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Determinant identities for the Catalan, Motzkin and Schröder numbers

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Abstract

In this paper, we find formulas for the determinants of several Hessenberg matrices whose nonzero entries are derived from the Catalan, Motzkin and Schröder number sequences. By a generalization of Trudi's formula, we obtain equivalent multi-sum identities involving sums of products of terms from these sequences. We supply both algebraic and combinatorial proofs of our results. For the latter, we draw upon the combinatorial interpretations of the Catalan, Motzkin and Schröder numbers as enumerators of certain classes of first-quadrant lattice paths. As a consequence of our results and the arguments used to establish them, one obtains both new formulas and combinatorial interpretations for some well-known integer sequences, including the central binomial coefficients, grand Motzkin numbers, Delannoy numbers and several entries from the On-Line Encyclopedia of Integer Sequences.

Keywords: Hessenberg matrix, Catalan number, Motzkin number, Schröder number, generalized Trudi's formula, lattice path, combinatorial proof.

Math. Subj. Class.: 05A19, 11C20, 15B05

1 Introduction

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n-th Catalan number for $n \geq 0$. Let M_n be the n-th Motzkin number, which is given recursively by

$$M_n = \frac{2n+1}{n+2}M_{n-1} + \frac{3(n-1)}{n+2}M_{n-2}, \qquad n \geq 2,$$

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with initial values $M_0 = M_1 = 1$. Let s_n be the n-th *small* Schröder number, which satisfies the recurrence

$$s_n = \frac{3(2n-1)}{n+1}s_{n-1} - \frac{n-2}{n+1}s_{n-2}, \qquad n \geq 2,$$

with $s_0 = s_1 = 1$. Let S_n denote the n-th *large* Schröder number given by $S_n = 2s_n$ for $n \geq 1$, with $S_0 = 1$. The first several terms of the sequences M_n and s_n for $n \geq 0$ are as follows:

$$\{M_n\}_{n\geq 0} = \{1, 1, 2, 4, 9, 21, 51, 127, 323, 835, 2188, \ldots\}$$

and

$$\{s_n\}_{n\geq 0} = \{1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots\}.$$

See entries A000108, A001006, A001003 and A006318 in [23] for further information on the sequences C_n, M_n, s_n and S_n, respectively.

We will make use of in our proofs the formulas for the (ordinary) generating functions given by

$$\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}, \qquad \sum_{n\geq 0} M_n x^n = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2},$$
$$\sum_{n\geq 0} s_n x^n = \frac{1+x-\sqrt{1-6x+x^2}}{4x}, \qquad \sum_{n\geq 0} S_n x^n = \frac{1-x-\sqrt{1-6x+x^2}}{2x}.$$

The sequences C_n, M_n, s_n and S_n (and their generalizations) are enumerators of important classes of first-quadrant lattice paths and several other related structures and as such arise in various contexts within algebraic and enumerative combinatorics (see, e.g., [1, 3, 5–8, 16–18, 22, 24, 25] and references contained therein). Here, we will be interested in some new combinatorial aspects of these sequences as it pertains to their occurrence in certain Hessenberg matrices.

Relations involving determinants of matrices with entries from these sequences and their generalizations have been an object of ongoing research. For example, Aigner [1] showed that the determinant of the Hankel matrix $\left(M_{i+j}\right)_{i,j=0}^{n-1}$ equals 1 for all $n \geq 1$, whereas $\det\left(M_{i+j+1}\right)_{i,j=0}^{n-1}$ is periodic with repeating block $1, 0, -1, -1, 0, 1$. These results were extended by Cameron and Yip [4] who used combinatorial methods to evaluate Hankel determinants for a sequence of sums of consecutive t-Motzkin numbers. The comparable determinant formulas for Schröder numbers are given by $\det\left(S_{i+j}\right)_{i,j=0}^{n-1} = 2^{\binom{n}{2}}$ and $\det\left(S_{i+j+1}\right)_{i,j=0}^{n-1} = 2^{\binom{n+1}{2}}$. Extensions of these results in terms of various families of Catalan-like sequences have been found; see, e.g., [9, 20] and references contained therein.

In [7], Deutsch obtained the following Catalan number determinant formula:

$$t_n = (-1)^{n-1}\begin{vmatrix} C_0 & 1 & 0 & \cdots & 0 \\ C_1 & C_0 & 1 & \cdots & 0 \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ C_{n-2} & C_{n-3} & C_{n-4} & \cdots & 1 \\ C_{n-1} & C_{n-2} & C_{n-3} & \cdots & C_0 \end{vmatrix}, \qquad n \geq 1,$$

where t_n denotes the n-th Fine number, a result which was later generalized by the authors (see [10, Formula 2.21]). In [14], some determinant formulas were proven for Toeplitz–Hessenberg matrices with Motzkin and Riordan number entries and direct counting arguments were provided. See [10–12] for comparable results involving the Catalan, Horadam and tribonacci numbers and also [2] for further results concerning permanents in the Horadam case. The determinant formulas involving the preceding sequences may also be rewritten equivalently as identities involving sums of products of terms from the sequence in question with multinomial coefficients. Finally, in [13], analogues of the results in the current paper were found for subsequences of the Fibonacci and Lucas numbers wherein the first column of a Toeplitz–Hessenberg matrix is allowed to contain terms from a different sequence.

The organization of this paper is as follows. In the next section, we prove by algebraic methods determinant identities for a certain class of Hessenberg matrices whose entries are derived from the Catalan, Motzkin and Schröder number sequences. These results may be expressed equivalently as identities involving sums of products (with multinomial coefficients) of terms from these sequences, by a generalization of Trudi's formula. In the third section, we provide combinatorial proofs of the aforementioned determinant identities, where we make use of the definition of a determinant as a signed sum over the set of permutations of $[n] = \{1, \dots, n\}$. We draw upon various counting techniques, including sign-reversing involutions, direct enumeration, recurrences and bijections between the relevant combinatorial structures (mostly classes of lattice paths where certain steps have been designated in some way). As a consequence of our results, we obtain new formulas in terms of determinants of several well-known integer sequences. Moreover, by extending our arguments, one can provide combinatorial explanations of a couple of related formulas involving the grand Motzkin numbers.

2 Catalan, Motzkin and Schröder determinant identities

An $n \times n$ matrix $H_n = (h_{ij})$ is said to be (*lower*) *Hessenberg* if its entries above the superdiagonal are all zero, i.e.,

$$H_n = \begin{pmatrix} h_{11} & h_{12} & 0 & \cdots & 0 \\ h_{21} & h_{22} & h_{23} & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ h_{n-1,1} & h_{n-1,2} & h_{n-1,3} & \cdots & h_{n-1,n} \\ h_{n1} & h_{n2} & h_{n3} & \cdots & h_{nn} \end{pmatrix}.$$

Consider the Hessenberg matrix of the form

$$K_n := K_n(a_0; a_1, \dots, a_n, k_1, \dots, k_n) = \begin{pmatrix} k_1 a_1 & a_0 & 0 & \cdots & 0 \\ k_2 a_2 & a_1 & a_0 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ k_{n-1} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ k_n a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}, \quad (2.1)$$

where $a_0 \neq 0$ and $a_i \neq 0$ for at least one $i > 0$. Such matrices have been studied, for example, in [15, 28]. Note that a *Toeplitz–Hessenberg* matrix is one that corresponds to the case $k_i = 1$ for all $i \geq 1$ in (2.1).

Expanding the determinant, first along the final row and then repeatedly along the final column, we obtain the recurrence

$$\det(K_n) = (-a_0)^{n-1}k_n a_n + \sum_{i=1}^{n-1}(-a_0)^{i-1}a_i \det(K_{n-i}), \qquad n \geq 2, \tag{2.2}$$

where $\det(K_1) = k_1 a_1$.

The case when $k_i = i$ for all i in (2.1) turns out to be of interest, especially from a combinatorial perspective. We investigate below several particular cases of K_n where $k_i = i$ and will denote such matrices by A_n. That is, A_n is given by

$$A_n := A_n(a_0; a_1, \dots, a_n) = \begin{pmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ 2a_2 & a_1 & a_0 & \cdots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ (n-1)a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0 \\ na_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{pmatrix}, \tag{2.3}$$

and our primary focus will be on cases of A_n when $a_0 = \pm 1$.

There is the following multinomial expansion of $\det(K_n)$, which can be shown by an inductive argument using (2.2).

Lemma 2.1 ([15]). *Let n be a positive integer. Then*

$$\det(K_n) = \sum_{\widetilde{s}=n} \frac{(-a_0)^{n-|s|}}{|s|} \binom{|s|}{s_1, \dots, s_n} \left(\sum_{i=1}^{n} s_i k_i\right) a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}, \tag{2.4}$$

where $\binom{|s|}{s_1,\dots,s_n} = \frac{|s|!}{s_1!\cdots s_n!}$, $|s| = s_1 + \cdots + s_n$, $\widetilde{s} = s_1 + 2s_2 + \cdots + ns_n$ and the sum is over all n-tuples $s = (s_1, \dots, s_n)$ of non-negative integers such that $\widetilde{s} = n$. In particular, when $k_i = i$ for all i, we have

$$\det(A_n) = n \sum_{\widetilde{s}=n} \frac{(-a_0)^{n-|s|}}{|s|} \binom{|s|}{s_1, \dots, s_n} a_1^{s_1} a_2^{s_2} \cdots a_n^{s_n}. \tag{2.5}$$

Remark 2.2. Note that [28, Equation 3] provides an alternative way of expressing (2.4) in terms of triangular matrices (for more details concerning the calculus of triangular matrices and its applications, see [27]). The case $k_i = 1$ of (2.4) is often called *Trudi's formula* (see, e.g., [19, Theorem 1] or [21, page 214]), and thus one may properly refer to (2.4) as a *generalized* Trudi's formula. Finally, note that (2.5) can be found in [21, page 228].

The next result provides a connection between the generating function for the sequence $\det(A_n)$ for $n \geq 1$ and that of the underlying sequence a_n.

Lemma 2.3. *Let $f(x) = \sum_{n\geq 1} \det(A_n)x^n$, where A_n is given by (2.3), and*

$$h(x) = -\frac{1}{a_0}\sum_{i\geq 1} a_i x^i.$$

Then we have

$$f(x) = \frac{-a_0 x h'(-a_0 x)}{1 - h(-a_0 x)}. \tag{2.6}$$

Proof. Consider the bivariate generating function

$$g(x,y)=\sum_{n\geq 1}x^n\sum_{\widetilde{s}=n}\binom{|s|}{s_1,\ldots,s_n}\left(\frac{-a_1}{a_0}\right)^{s_1}\cdots\left(\frac{-a_n}{a_0}\right)^{s_n}y^{|s|-1}.$$

Comparing coefficients of y, we have $g(x,y)=h(x)+h^2(x)y+h^3(x)y^2+\cdots$, and hence $g(x,y)=\frac{h(x)}{1-yh(x)}$. Note that

$$\int_0^1 g(x,y)dy=\big[-\ln(1-yh(x))\big]_{y=0}^{y=1}=-\ln(1-h(x)).$$

Thus, by (2.5), we have

$$\begin{aligned}f(x)=\sum_{n\geq 1}\det(A_n)x^n&=\sum_{n\geq 1}n(-a_0x)^n\sum_{\widetilde{s}=n}\frac{1}{|s|}\binom{|s|}{s_1,\ldots,s_n}\left(\frac{-a_1}{a_0}\right)^{s_1}\cdots\left(\frac{-a_n}{a_0}\right)^{s_n}\\&=z\frac{d}{dz}\big(-\ln(1-h(z))\big)\big|_{z=-a_0x}=\frac{-a_0xh'(-a_0x)}{1-h(-a_0x)},\end{aligned}$$

as desired. □

For the sake of brevity, we will write $D_\pm(a_1,a_2,\ldots,a_n)$ in place of $\det\left(A_n(\pm 1;a_1,a_2,\ldots,a_n)\right)$. We have the following determinant identity formulas involving the Catalan numbers.

Theorem 2.4. *If* $n\geq 1$, *then*

$$D_+(C_0,C_1,\ldots,C_{n-1})=\sum_{k=0}^{n-1}(-1)^k\binom{n+k-1}{k}, \tag{2.7}$$

$$D_-(C_0,C_1,\ldots,C_{n-1})=\binom{2n-1}{n}, \tag{2.8}$$

$$D_+(C_1,C_2,\ldots,C_n)=(-1)^{n-1}\binom{2n-1}{n}, \tag{2.9}$$

$$D_-(C_1,C_2,\ldots,C_n)=2^{2n-1}-\binom{2n-1}{n}, \tag{2.10}$$

$$D_+(C_2,C_3,\ldots,C_{n+1})=(-1)^{n-1}\binom{2n}{n}. \tag{2.11}$$

Proof. We find, more generally, a formula for the generating function of $\det(A_n)$ when $a_i=C_{i+m}$ for $i\geq 1$, where a_0 and m are arbitrary, from which (2.7) – (2.11) will follow as special cases. First suppose $m\geq 0$. Then $h(x)$ as defined in Lemma 2.3 is given in this case by

$$h(x)=-\frac{1}{a_0}\sum_{i\geq 1}C_{i+m}x^i=-\frac{1}{a_0x^m}\sum_{i\geq m+1}C_ix^i=-\frac{1}{a_0x^m}\left(C(x)-\sum_{i=0}^{m}C_ix^i\right),$$

where $C(x)=\sum_{i\geq 0}C_ix^i$. Hence,

$$h(-a_0x)=\frac{1}{2(-a_0)^{m+2}x^{m+1}}\left(1+2a_0x\sum_{i=0}^{m}C_i(-a_0x)^i-\sqrt{1+4a_0x}\right)$$

and

$$\begin{aligned} h'(-a_0x) &= \frac{mC(z) - zC'(z) + \sum\limits_{i=0}^{m}(i-m)C_iz^i}{a_0z^{m+1}}\Bigg|_{z=-a_0x} \\ &= \frac{-m\left(\frac{1-\sqrt{1+4a_0x}}{2a_0x}\right) + \frac{1+2a_0x-\sqrt{1+4a_0x}}{2a_0x\sqrt{1+4a_0x}} + \sum\limits_{i=0}^{m}(i-m)C_i(-a_0x)^i}{a_0^{m+2}(-x)^{m+1}} \\ &= \frac{m+1+(4m+2)a_0x + \left(2a_0x\sum\limits_{i=0}^{m}(i-m)C_i(-a_0x)^i - m - 1\right)\sqrt{1+4a_0x}}{2(-a_0)^{m+3}x^{m+2}\sqrt{1+4a_0x}}. \end{aligned}$$

By (2.6), we have that $f(x) = \sum_{n\geq 1}\det(A_n(a_0; C_{m+1}, \ldots, C_{m+n}))x^n$ is given by

$$\begin{aligned} f(x) &= \frac{-a_0xh'(-a_0x)}{1-h(-a_0x)} \\ &= \frac{\frac{m+1+(4m+2)a_0x+\left(2a_0x\sum\limits_{i=0}^{m}(i-m)C_i(-a_0x)^i - m-1\right)\sqrt{1+4a_0x}}{2(-a_0)^{m+2}x^{m+1}\sqrt{1+4a_0x}}}{1 - \frac{1}{2(-a_0)^{m+2}x^{m+1}}\left(1+2a_0x\sum\limits_{i=0}^{m}C_i(-a_0x)^i - \sqrt{1+4a_0x}\right)} \\ &= \frac{m+1+(4m+2)a_0x + \left(2a_0x\sum\limits_{i=0}^{m}(i-m)C_i(-a_0x)^i - m - 1\right)\sqrt{1+4a_0x}}{1+4a_0x + \left(2(-a_0)^{m+2}x^{m+1} - 1 - 2a_0x\sum\limits_{i=0}^{m}C_i(-a_0x)^i\right)\sqrt{1+4a_0x}}. \end{aligned} \tag{2.12}$$

Taking $a_0 = 1$, $m = 0$ in (2.12) gives

$$\begin{aligned} \sum_{n\geq 1} D_+(C_1, \ldots, C_n)x^n &= \frac{1+2x-\sqrt{1+4x}}{1+4x-\sqrt{1+4x}} = \frac{\sqrt{1+4x}-1}{2\sqrt{1+4x}} \\ &= \sum_{n\geq 1}(-1)^{n-1}\binom{2n-1}{n}x^n, \end{aligned}$$

which yields (2.9), where we have used [26, Equation 2.5.11] in the third equality, together with the fact $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$ for $n \geq 1$.

Taking $a_0 = -1$, $m = 0$ in (2.12) gives

$$\begin{aligned} \sum_{n\geq 1} D_-(C_1, \ldots, C_n)x^n &= \frac{1-2x-\sqrt{1-4x}}{(1-4x)\left(1-\sqrt{1-4x}\right)} = \frac{2x}{1-4x} - \frac{1-\sqrt{1-4x}}{2\sqrt{1-4x}} \\ &= \sum_{n\geq 1}\left(2^{2n-1} - \binom{2n-1}{n}\right)x^n, \end{aligned}$$

which yields (2.10).

Taking $a_0 = m = 1$ in (2.12) gives

$$\sum_{n\geq 1} D_+(C_2,\dots,C_{n+1})x^n = \frac{2+6x-2(1+x)\sqrt{1+4x}}{1+4x-(1+2x)\sqrt{1+4x}} = \frac{\sqrt{1+4x}-1}{\sqrt{1+4x}}$$
$$= \sum_{n\geq 1}(-1)^{n-1}\binom{2n}{n}x^n,$$

which yields (2.11).

Now let $a_i = C_{i-m}$, where we take $C_j = 0$ if $j < 0$ and $m \geq 1$ is fixed. In this case, we have

$$h(x) = -\frac{1}{a_0}\sum_{i\geq 1} C_{i-m}x^i = -\frac{x^m C(x)}{a_0},$$

so that

$$h(-a_0x) = \frac{(-a_0)^{m-2}x^{m-1}\left(1-\sqrt{1+4a_0x}\right)}{2}$$

and

$$h'(-a_0x) = \frac{(-a_0x)^{m-2}\left(m-1+(4m-2)a_0x-(m-1)\sqrt{1+4a_0x}\right)}{2a_0\sqrt{1+4a_0x}}.$$

Hence, by (2.6), we have

$$\begin{aligned} f(x) &= \sum_{n\geq 1}\det(A_n(a_0;C_{1-m},\dots,C_{n-m}))x^n \\ &= \frac{\frac{a_0^{m-2}(-x)^{m-1}\left(m-1+(4m-2)a_0x-(m-1)\sqrt{1+4a_0x}\right)}{2\sqrt{1+4a_0x}}}{1+\frac{a_0^{m-2}(-x)^{m-1}\left(1-\sqrt{1+4a_0x}\right)}{2}} \\ &= \frac{a_0^{m-2}(-x)^{m-1}\left(m-1+(4m-2)a_0x-(m-1)\sqrt{1+4a_0x}\right)}{\left(2+a_0^{m-2}(-x)^{m-1}\right)\sqrt{1+4a_0x}-a_0^{m-2}(-x)^{m-1}(1+4a_0x)}. \end{aligned} \tag{2.13}$$

Taking $a_0 = m = 1$ in (2.13) gives

$$\sum_{n\geq 1} D_+(C_0,\dots,C_{n-1})x^n = \frac{2x}{3\sqrt{1+4x}-1-4x} = \frac{x\left(3+\sqrt{1+4x}\right)}{2(2-x)\sqrt{1+4x}}$$
$$= \sum_{n\geq 1}x^n\sum_{k=0}^{n-1}(-1)^k\binom{n+k-1}{k},$$

which yields (2.7), where the third equality can be shown by using the recurrence for binomial coefficients to determine an equation satisfied by the generating function of the sequence.

Taking $a_0 = -1$, $m = 1$ in (2.13) gives

$$\sum_{n\geq 1} D_-(C_0,\dots,C_{n-1})x^n = \frac{2x}{1-4x+\sqrt{1-4x}} = \frac{1-\sqrt{1-4x}}{2\sqrt{1-4x}} = \sum_{n\geq 1}\binom{2n-1}{n}x^n,$$

which yields (2.8) and completes the proof. □

By Theorem 2.4 and (2.5), we have the following Catalan number identities.

Corollary 2.5. *If* $n \geq 1$, *then*

$$n \sum_{\tilde{s}=n} \frac{(-1)^{|s|}}{|s|} \binom{|s|}{s_1, \ldots, s_n} C_0^{s_1} C_1^{s_2} \cdots C_{n-1}^{s_n} = \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n+k-1}{k}, \tag{2.14}$$

$$n \sum_{\tilde{s}=n} \frac{1}{|s|} \binom{|s|}{s_1, \ldots, s_n} C_0^{s_1} C_1^{s_2} \cdots C_{n-1}^{s_n} = \binom{2n-1}{n}, \tag{2.15}$$

$$n \sum_{\tilde{s}=n} \frac{(-1)^{|s|-1}}{|s|} \binom{|s|}{s_1, \ldots, s_n} C_1^{s_1} C_2^{s_2} \cdots C_n^{s_n} = \binom{2n-1}{n}, \tag{2.16}$$

$$n \sum_{\tilde{s}=n} \frac{1}{|s|} \binom{|s|}{s_1, \ldots, s_n} C_1^{s_1} C_2^{s_2} \cdots C_n^{s_n} = 2^{2n-1} - \binom{2n-1}{n}, \tag{2.17}$$

$$n \sum_{\tilde{s}=n} \frac{(-1)^{|s|-1}}{|s|} \binom{|s|}{s_1, \ldots, s_n} C_2^{s_1} C_3^{s_2} \cdots C_{n+1}^{s_n} = \binom{2n}{n}. \tag{2.18}$$

Let G_n denote the n-th grand Motzkin and D_n the n-th central Delannoy number for $n \geq 0$; see $A002426[n]$ and $A001850[n]$, respectively. We have the following analogous results involving Motzkin and Schröder number determinants.

Theorem 2.6. *If* $n \geq 1$, *then*

$$D_+(M_0, M_1, \ldots, M_{n-1}) = (-1)^{n-1} A113682[n-1], \tag{2.19}$$

$$D_-(M_0, M_1, \ldots, M_{n-1}) = A055217[n-1], \tag{2.20}$$

$$D_+(M_1, M_2, \ldots, M_n) = (-1)^{n-1} G_n. \tag{2.21}$$

Theorem 2.7. *If* $n \geq 1$, *then*

$$D_-(s_0, s_1, \ldots, s_{n-1}) = D_{n-1}, \tag{2.22}$$

$$D_+(s_1, s_2, \ldots, s_n) = (-1)^{n-1} \left((n+1)S_n - D_n\right), \tag{2.23}$$

$$D_-(s_1, s_2, \ldots, s_n) = A271197[n-1], \tag{2.24}$$

$$D_+(S_0, S_1, \ldots, S_{n-1}) = (-1)^{n-1} D_{n-1}, \tag{2.25}$$

$$D_-(S_0, S_1, \ldots, S_{n-1}) = A002002[n], \tag{2.26}$$

$$D_+(S_1, S_2, \ldots, S_n) = (-1)^{n-1} A002003[n]. \tag{2.27}$$

Remark 2.8. Proofs comparable to the one presented above for Theorem 2.4 may be given for Theorems 2.6 and 2.7, upon making use of the Motzkin and Schröder number generating function formulas. Analogues of (2.14) – (2.18) for Motzkin and Schröder numbers may also be stated, which we omit. Rounding out the results from Theorems 2.6 and 2.7, we observe that the sequences corresponding to $D_-(M_1, \ldots, M_n)$, $D_+(s_0, \ldots, s_{n-1})$ and $D_-(S_1, \ldots, S_n)$ do not occur in the OEIS nor did we find simple closed form expressions in these cases. One can find however recursive formulas for these sequences based on combinatorial arguments comparable to those given for $D_-(M_0, \ldots, M_{n-1})$ and $D_-(s_1, \ldots, s_n)$, see proofs of (2.20) and (2.24) below.

3 Combinatorial proofs

In this section, we provide combinatorial proofs of the formulas in Theorems 2.4, 2.6 and 2.7 by making use of the definition of a determinant as

$$\det(A) = \sum_{\sigma \in \mathcal{S}_n} (-1)^{\operatorname{sgn}(\sigma)} a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}, \tag{3.1}$$

where $A = (a_{i,j})$ and $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ. Note that if A is Hessenberg, then one can restrict the sum in (3.1) to include only those permutations σ in which every cycle of σ consists of a set of consecutive integers in increasing order (where the smallest element is first in each cycle), as all other permutations are seen to contribute zero towards the sum.

Upon arranging cycles in increasing order of their smallest elements and identifying the (sequence of) various cycle lengths as parts, such σ are seen to be synonymous with compositions of n. Thus, one may regard the sum in (3.1) when A is of the form in (2.1) as being over the set of compositions of n that are weighted as follows. An initial part of size i receives weight $k_i a_i$, with all other parts of size i assigned the weight a_i. If $a_0 = 1$, then the weight of a composition is defined as the product of the weights of its parts, with its sign given by $(-1)^{n-m}$, where m denotes the number of parts. If $a_0 = -1$, then each composition is weighted just as described above, but where there is now no sign (as the sign of the associated permutation σ is cancelled out by the product of the superdiagonal -1's in the term corresponding to σ in (3.1)).

Suppose now that a_i, k_i are non-negative integers, with a_i enumerating a discrete structure Ω_i for each $i \geq 1$ whose members have size i in some sense. In these cases, consider overlaying every part of size i in a composition with a member of Ω_i, where a part of size i that starts a composition is designated in one of k_i ways. That is, if $\sigma = (\sigma_1, \ldots, \sigma_m)$ with $\sum_{i \geq 1} \sigma_i = n$ and $\sigma_i \geq 1$ for all i, then we overlay each σ_i with $\lambda_i \in \Omega_{\sigma_i}$, where λ_1 is designated in one of k_{σ_1} ways. Next, we concatenate the λ_i to obtain $\lambda = \lambda_1 \cdots \lambda_m$. Finally, we mark the final point of each component λ_i within λ, which has the effect of tracking the sequence of part sizes of σ while dividing λ into m identifiable sections. Let Υ_n denote the set of all λ (marked and designated as described) which arise in this manner as σ ranges over all compositions of n. Since each part σ_i for $i > 1$ within a composition $\sigma = (\sigma_1, \ldots, \sigma_m)$ is weighted by $a_{\sigma_i} = |\Omega_{\sigma_i}|$ (with σ_1 weighted by $k_{\sigma_1} a_{\sigma_1}$), the overall weight of σ, which is given by the product of the weights of the σ_i for $1 \leq i \leq m$, yields the cardinality of the subset of Υ_n for which there are exactly m (marked) components wherein the sequence of component sizes coincides with σ and the first component is designated in one of k_{σ_1} ways.

Thus, when $a_0 = -1$, summing over all compositions σ of n as in (3.1) implies

$$\det(K_n(-1; a_1, \ldots, a_n, k_1, \ldots, k_n)) = |\Upsilon_n|.$$

So given a sequence a_i of cardinalities of the components Ω_i and a non-negative sequence k_i, the task of computing $\det(K_n(-1; a_1, \ldots, a_n, k_1, \ldots, k_n))$ reduces to finding $|\Upsilon_n|$. Let $\Upsilon_{n,m}$ for $1 \leq m \leq n$ denote the subset of Υ_n whose members have exactly m marked sections. Note that members of $\Upsilon_{n,m}$ arise from the compositions of n with m parts in the sum (3.1), and hence they each receive a sign of $(-1)^{n-m}$ in the case when $a_0 = 1$. Thus, upon considering all possible m, we have that

$$\det(K_n) = \det(K_n(1; a_1, \ldots, a_n, k_1, \ldots, k_n))$$

gives the sum of signs of all members of $\Upsilon_n = \cup_{m=1}^{n} \Upsilon_{n,m}$.

We will apply the preceding strategy in several cases where the constituent components derived from the Ω_i correspond to various classes of lattice paths. By a *return* within a lattice path, we mean a step of any kind terminating on the x-axis. In forming a member of Υ_n from the Ω_i consisting of lattice paths, certain returns are to be marked, including always the final return, and a step to the left of (possibly including) the first marked return is to be designated (indicated by a particular step being circled in the proofs below). Further, one may impose various restrictions on the paths themselves which belong to the Ω_i, on the types of returns that can be marked and on the kind of steps that can be designated. Here, the Ω_i will be derived from various classes of Catalan, Motzkin and Schröder paths and we seek to determine the cardinality or find the sum of signs of the resulting set Υ_n in each case.

Throughout this section, let $\sigma(T)$ denote the sum of signs of a signed structure T. When $a_0 = 1$, to simplify the computation of $\sigma(\Upsilon_n)$, one can define a sign-reversing involution ψ on Υ_n as follows. Suppose that there is a non-terminal return of the type that can be marked (which we will refer to as a *changeable* return) occurring somewhere to the right of the circled step within $\lambda \in \Upsilon_n$. We then either mark the rightmost changeable return or remove the marking from it, leaving the rest of λ unchanged, to obtain $\psi(\lambda)$. Then ψ reverses the sign and it fails to be defined on the subset Υ'_n of Υ_n whose members contain only a single marked return (i.e., the final return) and where there is no unmarked changeable return occurring to the right of (and including) the circled step. Since $\Upsilon'_n \subseteq \Upsilon_{n,1}$, each member of Υ'_n has sign $(-1)^{n-1}$. It follows that $\det(K_n)$ is given by $(-1)^{n-1}|\Upsilon'_n|$. In this way, the task of computing a determinant is reduced to one of finding the cardinality of a certain class of restricted lattice paths.

There is another involution ϕ on Υ_n which provides a second expression for $\det(K_n)$. Suppose $\lambda \in \Upsilon_n$ has at least two marked returns with at least one changeable return occurring between the first and the last. Consider the leftmost changeable return of λ intermediate between the first and last marked returns and either mark it or remove the marking from it. Let $\phi(\lambda)$ denote the resulting member of Υ_n. Note that ϕ reverses the sign and does not result in a lattice path that fails to belong to Υ_n since the position of the first marked return is unaltered. The involution ϕ is not defined on $\Upsilon_{n,1}$ or on the subset $\Upsilon^*_{n,2}$ of $\Upsilon_{n,2}$ in which there are no (unmarked) changeable returns occurring between the two marked returns. This implies the formula

$$\det(K_n) = (-1)^{n-1}|\Upsilon_{n,1}| + (-1)^{n-2}|\Upsilon^*_{n,2}| = (-1)^{n-1}\left(|\Upsilon_{n,1}| - |\Upsilon^*_{n,2}|\right).$$

Though it possible to extend ϕ (by applying ψ) to $(\Upsilon_{n,1} - \Upsilon'_n) \cup \Upsilon^*_{n,2}$ and reach the same expression obtained using ψ above, there are cases in the proofs below where it is more convenient to enumerate the set $\Upsilon_{n,1} \cup \Upsilon^*_{n,2}$ than it is to enumerate Υ'_n.

We now introduce notation and recall some terms that will be used in the following proofs. Let $\mathcal{L}(n,m)$ denote the set of all lattice paths from $(0,0)$ to $(n+m, n-m)$ consisting of n up steps $u = (1,1)$ and m down steps $d = (1,-1)$ and let $\mathcal{L}(n) = \cup_{i=0}^{n}\mathcal{L}(i, n-i)$. Let $\mathcal{D}_n$ for $n \geq 1$ denote the subset of $\mathcal{L}(2n)$ whose members do not go below the x-axis at any point, with $\mathcal{D}_0$ consisting of the single empty path of length zero. Members of $\mathcal{D}_n$ are referred to as *Dyck* paths of semilength n and are enumerated by the Catalan number C_n. A *unit* within $\pi \in \mathcal{D}_n$ refers to a subpath of π between two adjacent returns (including the return step for the latter) or to the subpath of π consisting of all steps up to the first return. That is, a unit is a section of π of the form $u\pi'd$, where u starts and d

ends on the x-axis and π' is a possibly empty Dyck path. A member of $\mathcal{D}_n$ having only a single unit, i.e., one of the form $\pi = u\pi' d$, is said to be *primitive*.

We now provide combinatorial proofs of the formulas from Theorems 2.4, 2.6 and 2.7 above.

Proof of (2.7). Let $\mathcal{D}_{n,k}$ for $1 \leq k \leq n$ denote the subset of $\mathcal{D}_n$ whose members contain k units. Consider the set $\mathcal{D}^*_{n,k}$ of marked members of $\mathcal{D}_{n,k}$ wherein a u step belonging to the first unit is marked. Let $\mathcal{D}^*_n = \cup_{k=1}^n \mathcal{D}^*_{n,k}$ and define the sign of a member of $\mathcal{D}^*_{n,k}$ by $(-1)^{n-k}$. Then $D_+(C_0, \dots, C_{n-1})$ is seen to give $\sigma(\mathcal{D}^*_n)$, by the discussion above. To complete the proof, it suffices to demonstrate the equality

$$|\mathcal{D}^*_{n,k}| = \binom{2n-k-1}{n-1}, \qquad 1 \leq k \leq n, \tag{3.2}$$

for then we would have

$$D_+(C_0, \dots, C_{n-1}) = \sum_{k=1}^{n} (-1)^{n-k} \binom{2n-k-1}{n-1} = \sum_{k=0}^{n-1} (-1)^k \binom{n+k-1}{k},$$

as desired.

To show (3.2), first suppose $\rho \in \mathcal{D}^*_{n,k}$ is decomposed into units as $\rho = \rho^{(1)} \cdots \rho^{(k)}$, where $|\rho^{(j)}| = i_j$ for $1 \leq j \leq k$ and $|\pi|$ denotes the semilength of a Dyck path π. Note that there are C_{i_j-1} possibilities for $\rho^{(j)}$ if $j \geq 2$ and $i_1 C_{i_1-1} = \binom{2i_1-2}{i_1-1}$ possibilities for $\rho^{(1)}$ if $j = 1$. Since $\sum_{j=1}^k i_j = n$, it follows that $|\mathcal{D}^*_{n,k}|$ equals the coefficient of x^{n-k} in $\frac{1}{\sqrt{1-4x}} C(x)^{k-1}$. By [26, Eqn. 2.5.15], we have

$$\frac{1}{\sqrt{1-4x}} C(x)^{k-1} = \sum_{n \geq 0} \binom{2n+k-1}{n} x^n, \qquad k \geq 1,$$

which has a combinatorial proof in additional to an analytical one. Thus, we have

$$[x^{n-k}] \left(\frac{1}{\sqrt{1-4x}} C(x)^{k-1} \right) = \binom{2(n-k)+k-1}{n-k} = \binom{2n-k-1}{n-1},$$

which establishes (3.2) and completes the proof. □

Proof of (2.8) *and* (2.9). We first show (2.9). Let $\mathcal{A}_n$ denoted the set of marked, circled Dyck paths of semilength n in which (i) returns to the x-axis may be marked, (ii) the final return is always marked and (iii) some u prior to the first marked return is circled. Define the sign of $\lambda \in \mathcal{A}_n$ as $(-1)^{n-\mu(\lambda)}$, where $\mu(\lambda)$ denotes the number of marked returns. Then $D_+(C_1, \dots, C_n)$ is seen to give $\sigma(\mathcal{A}_n)$. We apply to $\mathcal{A}_n$ the general involution ψ defined above on Υ_n. Then the set $\mathcal{A}'_n$ of survivors of the involution in this case consists of those members of $\mathcal{A}_n$ in which only the final return is marked and the circled u lies in the last unit. Considering the semilength i of the last unit implies $|\mathcal{A}'_n| = \sum_{i=1}^n iC_{i-1}C_{n-i}$, with each member of $\mathcal{A}'_n$ having sign $(-1)^{n-1}$.

To establish (2.9), we show combinatorially the formula $\sum_{i=1}^n iC_{i-1}C_{n-i} = \binom{2n-1}{n}$ for $n \geq 1$. Upon subtracting $C_n = \sum_{i=1}^n C_{i-1}C_{n-i}$ from both sides, and re-indexing the resulting summation, we show equivalently

$$\sum_{i=1}^{n-1} iC_iC_{n-i-1} = \binom{2n-1}{n} - C_n, \qquad n \geq 2. \tag{3.3}$$

Let S be the set of ordered pairs (α, β), where $\alpha \in \mathcal{D}_i$ for some $i \in [n-1]$ wherein a u step within α is marked and $\beta \in \mathcal{D}_{n-i-1}$. Let T be the subset of $\mathcal{L}(n, n-1)$ whose members go below the x-axis at least once. Then $|S|$ and $|T|$ are seen to be given by the left and right sides of (3.3), respectively. So to show (3.3), it suffices to define a bijection f between S and T. We first transform α within $(\alpha, \beta) \in S$ as follows. Write $\alpha = \alpha' u \alpha''$, where the u indicated is the one that is marked. Given a lattice path π with u and d steps, let $\widetilde{\text{rev}}(\pi)$ be obtained from π by reading π backwards and replacing each u with d and each d with u. Let $g(\alpha) = \widetilde{\text{rev}}(\alpha') u \widetilde{\text{rev}}(\alpha'')$, where the indicated u is now no longer marked. Then $g(\alpha)$ is an arbitrary lattice path from $(0,0)$ to $(2i, 2)$, and the mapping g may be reversed by considering the minimum y-coordinate m of all the points on a path and then the rightmost point whose y-coordinate is m. Now define f by setting $f(\alpha, \beta) = \beta d g(\alpha)$. Note that the mapping f may be reversed by considering the x-coordinate $2(n-i)-1$, where $i \in [n-1]$, of the leftmost point whose y-coordinate is -1. Then it is seen that f provides the desired bijection between S and T, which establishes (3.3) and completes the proof of (2.9).

To show (2.8), consider overlaying each part of size j in a (weighted) composition appearing in the expansion of $D_-(C_0, \ldots, C_{n-1})$ with a primitive member of $\mathcal{D}_j$ for each $j \geq 1$. Then $D_-(C_0, \ldots, C_{n-1})$ equals the cardinality of the set of marked members of $\mathcal{D}_n$ wherein an up step belonging to the first unit is marked. Upon considering the semilength i of the first unit, one has that this cardinality is given by $\sum_{i=1}^{n} i C_{i-1} C_{n-i}$. By the combinatorial argument above, this was shown to equal $\binom{2n-1}{n}$, which implies (2.8). □

Proof of (2.10). Let $\mathcal{A}_n$ be as in the proof of (2.9). Then we have $D_-(C_1, \ldots, C_n) = |\mathcal{A}_n|$, so we need to show $|\mathcal{A}_n| = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right)$. Let $\mathcal{Y}_n$ denote the subset of $\mathcal{L}(2n)$ whose members terminate at a positive height. By symmetry, we have $|\mathcal{Y}_n| = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right)$, so to complete the proof of (2.10), it suffices to define a bijection between $\mathcal{A}_n$ and $\mathcal{Y}_n$. To do so, let $\lambda \in \mathcal{A}_n$ be decomposed as $\lambda = \lambda^{(1)} \cdots \lambda^{(k)}$, where $k \geq 1$, each $\lambda^{(j)}$ for $j \in [k]$ is a nonempty Dyck path whose final unit is marked, some u step in $\lambda^{(1)}$ is circled and the sum of the semilengths of the $\lambda^{(j)}$ is n. Consider applying the bijection g from the proof of (2.9) above to $\lambda^{(1)}$ to obtain $\rho = g(\lambda^{(1)}) \in \mathcal{L}(i+1, i-1)$, where $|\lambda^{(1)}| = i$.

To ρ, we apply a mapping h defined as follows. If ρ is first-quadrant (i.e., lies completely on or above the x-axis), then let $h(\rho) = \rho$. Otherwise, ρ achieves a minimum height of m for some $m \leq -1$ and consider the rightmost point on ρ whose y-coordinate is m. This leads to a decomposition of ρ as $\rho = \rho' u^2 \rho''$, where ρ' ends at height m and $u\rho''$ is first-quadrant (when positioned so that its starting point is the origin). Then define h in this case by setting $h(\rho) = \widetilde{\text{rev}}(\rho') u^2 \rho''$ and note that $h(\rho)$ has final height $2k := -2m+2 \geq 4$. Then h is seen to be a bijection between $\mathcal{L}(i+1, i-1)$ and the set of first-quadrant lattice paths with $2i$ steps whose final height is positive. Note that if such a path has final height $2k$, where $k \geq 2$, then h can be reversed by considering the position of the rightmost u having starting height $k-1$.

We now define a mapping ℓ between $\mathcal{A}_n$ and $\mathcal{Y}_n$. Let $\overline{\lambda}^{(j)}$ for $2 \leq j \leq k$ be obtained from $\lambda^{(j)}$ by reflecting the final unit of $\lambda^{(j)}$ in the x-axis, leaving all other units of $\lambda^{(j)}$ unchanged. Define $\ell(\lambda) = \overline{\lambda}^{(2)} \cdots \overline{\lambda}^{(k)} h(\rho)$, where the various lattice paths are understood to be concatenated. Note that $h(\rho) = hg(\lambda^{(1)})$ is first-quadrant with $2i$ steps and ending at a positive height, whence $\ell(\lambda) \in \mathcal{Y}_n$. To reverse ℓ, consider the position of the rightmost "negative" unit (if it exists) as well as the number of such units within $\pi \in \mathcal{Y}_n$. The remaining portion of π to the right of the last negative unit must be nonempty and determines the path $h(\rho)$, from which $\lambda^{(1)}$ can be obtained by reversing the bijections h and g. Thus, we have that ℓ yields the desired bijection between $\mathcal{A}_n$ and $\mathcal{Y}_n$, which completes the proof. □

Proof of (2.11). We may assume $n \geq 2$. Let $\mathcal{B}_{n,k}$ denote the set of marked ordered k-tuples $\lambda = (\lambda_1, \ldots, \lambda_k)$, where each λ_i is a Dyck path having semilength at least two such that $\sum_{i=1}^{k} |\lambda_i| = n + k$ wherein the r-th up step (from the left) of λ_1 is marked for some $1 \leq r \leq |\lambda_1| - 1$. Define the sign of $\lambda \in \mathcal{B}_{n,k}$ by $(-1)^{n-k}$ and let $\mathcal{B}_n = \cup_{k=1}^{n} \mathcal{B}_{n,k}$. Then we have that $\sigma(\mathcal{B}_n)$ is given by $D_+(C_2, \ldots, C_{n+1})$. We first define a sign-reversing involution on $\mathcal{B}_n$ based on the final component of λ as follows:

(a) $\lambda_k = \alpha\beta$, $|\alpha| \geq 2$ and β a unit $\leftrightarrow$ $\lambda_k = \alpha$, $\lambda_{k+1} = ud\beta$,

(b) $\lambda_k = u\alpha d$, $|\alpha| \geq 2 \leftrightarrow \lambda_k = \alpha$, $\lambda_{k+1} = u^2 d^2$,

where it is assumed $k \geq 2$ in both cases and all other components of λ remain unchanged. Note that since $k \geq 2$, the position of the marked u in the first component does not present an issue.

We can however extend the involution above to some cases when $k = 1$ as follows. Let $\lambda = (\lambda_1) \in \mathcal{B}_{n,1}$. If $\lambda_1 = \alpha\beta$, where $|\alpha| \geq 2$ and β is a unit and the r-th u step of λ_1 is marked for some $1 \leq r \leq |\alpha| - 1$, then one may apply (a) in this case where $\lambda_1 = \alpha$ on the right side is understood to have its r-th u marked. On the other hand, if λ_1 is primitive with $\lambda_1 = u\alpha d$, then we may apply (b) in this case, provided the r-th u of λ_1 is marked for some $1 \leq r \leq n - 1$.

The involution on $\mathcal{B}_n$ is then not defined in cases where $k = 1$ and (i) $\lambda_1 = \alpha\beta$, with $|\alpha| \geq 1$, β a unit and either the final u in α or one of the first $|\beta| - 1$ u's in β is marked, or (ii) $\lambda_1 = u\alpha d$, with the n-th u of λ_1 marked. This yields $\sum_{i=1}^{n} iC_{i-1}C_{n-i+1}$ possibilities in (i), where $i = |\beta|$, and C_n possibilities in (ii), where the sign of all members in both cases is $(-1)^{n-1}$. Thus, to complete the proof, we must show combinatorially

$$\sum_{i=1}^{n} iC_{i-1}C_{n-i+1} = \binom{2n}{n} - C_n, \qquad n \geq 1. \tag{3.4}$$

Above, it was shown $\sum_{i=1}^{n} iC_{i-1}C_{n-i} = \binom{2n-1}{n}$, i.e.,

$$\sum_{i=1}^{n-1} iC_{i-1}C_{n-i} = \binom{2n-1}{n} - nC_{n-1}.$$

Upon replacing n with $n-1$ in (3.4), we then need to show

$$\binom{2n-2}{n-1} - C_{n-1} = \binom{2n-1}{n} - nC_{n-1}.$$

However, we have

$$\binom{2n-1}{n} - nC_{n-1} = \binom{2n-1}{n} - \binom{2n-2}{n-1} = \binom{2n-2}{n} = \binom{2n-2}{n-1} - C_{n-1},$$

with each of the preceding three equalities understood combinatorially by standard arguments, which completes the proof. □

Before proving the results of Theorem 2.6 concerning Motzkin number determinants, we introduce some further terminology and notation. Let $\mathcal{M}_n$ denote the set of lattice paths from $(0,0)$ to $(n,0)$ using u, d and $h=(1,0)$ steps that never dip below the x-axis. Members of $\mathcal{M}_n$ are referred to as *Motzkin* paths of length n and are enumerated by M_n. A *low* h step is one joining the points $(i-1,0)$ and $(i,0)$ along the x-axis for some $i>0$. Let $\mathcal{R}_n$ denote the subset of $\mathcal{M}_n$ consisting of those paths which contain no low h steps. Members of $\mathcal{R}_n$ are referred to as *Riordan* paths and are enumerated by the n-th Riordan number R_n; see, e.g., $A005043$ in [23].

A *unit* within $\pi\in\mathcal{M}_n$ will refer to either a single low h or a subpath of π of the form $u\pi'd$, where u starts and d ends on the x-axis and π' is a possibly empty Motzkin path. Note that a low h step is implicitly also considered as a return to the x-axis. Let $\mathcal{G}_n$ denote the set of all lattice paths from $(0,0)$ to $(n,0)$ consisting of u, d and $h=(1,0)$ steps where one is allowed to go below the x-axis. Note that $|\mathcal{G}_n|=G_n$ for all $n\geq 0$, with members of $\mathcal{G}_n$ being referred to as *grand Motzkin* paths.

Proof of (2.19) *and* (2.20). Let $\mathcal{E}_n$ denote the set of marked, circled members of $\mathcal{M}_n$ in which low h's may be marked, the final step is a marked low h and some step to the left of (and including) the first marked low h is circled. Let $\mathcal{E}_{n,k}$ for $1\leq k\leq n$ denote the subset of $\mathcal{E}_n$ whose members contain exactly k marked low h's. Define the sign of a member of $\mathcal{E}_{n,k}$ as $(-1)^{n-k}$. Then we have $D_-(M_0,\ldots,M_{n-1})=|\mathcal{E}_n|$, whereas $D_+(M_0,\ldots,M_{n-1})=\sigma(\mathcal{E}_n)$. Let $u_n=|\mathcal{E}_n|$ for $n\geq 1$. To show $u_n=A055217[n-1]$, we demonstrate that the two sequences have the same generating function, where the latter is defined as the coefficients in a certain infinite series expansion, namely,

$$\sum_{n\geq 0}A055217[n]x^n=\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1-2x-3x^2)}.$$

To do so, first note that u_n satisfies the recurrence

$$u_n=nM_{n-1}+\sum_{i=1}^{n-1}M_{i-1}u_{n-i},\qquad n\geq 1, \tag{3.5}$$

upon considering whether a member of $\mathcal{E}_n$ has one or more marked low h's and, if there is more than one, the number of steps $i-1$ where $i\in[n-1]$ between the final two marked low h's. Note that

$$\sum_{n\geq 1}M_{n-1}x^n=\frac{1-x-\sqrt{1-2x-3x^2}}{2x},$$

and hence

$$\sum_{n\geq 1}nM_{n-1}x^n=\frac{1-x-\sqrt{1-2x-3x^2}}{2x\sqrt{1-2x-3x^2}}.$$

Multiplying both sides of (3.5) by x^n, summing over $n\geq 1$ and solving for $\sum_{n\geq 1}u_nx^n$ then yields

$$\begin{aligned}\sum_{n\geq 1}u_nx^n&=\frac{1-x-\sqrt{1-2x-3x^2}}{1-2x-3x^2+(3x-1)\sqrt{1-2x-3x^2}}=\frac{1+x-\sqrt{1-2x-3x^2}}{2(1-2x-3x^2)}\\&=\sum_{n\geq 1}A055217[n-1]x^n,\end{aligned}$$

which implies (2.20).

To show (2.19), first consider applying the general involution ϕ defined above to $\mathcal{E}_n$. The set of survivors of this involution consists of $\mathcal{E}_{n,1}$ along with members of $\mathcal{E}_{n,2}$ in which there are no (unmarked) low h's occurring between the two marked low h's. This implies

$$D_+(M_0,\ldots,M_{n-1}) = (-1)^{n-1}\left(nM_{n-1} - \sum_{i=1}^{n-1} iM_{i-1}R_{n-i-1}\right),$$

upon considering the position i of the first marked low h to enumerate the members of $\mathcal{E}_{n,2}$ in question.

Recall that one of the combinatorial properties of the sequence $A113682[n]$ is that it is given explicitly as $A113682[n] = \sum_{k=0}^{n} R_k G_{n-k}$ for all $n \geq 0$. Thus, to complete the proof of (2.19), it suffices to show

$$A113682[n-1] = nM_{n-1} - \sum_{i=1}^{n-1} iM_{i-1}R_{n-i-1},$$

i.e.,

$$(n+1)M_n = \sum_{i=0}^{n-1}(i+1)M_iR_{n-i-1} + \sum_{k=0}^{n} R_kG_{n-k}, \qquad n \geq 1. \tag{3.6}$$

To prove (3.6), we first develop a combinatorial interpretation for the sequence $(n+1)M_n$ in terms of grand Motzkin paths as follows. Let $\mathcal{G}_n^*$ denote the subset of $\mathcal{G}_n$ whose members end in u. Given $\pi \in \mathcal{M}_n$ and $i \in [n]_0 = [n] \cup \{0\}$, write $\pi = \pi'\pi''$, where π' is the subpath of π consisting of its first i steps (note $\pi' = \varnothing$ if $i = 0$). Let $\pi^* = \widetilde{\text{rev}}(\pi')d\widetilde{\text{rev}}(\pi'')u$, where $\widetilde{\text{rev}}(\rho)$ for a subpath ρ of a Motzkin path is obtained from ρ by reading the sequence of steps in ρ backwards and replacing each u with d and d with u, leaving all h steps unchanged. It is seen that the mapping $(\pi, i) \mapsto \pi^*$ may be reversed by considering the position of the leftmost minimum point within a member of $\mathcal{G}_{n+2}^*$, and hence it defines a bijection from $\mathcal{M}_n \times [n]_0$ to $\mathcal{G}_{n+2}^*$.

Using this interpretation for $(n+1)M_n$, one can now readily explain (3.6) combinatorially. First suppose $\lambda \in \mathcal{G}_{n+2}^*$ can be decomposed as $\lambda = \lambda' h \lambda''$, where $\lambda' \in \mathcal{R}_{n-i-1}$ and $\lambda'' \in \mathcal{G}_{i+2}^*$ for some $i \in [n-1]_0$. Considering all possible i implies that the first sum on the right side of (3.6) counts all members of $\mathcal{G}_{n+2}^*$ in which a low h occurs prior to the first negative step. Otherwise, $\lambda \in \mathcal{G}_{n+2}^*$ can be expressed as $\lambda = \lambda' d \lambda'' u$ for some $\lambda' \in \mathcal{R}_k$ and $\lambda'' \in \mathcal{G}_{n-k}$. Considering all possible $k \in [n]_0$ gives the second sum on the right side of (3.6), which completes the proof. □

Proof of (2.21). Let $\mathcal{F}_n$ denote the set of marked, circled members of $\mathcal{M}_n$ in which units (including low h's) may be marked, the last unit is marked and some step to the left of or belonging to the first marked unit is circled. Let $\mathcal{F}_{n,k}$ denote the subset of $\mathcal{F}_n$ whose members contain k marked units. Define the sign of $\pi \in \mathcal{F}_{n,k}$ by $(-1)^{n-k}$ for $1 \leq k \leq n$ and it is seen $D_+(M_1,\ldots,M_n) = \sigma(\mathcal{F}_n)$. Let $\mathcal{F}_n'$ denote the set of survivors in $\mathcal{F}_n$ when the general involution ψ above is applied to $\mathcal{F}_n$. Note that $\mathcal{F}_n'$ consists of those $\pi \in \mathcal{F}_{n,1}$ expressible as $\pi = \pi^*\alpha$, where α is a unit in which one of the steps is circled. We define a mapping ρ between $\mathcal{F}_n'$ and $\mathcal{G}_n$ as follows. If the final step of α is circled (note this covers the case when α is a single low h), then let $\rho(\pi) = \pi$. So assume some step of α other than the last is circled and write $\alpha = u\beta\gamma d$, where the final step of the subpath $u\beta$ is circled and β or γ (possibly both) may be empty. Note that $\beta = \varnothing$ corresponds to the case when the first step of α is circled, whereas $\gamma = \varnothing$ implies the penultimate step of α is circled. Define $\rho(\pi) = \pi^* d\widetilde{\text{rev}}(\beta) u\widetilde{\text{rev}}(\gamma)$ for $\pi = \pi^* u\beta\gamma d$ as described. Then $\rho(\pi) \in \mathcal{G}_n - \mathcal{M}_n$ and ρ may be reversed by considering the rightmost occurrence of the minimum (negative) height since α is a unit. One may then verify that the (composite) mapping ρ is a bijection between $\mathcal{F}'_n$ and $\mathcal{G}_n$, which implies (2.21). □

We introduce now some further terms before proving the final set of identities involving Schröder numbers. Let $\mathcal{P}_n$ denote the set of lattice paths from $(0,0)$ to $(2n,0)$ using u, d and $h = (2,0)$ steps that do not go below the x-axis at any point. Members of $\mathcal{P}_n$ are referred to as *Schröder* paths of semilength n and are enumerated by the large Schröder number S_n. Note that the semilength of a member of $\mathcal{P}_n$ equals half the sum of the numbers of u and h steps (or, equivalently, of d and h steps). The terms *unit* and *low* h will be used in the same way with regard to Schröder as they were before with regard to Motzkin paths. Let $\mathcal{Q}_n$ denote the subset of $\mathcal{P}_n$ whose members contain no low h steps, which are enumerated by the small Schröder number s_n. Finally, the set of all lattice paths from $(0,0)$ to $(2n,0)$ using u, d and $h = (2,0)$ steps will be denoted here by $\mathcal{N}_n$, the members of which are called *Delannoy* paths. Recall $|\mathcal{N}_n| = D_n$ for all $n \geq 0$; see, e.g., A001850 in [23].

Proof of (2.22), (2.23) *and* (2.25). Let $\mathcal{H}_n$ be the set of marked Schröder paths of semilength n ending in h in which some u or h step to the left of (and including) the leftmost low h is marked. Upon overlaying each part of size m in a composition of n with a member of $\mathcal{Q}_{m-1}$ followed by h, it is seen $D_-(s_0, \ldots, s_{n-1}) = |\mathcal{H}_n|$. Suppose the leftmost low h terminates at $(2i, 0)$ for some i within a member of $\mathcal{H}_n$. Allowing i to vary over $[n]$ then implies

$$|\mathcal{H}_n| = ns_{n-1} + \sum_{i=1}^{n-1} is_{i-1}S_{n-i-1}, \qquad n \geq 1,$$

upon considering separately the cases $i = n$ or $i < n$. Replacing n with $n+1$, we thus need to show

$$(n+1)s_n + \sum_{i=0}^{n-1}(i+1)s_iS_{n-i-1} = D_n, \qquad n \geq 0. \tag{3.7}$$

Note that $S_n = s_n + \sum_{i=0}^{n-1} s_iS_{n-i-1}$, upon considering the position of the first low h (if it exists) within a member of $\mathcal{P}_n$. Upon subtracting S_n from both sides of (3.7), to complete the proof of (2.22), one can show alternatively

$$ns_n + \sum_{i=1}^{n-1} is_iS_{n-i-1} = D_n - S_n, \qquad n \geq 1. \tag{3.8}$$

To establish (3.8), we argue that the left-hand side enumerates $\mathcal{N}_n - \mathcal{P}_n$, i.e., the subset of $\mathcal{N}_n$ whose members contain negative steps. Note that $\lambda \in \mathcal{N}_n - \mathcal{P}_n$ implies that it can be expressed as $\lambda = \alpha\tau\beta$, where α and β (each possibly empty) consist of units of the form $u\gamma d$ with γ allowed to empty and τ is nonempty and starts with d or h, ends with u or h and goes below the x-axis. We will refer to τ as the *central section* of λ. Given $n \geq 2$ and $i \in [n-1]$, let $\mathcal{X}_{n,i}$ denote the set of ordered triples $\mu = (\rho, \sigma, r)$, where $\rho \in \mathcal{Q}_i$, $\sigma \in \mathcal{P}_{n-i-1}$ and $r \in [i]$, and hence $|\cup_{i=1}^{n-1} \mathcal{X}_{n,i}| = \sum_{i=1}^{n-1} is_iS_{n-i-1}$. We transform triples in $\mathcal{X}_{n,i}$ into members of $\mathcal{N}_n - \mathcal{P}_n$ as follows. We first mark the r-th u or h step (from the left) within $\rho \in \mathcal{Q}_i$. Consider the unit ρ' of ρ in which the marked step lies and we express ρ as $\rho = \alpha\rho'\beta$. Further, we decompose ρ' as $u\gamma\delta d$, where the last step of the subpath $u\gamma$ is marked, with γ or δ possibly empty.

We now convert μ to the Delannoy path μ^* given by

$$\mu^* = \alpha\widetilde{\text{rev}}(\gamma)d\sigma u\widetilde{\text{rev}}(\delta)h\beta,$$

where $\widetilde{\text{rev}}(\pi)$ is defined for a Schröder path π in the same way as it was for a Motzkin path. One may verify $\mu^* \in \mathcal{N}_n - \mathcal{P}_n$ with central section $\widetilde{\text{rev}}(\gamma)d\sigma u\widetilde{\text{rev}}(\delta)h$. Note that σ in the decomposition above coincides with the subpath of μ^* starting with the first and ending with the last global minimum point. Then the mapping $\mu \mapsto \mu^*$ may be reversed by considering the positions of the first and last global minimum points within a member of $\mathcal{N}_n - \mathcal{P}_n$ whose central section ends in h. Thus, the mapping $\mu \mapsto \mu^*$ is a bijection from $\cup_{i=1}^{n-1}\mathcal{X}_{n,i}$ to members of $\mathcal{N}_n - \mathcal{P}_n$ whose central section ends in h, and hence there are $\sum_{i=1}^{n-1} is_iS_{n-i-1}$ such members of $\mathcal{N}_n - \mathcal{P}_n$.

Let $\mathcal{X}_{n,n}$ denote the set of marked $\rho \in \mathcal{Q}_n$ wherein some u or h step is marked, and hence $|\mathcal{X}_{n,n}| = ns_n$. Suppose again that the marked step lies within the unit ρ' of ρ. We then decompose ρ and ρ' as before and define

$$\rho^* = \alpha\widetilde{\text{rev}}(\gamma)d\widetilde{\text{rev}}(\delta)u\beta.$$

Similar to before, one may verify that the mapping $\rho \mapsto \rho^*$ is a bijection from $\mathcal{X}_{n,n}$ to members of $\mathcal{N}_n - \mathcal{P}_n$ whose central section ends in u. Combining the mappings $\mu \mapsto \mu^*$ and $\rho \mapsto \rho^*$ then yields a bijection between $\cup_{i=1}^{n}\mathcal{X}_{n,i}$ and $\mathcal{N}_n - \mathcal{P}_n$, which implies (3.8) and completes the proof of (2.22).

To show (2.23), let $\mathcal{Q}_n^*$ be obtained from the members of $\mathcal{Q}_n$ by marking some subset of the returns, including the final return, and circling a u or h step to the left of the first marked return. Let $\mathcal{Q}_{n,k}^*$ denote the subset of $\mathcal{Q}_n^*$ whose members contain k marked returns and define the sign of a member of $\mathcal{Q}_{n,k}^*$ by $(-1)^{n-k}$. Then $D_+(s_1,\dots,s_n) = \sigma(\mathcal{Q}_n^*)$ and consider applying the involution ϕ defined above to $\mathcal{Q}_n^*$. Note that the set of survivors consists of $\mathcal{Q}_{n,1}^*$, together with the members of $\mathcal{Q}_{n,2}^*$ in which there are no returns between the two marked returns. The sum of the signs of the survivors is then given by

$$\begin{aligned}(-1)^{n-1}ns_n + (-1)^{n-2}\sum_{i=1}^{n-1} is_iS_{n-i-1} &= (-1)^{n-1}\left(ns_n - (D_n - S_n - ns_n)\right)\\ &= (-1)^{n-1}((n+1)S_n - D_n),\end{aligned}$$

by (3.8), which implies (2.23). Note that the fact $S_n = 2s_n$ for $n \geq 1$, which was used in the second equality, may be realized bijectively from the combinatorial definition of the two sequences.

Finally, for (2.25), let $\mathcal{P}_n^*$ be obtained from the members of $\mathcal{P}_n$ that end in h by marking some subset of the low h's, including the terminal h, and then circling a u or h to the left (and including) the first marked low h. Let $\mathcal{P}_{n,k}^*$ denote the subset of $\mathcal{P}_n^*$ whose members contain k marked low h steps and define the sign of a member of $\mathcal{P}_{n,k}^*$ by $(-1)^{n-k}$. Then we have $D_+(S_0,\dots,S_{n-1}) = \sigma(\mathcal{P}_n^*)$ and consider applying the involution ψ to $\mathcal{P}_n^*$. The survivors of the involution each have sign $(-1)^{n-1}$ and can be expressed as either $\pi = \pi'h$, where $\pi' \in \mathcal{Q}_{n-1}$, or as $\pi = \alpha h\pi'h$, where α and π' are Schröder paths, π' contains no low h's and either may be empty. Further, in both cases, some u or h step within $\pi'h$ is circled. Then the mapping from the set of survivors of ψ which leaves π in the first case unchanged and replaces π in the second case by $\pi' h\alpha h$ is a bijection with $\mathcal{H}_n$, and hence the survivors number D_{n-1}, which yields (2.25). □

Proof of (2.24) *and* (2.26). First note that $D_-(s_1,\ldots,s_n)=|\mathcal{Q}_n^*|$ and

$$D_-(S_0,\ldots,S_{n-1})=|\mathcal{P}_n^*|$$

for all $n\geq 1$, where $\mathcal{Q}_n^*$ and $\mathcal{P}_n^*$ are as in the preceding proof. Let $u_n=|\mathcal{Q}_n^*|$ and observe that u_n satisfies the recurrence

$$u_n = ns_n + \sum_{i=1}^{n-1} s_i u_{n-i}, \qquad n\geq 1,$$

upon considering the position of the first marked return within a member of $\mathcal{Q}_n^*$. Proceeding as in the proof of (2.20) above, and making use of the generating function formula

$$\sum_{n\geq 1} s_n x^n = \frac{1-3x-\sqrt{1-6x+x^2}}{4x},$$

we have

$$\sum_{n\geq 1} u_n x^n = \frac{1-3x-\sqrt{1-6x+x^2}}{1-6x+x^2+(7x-1)\sqrt{1-6x+x^2}} = \sum_{n\geq 1} A271197[n-1]x^n,$$

which implies (2.24).

To show (2.26), first recall $A002002[n]=\frac{1}{2}(D_n-D_{n-1})$ for $n\geq 1$. Thus, by symmetry, we have that $A002002[n]$ gives the cardinality of the subset $\mathcal{N}_n'$ of $\mathcal{N}_n$ whose members start with u. To establish (2.26), we then define a bijection p between $\mathcal{P}_n^*$ and $\mathcal{N}_n'$. Assume in this proof that some d or h step to the left of (and including) the first low h within a member of $\mathcal{P}_n^*$ is circled, instead of a u or h step (note the resulting set is of the same cardinality, which we again call $\mathcal{P}_n^*$, by a slight abuse of notation). Let $\lambda\in\mathcal{P}_n^*$, which we decompose as $\lambda=\lambda'\underline{h}\lambda^{(1)}\underline{h}\cdots\lambda^{(k)}\underline{h}$, where λ' and the $\lambda^{(i)}$ are Schröder paths, each marked low h step is underlined, some d or h within $\lambda'\underline{h}$ is circled and $k=0$ is possible (in which case $\lambda=\lambda'\underline{h}$). First suppose that the circled step is the leftmost marked low h (i.e., the first underlined h in the decomposition of λ above). Then let $p(\lambda)$ in this case be given by $p(\lambda)=u\lambda' du\lambda^{(1)}d\cdots u\lambda^{(k)}d$.

On the other hand, if the circled step lies within λ', then write $\lambda'=\rho\textcircled{x}\tau$, where $x=d$ or h and ρ or τ may be empty. Define $p(\lambda)$ in this case by

$$p(\lambda)=u\widetilde{\text{rev}}(\rho)d\widetilde{\text{rev}}(\tau)yu\lambda^{(1)}d\cdots u\lambda^{(k)}d,$$

where $y=u$ or h depending on whether $x=d$ or h. By considering the semilength of the first unit within a member of $\mathcal{N}_n'\cap\mathcal{Q}_n$ or the position of the rightmost negative step or low h (as well as the position of the first minimum point) within a member of $\mathcal{N}_n'-\mathcal{Q}_n$, one can reverse both cases of the mapping p. Since each case of p is seen to be onto its respective codomain, we have that p yields the desired bijection between $\mathcal{P}_n^*$ and $\mathcal{N}_n'$, which completes the proof. □

Proof of (2.27). Let $\mathcal{J}_n$ denote the set of marked, circled Schröder paths of semilength n in which some subset of the returns is marked, including the final return, and a d or h step to the left of the first marked return is circled. Define the sign of a member $\pi \in \mathcal{J}_n$ as $(-1)^{n-\mu(\pi)}$, where $\mu(\pi)$ denotes the number of marked returns of π. We have $D_+(S_1, \ldots, S_n) = \sigma(\mathcal{J}_n)$ and consider applying the involution ψ to $\mathcal{J}_n$. Then the members of the set $\mathcal{J}_n'$ of survivors of ψ each have sign $(-1)^{n-1}$ and are expressible as $\pi = \tau\pi^*$, where τ is a possibly empty Schröder path, π^* is a unit and some d or h step in π^* is circled. From the generating function formulas, it is seen $A002002[n] + A002003[n] = D_n$ for $n \geq 1$, and hence $A002003[n]$ enumerates the subset of $\mathcal{N}_n$ whose members start with d or h. By symmetry, this is equivalent to counting members of $\mathcal{N}_n$ that end in d or h, the set of which we denote by $\widehat{\mathcal{N}}_n$.

To establish (2.27), we define a bijection q between $\mathcal{J}_n'$ and $\widehat{\mathcal{N}}_n$. If the final step of the unit π^* is circled (which covers the case when π^* consists of a single low h), then let $q(\pi) = \pi$. Otherwise, write $\pi^* = u\alpha\textcircled{x}\beta d$, where $x = d$ or h and α or β may be empty. In this case, let $q(\pi) = \tau d\beta u\alpha x$, where x is the same as the circled step in π^*. Note that q may be reversed by considering whether or not a member of $\widehat{\mathcal{N}}_n$ contains a negative step, and if it does, considering further the positions of the leftmost negative step and the rightmost point on a path for which the y-coordinate is a minimum. Hence, q yields the desired bijection between $\mathcal{J}_n'$ and $\widehat{\mathcal{N}}_n$, which completes the proof. □

Remark 3.1. From the preceding arguments, one obtains new combinatorial interpretations of such sequences as the central binomial coefficients, grand Motzkin numbers, Delannoy numbers, $A113682[n]$, $A002002[n]$ and $A002003[n]$ as well as perhaps the first such interpretation of the sequences $A055217[n]$ and $A271197[n]$.

Extending the arguments above yields combinatorial proofs of a couple of relations between Motzkin and grand Motzkin numbers which do not seem to have been previously given.

Combinatorial proofs of two related identities

We provide proofs of the following formulas:

$$2G_n = (n+2)M_n - nM_{n-1}, \qquad n \geq 1, \tag{3.9}$$

$$(n+1)M_n = G_{n+1} - R_{n+1}, \qquad n \geq 1. \tag{3.10}$$

To show (3.9), we first write it more strategically as

$$nM_n = G_n + (n-1)M_{n-1} + G_n - M_n - (M_n - M_{n-1}), \tag{3.11}$$

where we may assume $n \geq 3$. Let $\mathcal{M}_n^*$ denote the set of marked members of $\mathcal{M}_n$ wherein some step is marked. We argue that both sides of (3.11) count the members of $\mathcal{M}_n^*$, the left clearly doing so. For the right, note first that there are G_n members of $\mathcal{M}_n^*$ where the marked step lies in the final unit, by the bijection ρ used in the proof of (2.21) above. Further, there are clearly $(n-1)M_{n-1}$ members of $\mathcal{M}_n^*$ ending in h whose marked step is not the final h. So suppose $\pi \in \mathcal{M}_n^*$ does not end in h with its marked step not lying in the final unit of π. Upon considering the length $n-i$ of the final unit of π where $1 \leq i \leq n-2$, it follows that there are $\sum_{i=1}^{n-2} iM_iM_{n-i-2}$ such members of $\mathcal{M}_n^*$. To complete the proof of (3.11), we then establish via a combinatorial argument the formula

$$\sum_{i=1}^{n-2} iM_iM_{n-i-2} = G_n - M_n - (M_n - M_{n-1}), \qquad n \geq 3. \tag{3.12}$$

To do so, consider the set S of ordered pairs (α, β), where $\alpha \in \mathcal{M}_i$, the j-th step of α is marked for some $j \in [i]$, $\beta \in \mathcal{M}_{n-i-2}$ and i can be any element of $[n-2]$. Then clearly S is enumerated by the left-hand side of (3.12) and we define a mapping f between S and $\mathcal{G}_n - \mathcal{M}_n$. Suppose $(\alpha, \beta) \in S$ and that α can be decomposed as $\alpha = \alpha' s \alpha''$, where s is the marked step of α. Let $f(\alpha, \beta) = \widetilde{\text{rev}}(\alpha' s) d \widetilde{\text{rev}}(\alpha'') u \beta$, which is seen to belong to $\mathcal{G}_n - \mathcal{M}_n$ since the middle d step must terminate at a negative height. Further, the mapping f can be reversed upon considering the first occurrence of the minimum height $m < 0$ and the position of the rightmost negative step. Note that all members of $\mathcal{G}_n - \mathcal{M}_n$ belong to the range of f except for those starting with d for which $m = -1$, i.e., those expressible as $\upsilon = d\sigma u \tau$, where σ and τ are Motzkin paths such that $|\sigma| + |\tau| = n - 2$. Let $\upsilon' = u\sigma d\tau$ and the mapping $\upsilon \mapsto \upsilon'$ is a bijection with the subset of $\mathcal{M}_n$ whose members start with u, which number $M_n - M_{n-1}$, by subtraction. Thus, it has been demonstrated that there is a bijection between S and a subset of $\mathcal{G}_n - \mathcal{M}_n$ whose cardinality is given by the right-hand side of (3.12), which completes the proof of (3.9).

Next observe that the right side of (3.10) is the cardinality of $\mathcal{G}_{n+1} - \mathcal{R}_{n+1}$, which is the set of grand Motzkin paths of length $n+1$ that contain a negative or low h step (possibly both). To prove (3.10), it suffices to show that the right side of (3.6) also enumerates $\mathcal{G}_{n+1} - \mathcal{R}_{n+1}$. Let $\pi \in \mathcal{G}_{n+1} - \mathcal{R}_{n+1}$. First suppose π is expressible as $\pi = \pi' h \pi''$, where $\pi' \in \mathcal{R}_{n-i}$ and $\pi'' \in \mathcal{G}_i$. Considering all possible $0 \leq i \leq n$ then yields the second sum on the right side of (3.6), upon replacing i with $n-k$. Note that this accounts for all members of $\mathcal{G}_{n+1} - \mathcal{R}_{n+1}$ in which a low h occurs earlier than any negative steps. Now suppose a negative step occurs in π earlier than any low h. Then we have $\pi = \pi'\pi''$, where $\pi' \in \mathcal{R}_{n-i-1}$, $\pi'' \in \mathcal{G}_{i+2}$ starts with d and $0 \leq i \leq n-1$. In the proof of (2.19), it was shown that there are $(i+1)M_i$ members of $\mathcal{G}_{i+2}$ that end with u, and hence the same number that start with d, upon applying the $\widetilde{\text{rev}}$ operation. Considering all possible i then accounts for the first sum on the right side of (3.6) and completes the proof of (3.10). □

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