)}{2} > 0$.

Verify that the ball with respect to ρ with the center (x_0, y_0, a_0) and the radius ε is contained in $X \times X \times [0, M] \setminus \operatorname{sub} d$. If $\rho\big((x_0, y_0, a_0), (x, y, a)\big) < \varepsilon$, then $d(x_0, x) \leq \hat{d}(x_0, x) < \varepsilon$, $d(y_0, y) \leq \hat{d}(y_0, y) < \varepsilon$, hence $d(x, y) < d(x_0, y_0) + \varepsilon$, $a > a_0 - \varepsilon$. Taking into account $d(x_0, y_0) + \varepsilon = a_0 - \varepsilon$, we obtain $d(x, y) < a$, i.e., $(x, y, a) \notin \operatorname{sub} d$, which completes the proof. □

Lemma 2. *For a compact pseudoultrametric* $d \in \hat{d}\downarrow$ *the following conditions hold:*

1) $X \times X \times \{0\} \subset \operatorname{sub} d$;

2) *for all* $(x, y, b) \in \operatorname{sub} d$ *and* $a \in [0, b]$ *we have* $(x, y, a) \in \operatorname{sub} d$;

3) *if* $(x, y, a) \in \operatorname{sub} d$, *then* $(y, x, a) \in \operatorname{sub} d$;

4) *if* $(x, x, a) \in \operatorname{sub} d$, *then* $a = 0$;

5) *for all* $x, y, z \in X$, *if* $(x, z, c) \in \operatorname{sub} d$, *then* $(x, y, a) \in \operatorname{sub} d$ *or* $(y, z, b) \in \operatorname{sub} d$.

Proof. 1) By the definition, $d(x, y) \geq 0$ for all $x, y \in X$, therefore $(x, y, 0) \in \operatorname{sub} d$.

2) If $a \in [0, b]$ and $(x, y, b) \in \operatorname{sub} d$, then $0 \leq a \leq b \leq d(x, y) \in \operatorname{sub} d$, hence $(x, y, a) \in \operatorname{sub} d$.

3) The equality $d(x, y) = d(y, x)$ implies $a \leq d(x, y) \iff a \leq d(y, x)$, therefore $(x, y, a) \in \operatorname{sub} d$ is equivalent to $(y, x, a) \in \operatorname{sub} d$.

4) By the definition, $(x, x, a) \in \operatorname{sub} d \iff 0 \leq a \leq d(x, x) = 0$, i.e., $a = 0$.

5) Let $(x, y, c) \notin \operatorname{sub} d$ and $(y, z, c) \notin \operatorname{sub} d$, then $d(x, y) < c$, $d(y, z) < c$, and, by triangle inequality, $d(x, z) \leq \max\big\{d(x, y), d(y, z)\big\} < \max\{c, c\} = c$, which contradicts to the assumption $(x, z, c) \in \operatorname{sub} c$, i.e., $c \leq d(x, z)$. □

Observe that 1)–5) are valid for the subgraph of $\hat{d}$ itself, and recall $\operatorname{sub} d \subset \operatorname{sub} \hat{d}$ for all $d \in \hat{d}\downarrow$. Now we describe the subsets that are subgraphs of compact pseudoultrametric.

Proposition 1. *Let* $F \subset X \times X \times [0, M]$*. The set* F *satisfies:*

1) F *is contained in* $\operatorname{sub} \hat{d}$ *and closed in* $X \times X \times [0, M]$ *with respect to* ρ;

2) $X \times X \times \{0\} \subset F$;

3) *if* $(x, x, a) \in F$, *then* $a = 0$;

4) *if* $(x, y, b) \in F$, *then* $(x, y, a) \in F$ *for all* $a \in [0, b]$;

5) *if* $(x, y, a) \in F$, *then* $(y, x, a) \in F$;

6) *for all* $x, y, z \in X$, *if* $(x, z, c) \in F$, *then* $(x, y, a) \in F$ *or* $(y, z, b) \in F$;

if and only if there is a compact pseudoultrametric $d \leq \hat{d}$ *such that* $F = \operatorname{sub} d$*.*

Proof. Necessity of 1)–6) has been proved above. Now show sufficiency. Given a subgraph $\operatorname{sub} d$, the pseudoultrametric can be recovered as follows: $d(x,y) = \max\{a \in [0;+\infty) : (x,y,a) \in \operatorname{sub} d\}$. Hence the unique d with the subgraph F (if it exists) should be equal to

$$d(x,y) = \sup\{a \in [0;M] : (x,y,a) \in F\}.$$

Show that such d is a pseudoultrametric. Verify the definition.

The existence and non-negativity of the supremum is guaranteed by $X \times X \times \{0\} \subset F$, hence $\{a \in [0;M] : (x,y,a) \in F\} \supset \{0\}$, and the set is non-empty and bounded from above by M.

The closedness of F implies that $(x,y,b) \in F$ for $b = \sup\{a \in [0;M] : (x,y,a) \in F\}$, i.e., the least upper bound is attained here, so we can write $d(x,y) = \max\{a \in [0;M] : (x,y,a) \in F\}$.

Calculate $d(x,x) = \sup\{a \in [0;M] : (x,x,a) \in F\}$. By 3) the only a here is $a = 0$, hence $d(x,x) = 0$.

To verify symmetry, observe that 5) implies

$$d(x,y) = \sup\{a \in [0;M] : (x,y,a) \in F\} = \{a \in [0;M] : (y,x,a) \in F\} = d(y,x).$$

To show triangle inequality, let $a = d(x,y)$, $b = d(y,z)$, $c = d(x,z)$. By 6), $(x,z,c) \in F$ implies $(x,y,c) \in F$ or $(y,z,c) \in F$, then $c \le a$ or $c \le b$. Thus $c \le \max\{a,b\}$, which completes the proof that d is a pseudoultrametric.

By the assumption of the lemma, $d(x,y) \le \hat{d}(x,y)$ for all $x,y \in X$. Therefore d is continuous with respect to the compact pseudoultrametric $\hat{d}$, hence it is a compact pseudoultrametric as well.

By the construction, $\big(x,y,d(x,y)\big) \in F$ for all $x,y \in X$, and 4) impies $(x,y,a) \in F$ for all $a \in \big[0,d(x,y)\big]$. Therefore $\operatorname{sub} d \subseteq F$.

On the other hand, the definition $d(x,y) = \sup\{a \in [0;M] : (x,y,a) \in F\}$ and 4) imply $F \subseteq \operatorname{sub} d$. This completes the proof that, under the assumptions 1–6) we have $F = \operatorname{sub} d$. □

Remark 1. *The item 3) above is a corollary of $F \subset \operatorname{sub}\hat{d}$ in 1) and of 3) for $\operatorname{sub}\hat{d}$, therefore its verification can be omitted.*

3 Metrization via subgraphs

Recall that the set of all non-empty closed subsets of a (pseudo-)metric space (X,d) is called the hyperspace of this space and denoted with $\exp X$. It can be metrizable with Hausdorff metric

$$d_H(A,B) = \max\Big\{\sup\{d(a,B) : a \in A\}, \sup\{d(b,A) : b \in B\}\Big\}, \quad A,B \in \exp X.$$

It is known that, for a compact (pseudo-)metric d, the metric space $(\exp X, d_H)$ is compact as well. Hence, for any $\hat{d} \in CPsU(X)$ and $M \ge \max\hat{d}$, compactness of the product $X \times X \times [0,M]$ with respect to the pseudometric

$$\rho\big((x_1,y_1,a_1),(x_2,y_2,a_2)\big) = \max\{\hat{d}(x_1,x_2), \hat{d}(y_1,y_2), |a_1 - a_2|\}$$

implies that the set $\exp\big(X \times X \times [0,M]\big)$ of non-empty closed subsets of $X \times X \times [0,M]$ with the metric ρ_H is compact.

For it has been shown that there is a one-to-one correspondence between compact pseudoultrametrics and their subgraphs, we can define a distance (a metric) between $d_1, d_2 \in \hat{d}\downarrow$ as the Hausdorff distance between their subgraphs: $D_H^{\hat{d}}(d_1, d_2) = \rho_H(\operatorname{sub} d_1, \operatorname{sub} d_2)$, i.e.,

$$D_H^{\hat{d}}(d_1, d_2) = \max\big\{ \sup_{u \in \operatorname{sub} d_1} \rho(u, \operatorname{sub} d_2),\ \sup_{v \in \operatorname{sub} d_2} \rho(v, \operatorname{sub} d_1) \big\}$$

where $\rho(u, \operatorname{sub} d_2) = \inf_{v \in \operatorname{sub} d_2} \rho(u, v)$, and similarly for $\rho(v, \operatorname{sub} d_1)$.

Lemma 3. *The set* $S = \{\operatorname{sub} d : d \le \hat{d}, d$ *is a compact pseudoultrametric on* $X\}$ *is closed in the hyperspace* $\exp\big(X \times X \times [0, M]\big)$.

Proof. We are going to prove that the set of all $F \in \exp\big(X \times X \times [0, M]\big)$, that satisfy the conditions of the latter Proposition, is closed.

If $F \not\subset \operatorname{sub} \hat{d}$, then there is a point $(x_0, y_0, a_0) \in \big(X \times X \times [0, M]\big) \setminus \operatorname{sub} \hat{d}$, and due to the closedness of $\operatorname{sub} \hat{d}$ there is a ball $B_\delta\big((x_0, y_0, a_0)\big)$ that has an empty intersection with $\operatorname{sub} \hat{d}$. Therefore for any $G \in \exp\big(X \times X \times [0, M]\big)$ such that $\rho_H(G, F) < \delta$, we have $G \cap B_\delta\big((x_0, y_0, a_0)\big) \neq \varnothing$, hence $G \not\subset \operatorname{sub} \hat{d}$. Thus the set of all $F \in \exp\big(X \times X \times [0, M]\big)$ such that $F \subset \operatorname{sub} \hat{d}$, i.e., 1) holds, is closed. Recall that 3) is valid as well for all these F.

Similarly, if $X \times X \times \{0\} \not\subset F$, then there is $(x_0, y_0, 0) \in \big(X \times X \times [0, M]\big) \setminus F$, and therefore a ball $B_\delta\big((x_0, y_0, a_0)\big)$ that has an empty intersection with F. Then for $G \in \exp\big(X \times X \times [0, M]\big)$ the inequality $\rho_H(G, F) < \delta$ implies $(x_0, y_0, 0) \notin G$, hence G fails to satisfy 2) as well. Thus 2) selects a closed subset in $\exp\big(X \times X \times [0, M]\big)$.

If 4) does not hold for $F \in \exp\big(X \times X \times [0, M]\big)$, then there are $0 \le a_0 < b_0 \le M$ and $x_0, y_0 \in X$ such that $(x_0, y_0, b_0) \in F$ but $(x_0, y_0, a_0) \notin F$. Choose $\delta > 0$ such that $B_\delta\big((x_0, y_0, a_0)\big)$ that has an empty intersection with F. If $G \in \exp\big(X \times X \times [0, M]\big)$ and $\rho_H(G, F) < \delta/2$, then there is $(x, y, b) \in G$ such that $\rho\big((x, y, b), (x_0, y_0, b_0)\big) < \delta/2$.

Denote $a = \max\big\{0, a_0 + (b - b_0)\big\}$, and observe that $0 \le a \le b$ and $\rho\big((x, y, a_0), (x_0, y_0, a)\big) \le \rho\big((x, y, b), (x_0, y_0, b_0)\big)$, hence $\rho\big((x, y, a), (x_0, y_0, a_0)\big) < \delta/2$. Then

$$\rho\big((x, y, a), F\big) \ge \rho\Big((x, y, a), \big(X \times X \times [0, M]\big) \setminus B_\delta\big((x_0, y_0, a_0)\big)\Big) > \delta/2,$$

hence $(x, y, a) \notin G$. Taking into account $(x, y, b) \in G$, we see that all $G \in \exp\big(X \times X \times [0, M]\big)$ such that $\rho_H(G, F) < \delta/2$ fails to satisfy 4).

Assume F does not satisfy 5), i.e., $(x_0, y_0, a_0) \in F$ but $(y_0, x_0, a_0) \notin F$. The closedness of F implies that there exists $\delta > 0$ such that $B_\delta\big((y_0, x_0, a_0)\big) \bigcap F = \varnothing$. Let $G \in \exp\big(X \times X \times [0, M]\big)$ be such that $\rho_H(G, F) < \delta/2$. Then, on the one hand, there is $(x, y, a) \in G$ such that $\rho\big((x, y, a), (x_0, y_0, a_0)\big) < \delta/2$. On the other hand, $\rho\big((y, x, a), (y_0, x_0, a_0)\big) < \delta/2$ implies

$$\rho\big((y, x, a), F\big) \ge \rho\Big((y, x, a), \big(X \times X \times [0, M]\big) \setminus B_\delta\big((y_0, x_0, a_0)\big)\Big) > \delta/2,$$

hence $(y, x, a) \notin G$. Thus 5) fails for all G such that $\rho_H(G, F) < \delta/2$, and the set of all $F \in \exp\big(X \times X \times [0, M]\big)$ such that 5) is valid, is closed.

Let 6) fails for $F \in \exp\big(X \times X \times [0, M]\big)$, i.e., there are $x_0, y_0, z_0 \in X$ and $c_0 \in [0, M]$ such that $(x_0, z_0, c_0) \in F$ but $(x_0, y_0, c_0) \notin F$, $(y_0, z_0, c_0) \notin F$. Choose $\delta > 0$ such that $B_\delta(x_0, y_0, c_0)) \bigcap F = B_\delta(y_0, z_0, c_0)) \bigcap F = \varnothing$. If $G \in \exp(X \times X \times [0, M])$, $\rho_H(G, F) < \delta/2$, then there is $(x, z, c) \in G$ such that $\rho\big((x, z, c), (x_0, z_0, c_0)\big) < \delta/2$. This implies

$$\rho\big((x, y_0, c), (x_0, y_0, c_0)\big) < \delta/2, \quad \rho\big((y_0, z, c), (y_0, z_0, c_0)\big) < \delta/2,$$

therefore

$$\rho\big((x, y_0, c), G\big) > \delta/2, \quad \rho\big((y_0, z, c), G\big) > \delta/2,$$

hence $(x, y_0, c) \notin G$, $(y_0, z, c) \notin G$. Thus such G does not satisfy 6), which completes the proof that S is a closed set in $\exp\big(X \times X \times [0, M]\big)$. □

A closed subset of a metric compactum is a metric compactum as well, therefore we immediately obtain the following assertion.

Corollary 1. *The set $\hat{d}{\downarrow} \subset CPsU(X)$ with the metric $D_H^{\hat{d}}$ is a compact metric space.*

Thus we have obtained a compact metric on the subset $\hat{d}{\downarrow}$. Show that least upper bounds in the lattice $\hat{d}{\downarrow}$ are continuous with respect to this metric.

It is straightforward to observe that $\operatorname{sub}\sup\{d_1, d_2\} = \operatorname{sub} d_1 \cup \operatorname{sub} d_2$ for any $d_1, d_2 \in \hat{d}{\downarrow}$. Moreover, the following lemma is true.

Lemma 4. *If $\{d_\alpha : \alpha \in A\} \subset \hat{d}{\downarrow}$ is non-empty, then $\sup\{d_\alpha : \alpha \in A\}$ exists, and the equality $\operatorname{sub}\sup\{d_\alpha : \alpha \in A\} = \operatorname{Cl}\bigcup_{\alpha \in A} \operatorname{sub} d_\alpha$ is valid for its subgraph.*

Proof. By the above observation, we can assume without loss of generality that the set $\{d_\alpha : \alpha \in A\}$ is directed, then the set $\{\operatorname{sub} d_\alpha : \alpha \in A\} \subset \exp\big(X \times X \times [0, M]\big)$ of subgraphs is directed as well, and all its elements $F_\alpha = \operatorname{sub} d_\alpha$ satisfy conditions 1)–6) above. It is straightforward to verify that the set $F = \operatorname{Cl}(\bigcup_{\alpha \in A} F_\alpha)$ satisfies 1)–6) as well, hence is the subgraph of a unique compact pseudoultrametric $d \in \hat{d}{\downarrow}$. Obviously, for any $d' \in \hat{d}{\downarrow}$ we have

$$\begin{aligned} d' \geq d_\alpha \text{ for all } \alpha \in A &\iff \operatorname{sub} d' \supset \operatorname{sub} d_\alpha \text{ for all } \alpha \in A \\ &\iff \operatorname{sub} d' \supset \operatorname{Cl}\bigcup_{\alpha \in A} \operatorname{sub} d_\alpha = \operatorname{sub} d \\ &\iff d' \geq d, \end{aligned}$$

i.e., d is the least upper bound of all d_α. □

Remark 2. *If a set $\{d_\alpha : \alpha \in A\}$ is non-empty and closed in $\hat{d}{\downarrow}$ with respect to $D_H^{\hat{d}}$, i.e., the set of subgraphs is closed with respect to the Hausdorff distance, then the union $\bigcup_{\alpha \in A} \operatorname{sub} d_\alpha$ is closed and is the subgraph of the supremum d in question.*

So we obtain the mapping $\sup : \exp(\hat{d}{\downarrow}) \to \hat{d}{\downarrow}$, which, when passing to subgraphs, acts simply as the union of a closed family of closed sets. It is well known and easy to verify that such union operation is continuous with respect to the Hausdorff metric. Recall that a compact Hausdorff (hence complete) topological upper semilattice S such that the mapping $\exp S \to S$ is continuous with respect to the Vietoris topology on the hyperspace $\exp S$, is called a Lawson compact Hausdorff upper semilattice [1]. As the Vietoris topology on the hyperspace of a metric compactum is induced with the Hausdorff metric, we arrive at a conclusion.

Corollary 2. *The semilattice $\hat{d}{\downarrow}$ with the topology induced by $D_H^{\hat{d}}$ is a Lawson compact Hausdorff upper semilattice.*

The proposed method of metrization has a substantial drawback: it depends on the choice of a pseudoultrametric $\hat{d}$ "above" the considered pseudoultrametrics. Later we will show that for any $\hat{d}, \bar{d} \in CPsU(X)$ the distances $D_H^{\hat{d}}$ and $D_H^{\bar{d}}$ induce the same topology $\hat{d}{\downarrow} \cap \bar{d}{\downarrow}$.

4 Uniform convergence metric

All $d_1, d_2 \in \hat{d}\downarrow \subset CPsU(X)$ are bounded functions on X, hence we can use the uniform convergence metric

$$D_u(d_1, d_2) = \sup\Big\{ \big|d_1(x,y) - d_2(x,y)\big| : x, y \in X \Big\}.$$

Moreover, due to compactness the least upper bound is attained here, so we can write max instead of sup. This metric appears to be very close to the previously defined "Hausdorff-like" metric.

Proposition 2. *Let d, d_1, d_2 be compact pseudoultrametrics on a set X, and $d_1 \le d$, $d_2 \le d$. Then*

$$D_H^d(d_1, d_2) \le D_u(d_1, d_2) \le 2 \cdot D_H^d(d_1, d_2).$$

Proof. For any $\varepsilon > 0$ the inequality $D_H^d(d_1, d_2) \le \varepsilon$ is equivalent to the following: for all $(x, y, \alpha) \in \operatorname{sub} d_1$ there is $(x', y', \alpha') \in \operatorname{sub} d_2$ such that $d(x, x') \le \varepsilon$, $d(y, y') \le \varepsilon$, $|\alpha - \alpha'| \le \varepsilon$, and vice versa for all $(x', y', \alpha') \in \operatorname{sub} d_2$.

The first condition can then be formulated equivalently as $\mathbf{A_1}(\varepsilon)$: for all $x, y \in X$ there are $x', y' \in X$ such that

$$\begin{cases} d(x, x') \le \varepsilon,\ d(y, y') \le \varepsilon, \\ d_2(x', y') \ge d_1(x, y) - \varepsilon, \end{cases}$$

and similarly $\mathbf{A_2}(\varepsilon)$ for the second one: for all $x', y' \in X$ there are $x, y \in X$ such that

$$\begin{cases} d(x, x') \le \varepsilon,\ d(y, y') \le \varepsilon, \\ d_1(x, y) \ge d_2(x', y') - \varepsilon. \end{cases}$$

What is required is to show that, for any fixed $\varepsilon \ge 0$, the inequality $D_u(d_1, d_2) \le \varepsilon$ holds, i.e., $\big|d_1(x,y) - d_2(x,y)\big| \le \varepsilon$ for all $x, y \in X$, which we denote with $\mathbf{B}(\varepsilon)$, implies the above pair of conditions $\mathbf{A_1}(\varepsilon)$, $\mathbf{A_2}(\varepsilon)$, which, in turn, imply $\mathbf{B}(2\varepsilon)$.

If $\mathbf{B}(\varepsilon)$ is valid, then for all $x, y \in X$ we can put $x' = x$, $y' = y$, and obviously $\mathbf{A_1}(\varepsilon)$ $\big($and similarly$\mathbf{A_2}(\varepsilon)\big)$ holds.

Assume $\mathbf{A_1}(\varepsilon)$+$\mathbf{A_2}(\varepsilon)$ but the existence of $x, y \in X$ such that $d_2(x, y) < d_1(x, y) - 2\varepsilon$, hence $d_1(x, y) > 2\varepsilon$. Using $\mathbf{A_1}(\varepsilon)$, choose $x', y' \in X$ such that

$$\begin{cases} d(x, x') \le \varepsilon,\ d(y, y') \le \varepsilon, \\ d_2(x', y') \ge d_1(x, y) - \varepsilon > \varepsilon. \end{cases}$$

Taking into account $d_2(x, x') \le \varepsilon$, $d_2(y, y') \le \varepsilon$, by the triangle inequality for pseudoultrametrics we obtain

$$d_2(x', y') = d_2(x', y) = d_2(x, y) \implies d_2(x, y) \ge d_1(x, y) - \varepsilon,$$

which contradicts to $d_2(x, y) < d_1(x, y) - 2\varepsilon$. Hence the latter inequality is impossible for all $x, y \in X$, and $d_2(x, y) \ge d_1(x, y) - 2\varepsilon$.

Analogously deduce $d_1(x, y) \ge d_2(x, y) - 2\varepsilon$ from $\mathbf{A_2}(\varepsilon)$, and obtain $\big|d_1(x,y) - d_2(x,y)\big| \le 2\varepsilon$, i.e., $\mathbf{B}(2\varepsilon)$. ☐

Remark 3. *The factor 2 in the above inequality cannot be reduced, which is shown by the following example: consider $X = \{a_1, a_2, b_1, b_2\}$, and, for all $x, y \in X$,*

$$d_1(x,y) = \begin{cases} 0, & x = y, \\ 1, & \{x,y\} = \{a_1,a_2\} \text{ or } \{x,y\} = \{b_1,b_2\}, \\ 2 & \text{otherwise,} \end{cases}$$

$$d_2(x,y) = \begin{cases} 0, & x = y \text{ or } \{x,y\} = \{a_1,b_1\}, \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $d_2 \leq d_1$, $D_u(d_1,d_2) = d_1(a_1,b_1) - d_2(a_1,b_1) = 2 - 0 = 2$, but it is straightforward to verify that $D^d_H(d_1,d_2) = 1$.

This proposition implies that the metrics D_u and D^d_H induce the same topology on the subset $d{\downarrow} \subset CPsU(X)$. Moreover, we easily obtain the following result.

Corollary 3. *If $\hat{d} \leq \bar{d}$ holds for compact pseudoultrametrics on X, then the inclusion of $(\hat{d}{\downarrow}, D^{\hat{d}}_H)$ into $(\bar{d}{\downarrow}, D^{\bar{d}}_H)$ is a topological embedding.*

Therefore for all $\hat{d}, \bar{d} \in CPsU(X)$ the topologies induced by $D^{\hat{d}}_H$ and $D^{\bar{d}}_H$ on $\hat{d}{\downarrow} \cap \bar{d}{\downarrow}$ agree, and all $(\hat{d}{\downarrow}, D^{\hat{d}}_H) \cong (\hat{d}{\downarrow}, D_u)$ are topological subspaces of $(CPsU(X), D_u)$.

Consider the space $(\hat{d}{\downarrow}, D_u)$ for a particular $\hat{d} \in CPsU(X)$.

Lemma 5. *Any element $d \in \hat{d}{\downarrow}$ is a non-expanding function $X \times X \to \mathbb{R}$ with respect to the pseudoultrametric $\hat{d}_\times$ on $X \times X$ defined by the formula*

$$\hat{d}_\times\left((x,y),(x',y')\right) = \max\left\{\hat{d}(x,x'), \hat{d}(y,y')\right\}.$$

Proof. Let $\hat{d}_\times\left((x,y),(x',y')\right) \leq \varepsilon$, then $\hat{d}(x,x') \leq \varepsilon$, $\hat{d}(y,y') \leq \varepsilon$. Taking into account $d(x,x') \leq \hat{d}(x,x')$, $d(y,y') \leq \hat{d}(y,y')$, we obtain $d(x,x') \leq \varepsilon$, $d(y,y') \leq \varepsilon$. Then by the triangle inequality

$$d(x',y') \leq \max\left\{d(x',x), d(x,y), d(y,y')\right\} \leq \max\left\{d(x,y), \varepsilon\right\} \leq d(x,y) + \varepsilon,$$

and, analogously, $d(x,y) \leq d(x',y') + \varepsilon$, which yields $\left|d(x,y) - d(x',y')\right| \leq \varepsilon$. This completes the proof. □

Recall that, by Arzelà-Ascoli theorem, a set $\mathcal{F}$ of continuous functions on a compact (pseudo-)metric space is relatively compact with respect to the uniform convergence metric (i.e., its closure is compact) if and only if $\mathcal{F}$ is uniformly equicontinuous and pointwise bounded. The elements of $\hat{d}{\downarrow}$ are non-expanding functions on $(X \times X, \hat{d}_\times)$, hence $\hat{d}{\downarrow}$ is uniformly equicontinuous. It is bounded with any $M \geq \max \hat{d}$, and it is straightforward to verify that the limit of a uniformly convergent sequence d_n in $\hat{d}{\downarrow}$ is a pseudoultrametric in $\hat{d}{\downarrow}$. Thus $\hat{d}{\downarrow}$ is closed, and we obtain is an alternative proof that $(\hat{d}{\downarrow}, D^{\hat{d}}_H) \cong (\hat{d}{\downarrow}, D_u)$ is compact.

Let $\hat{d} \leq \bar{d}$, then on $X \times X \times [0, M]$, with $M \geq \sup \bar{d} \geq \sup \hat{d}$, we have pseudometrics:

$$\hat{\rho}\big((x_1,y_1,a_1),(x_2,y_2,a_2)\big) = \max\left\{\hat{d}(x_1,x_2), \hat{d}(y_1,y_2), |a_1 - a_2|\right\}$$

and

$$\bar{\rho}\big((x_1,y_1,a_1),(x_2,y_2,a_2)\big) = \max\left\{\bar{d}(x_1,x_2), \bar{d}(y_1,y_2), |a_1 - a_2|\right\}.$$

Obviously $\hat{\rho} \le \bar{\rho}$, hence $\hat{\rho}_H \le \bar{\rho}_H$, which for all $d_1, d_2 \in \hat{d}{\downarrow} \subset \bar{d}{\downarrow}$ implies

$$D_H^{\hat{d}}(d_1, d_2) = \hat{\rho}_H(\operatorname{sub} d_1, \operatorname{sub} d_2) \le \bar{\rho}_H(\operatorname{sub} d_1, \operatorname{sub} d_2) = D_H^{\bar{d}}(d_1, d_2),$$

hence the more is $\hat{d}$, the more is the metric $D_H^{\hat{d}}$.

Recall that a net is a collection $(x_\alpha)_{\alpha \in (A, \preceq)}$ of elements x_α indexed with a directed poset $(A, \preceq)$. If all x_α are elements of a poset itself, we call the net $(x_\alpha)_{\alpha \in (A, \preceq)}$ non-decreasing if $\alpha \preceq \beta$ in A implies $x_\alpha \le x_\beta$.

Definition 3. *We say that a non-decreasing net $(d_\alpha)_{\alpha \in (A, \preceq)}$ in $CPsU(X)$ grinds a compact pseudoultrametric d on X if for all $x_0 \in X$ and $r > 0$ the diameters with respect to d of the balls $B_r^\alpha(x_0) = \{x \in X : d_\alpha(x, x_0) < r\}$ converge to 0.*

It is equivalent to the statement that for all $x_0 \in X$, $r > 0$, $\varepsilon > 0$ there is $\alpha \in A$ such that $d(x, x_0) < \varepsilon$ for all $x \in X$ such that $d_\alpha(x, x_0) < r$. It is easy to observe that this implies $\lim_{\alpha \in (A, \preceq)} \sup d_\alpha = +\infty$.

Proposition 3. *Let $(d_\alpha)_{\alpha \in (A, \preceq)}$ be a non-decreasing net in $CPsU(X)$ that grinds both $d, d' \in CPsU(X)$, and assume that $\beta \in A$ exists such that $d \le d_\beta$, $d' \le d_\beta$. Then*

$$\lim_{\alpha \in (A, \preceq)} D_H^{d_\alpha}(d, d') = D_u(d, d').$$

Here we ignore the "missing" elements of the net for $\alpha \not\succeq \beta$ when taking the latter limit.

Proof. Without loss of generality we can assume that $d \le d_\alpha$, $d' \le d_\alpha$ for all $\alpha \in A$. The net $\left(D_H^{d_\alpha}(d, d')\right)_{\alpha \in (A, \preceq)}$ in $\mathbb{R}$ is non-decreasing and bounded from above by $D_u(d, d')$, hence has a limit, which we denote by C. Clearly $C \le D_u(d, d')$, and what is left to show is that $C < D_u(d, d')$ is impossible.

Assuming the contrary, we obtain the existence of $x, y \in X$ such that

$$\left|d(x, y) - d'(x, y)\right| > C \ge D_H^{d_\alpha}(d, d')$$

for all $\alpha \in A$. Let, e.g., $d(x, y) = a$, $d'(x, y) = b$, $b - a > C$. Then $(x, y, b) \in \operatorname{sub} d_2 \setminus \operatorname{sub} d$, and, for the pseudometric ρ_α on $X \times X \times [0, \sup d_\alpha]$ determined with d_α in the way described above, $\rho_\alpha((x, y, b), \operatorname{sub} d) \le C$ for all $\alpha \in A$. This implies the existence of $(x_\alpha, y_\alpha, a_\alpha) \in \operatorname{sub} d$ such that

$$\max\left\{d_\alpha(x_\alpha, x), d_\alpha(y_\alpha, y), |a_\alpha - b|\right\} \le C,$$

hence $d_\alpha(x_\alpha, x) \le C$, $d_\alpha(y_\alpha, y) \le C$, $a_\alpha \ge b - C$, therefore $d(x_\alpha, y_\alpha) \ge d'(x, y) - C = b - C$.

On the other hand, there is $\alpha \in A$ such that for all $x', y' \in X$ the inequalities $d_\alpha(x, x') < C$, $d_\alpha(y, y') < C$ imply $d(x, x') < b - C$, $d(y, y') < b - C$. Hence, $d(x_\alpha, x) \le b - C$, $d(y_\alpha, y) \le b - C$, and, taking into account $d(x_\alpha, y_\alpha) \ge b - C$ and the triangle inequality, we obtain $d(x, y) = a \ge b - C$, which contradicts to the assumption $b - a > C$. This completes the proof. □

Remark 4. *Probably the easiest way to obtain a net that satisfy the above requirements for given d, d' is to choose an arbitrary $\hat{d} \ge d$, $\hat{d} \ge d'$ (e.g., $\sup\{\hat{d}, d'\}$), and to put $A = \mathbb{N}$, $d_n(x, y) = n \cdot \hat{d}(x, y)$.*

5 Conclusions and future work

We have shown that, for a fixed compact pseudoultrametric $\hat{d}$ on a set X, the set $\hat{d}\downarrow$ of all compact pseudoultrametric on X less or equal to $\hat{d}$ can be metrized in two distinct but topologically equivalent ways, and with the induced topology it is a Lawson compact Hausdorff upper semilattice. By Fundamental Theorem on Compact Semilattices [1] this topology is uniquely determined with the partial order on $\hat{d}\downarrow$, namely with the "way above" relation. We have already described the dual "way below" relation in [6], and in our upcoming publication a similar characterization of "way above" will be presented. Then it will be shown that $\hat{d}\downarrow$ is a (rather rare) example of a bicontinuous lattice [1], and its properties will be further investigated. We are also going to consider subspaces of compact pseudoultrametrics with the values in a closed subset of $[0, +\infty)$.

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Никорович С., Никифорчин О. *Метрика і топологія на частково упорядкованій множині компактних псевдоультраметрик* // Карпатські матем. публ. — 2023. — Т.15, №2. — С. 321–330.

Двома способами ми впроваджуємо метрики на множині всіх псевдоультраметрик, що не перевищують даної компактної псевдоультраметрики на деякій фіксованій множині, і доводимо, що отримані метрики компактні і топологічно еквівалентні. Для цього ми характеризуємо множини, які є підграфіками вказаних псевдоультраметрик, і до їх сукупності застосовуємо метрику Гаусдорфа. Доведено, що метрика рівномірної збіжності є граничним випадком метрик, означених через підграфіки. Показано, що множина всіх псевдоультраметрик, що не перевищують даної компактної псевдоультраметрики, з індукованою топологією є лоусоновою компактною гаусдорфовою верхньою напівґраткою.

Ключові слова і фрази: псевдоультраметрика, метризація, компакт, підграфік.