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RIGHT-TOPOLOGICAL SEMIGROUP OPERATIONS ON INCLUSION HYPERSPACES

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We show that for any discrete semigroup X the semigroup operation can be extended to a right-topological semigroup operation on the space $G(X)$ of inclusion hyperspaces on X. We detect some important subsemigroups of $G(X)$, study the minimal ideal, the (topological) center, left and right cancelable elements of $G(X)$.

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Доказано, что каждую полугрупповую операцию на дискретном пространстве X можно продолжить до право-топологической полугрупповой операции на пространстве $G(X)$ гиперпространств включения на X. Изучаются некоторые важные подполугруппы в $G(X)$, описывается минимальный идеал, (топологический) центр, сократимые элементы $G(X)$.

Introduction. After the topological proof of Hindman theorem [6] given by Galvin and Glazer (unpublished, see [8, p.102], [7]) topological methods become a standard tool in the modern combinatorics of numbers, see [8], [11]. The crucial point is that the semigroup operation $*$ defined on any discrete space S can be extended to a right-topological semigroup operation on βS, the Stone-Čech compactification of S. The product of two ultrafilters $\mathcal{U}, \mathcal{V} \in \beta S$ can be found in two steps: firstly for every element $a \in S$ of the semigroup we extend the left shift $L_a\colon S \to S$, $L_a\colon x \mapsto a*x$, to a continuous map $\beta L_a\colon \beta S \to \beta S$. In such a way, for every $a \in S$ we define the product $a * \mathcal{V} = \beta L_a(\mathcal{V})$. Then, extending the function $R_{\mathcal{V}}\colon S \to \beta S$, $R_{\mathcal{V}}\colon a \mapsto a * \mathcal{V}$, to a continuous map $\beta R_{\mathcal{V}}\colon \beta S \to \beta S$, we define the product $\mathcal{U} \circ \mathcal{V} = \beta R_{\mathcal{V}}(\mathcal{U})$. This product can be also defined directly: this is an ultrafilter with the base $\bigcup_{x\in U} x * V_x$ where $U \in \mathcal{U}$ and $\{V_x\}_{x\in U} \subset \mathcal{V}$. Endowed with so-extended operation the Stone-Čech compactification βS becomes a compact Hausdorff right-topological semigroup. Because of the compactness the semigroup βS has idempotents, minimal (left) ideals, etc., whose existence has many important combinatorial consequences.

The Stone-Čech compactification βS can be considered as a subset of the double power-set $\mathcal{P}(\mathcal{P}(S))$. The power-set $\mathcal{P}(X)$ of any set X (in particular, $X = \mathcal{P}(S)$) carries a natural compact Hausdorff topology inherited from the Cantor cube $\{0,1\}^X$ after identification of each subset $A \subset X$ with its characteristic function. The power-set $\mathcal{P}(X)$ is a complete distributive lattice with respect to the operations of union and intersection.

The smallest complete sublattice of $\mathcal{P}(\mathcal{P}(S))$ containing βS coincides with the space $G(S)$ of inclusion hyperspaces, a well-studied object in Categorial Topology. By definition,

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a family $\mathcal{A} \subset \mathcal{P}(S)$ of non-empty subsets of S is called an *inclusion hyperspace* if together with each set $A \in \mathcal{A}$ the family $\mathcal{A}$ contains all supersets of A in S. In [4] it is shown that $G(S)$ is a compact Hausdorff lattice with respect to the operations of intersection and union.

Our principal observation is that the algebraic operation of the semigroups S can be extended not only to βS but also to the complete lattice hull $G(S)$ of βS in $\mathcal{P}(\mathcal{P}(S))$. Endowed with the so-extended operation, the space of inclusion hyperspaces $G(S)$ becomes a compact Hausdorff right-topological semigroup containing βS as a closed subsemigroup. Besides βS, the semigroup $G(S)$ contains many other important spaces as closed subsemigroups: the superextension λS of S, the space $N_k(S)$ of k-linked inclusion hyperspaces, the space $\mathrm{Fil}(S)$ of filters on S (which contains an isomorphic copy of the global semigroup $\Gamma(S)$ of S), etc.

We shall study some properties of the semigroup operation on $G(S)$ and its interplay with the lattice structure of $G(S)$. We expect that studying the algebraic structure of $G(S)$ will have some combinatorial consequences that cannot be obtained with help of ultrafilters, see [2] for further development of this subject.

1. Inclusion hyperspaces. In this section we recall some basic information about inclusion hyperspaces. More detail information can be found in the paper [4].

1.1. General definition and reduction to the compact case. For a topological space X by $\exp(X)$ we denote the space of all non-empty closed subspaces of X endowed with the Vietoris topology. By an *inclusion hyperspace* we mean a closed subfamily $\mathcal{F} \subset \exp(X)$ that is *monotone* in the sense that together with each set $A \in \mathcal{F}$ the family $\mathcal{F}$ contains all closed subsets $B \subset X$ that contain A. By [4], the closure of each monotone family in $\exp(X)$ is an inclusion hyperspace. Consequently, each family $\mathcal{B} \subset \exp(X)$ generates an inclusion hyperspace

$$\mathrm{cl}_{\exp(X)}\{A \in \exp(X) \colon \exists B \in \mathcal{B} \text{ with } B \subset A\}$$

denoted by $\langle \mathcal{B} \rangle$.[1] In this case $\mathcal{B}$ is called a *base* of $\mathcal{F} = \langle \mathcal{B} \rangle$. An inclusion hyperspace $\langle x \rangle$ generated by a singleton $\{x\}$, $x \in X$, is called *principal.*

If X is discrete, then each monotone family in $\exp(X)$ is an inclusion hyperspace, see [4].

Denote by $G(X)$ the space of all inclusion hyperspaces with the topology generated by the subbase

$$U^+ = \{\mathcal{A} \in G(X) \colon \exists B \in \mathcal{A} \text{ with } B \subset U\}, \quad U^- = \{\mathcal{A} \in G(X) \colon \forall B \in \mathcal{A} \quad B \cap U \neq \varnothing\},$$

where U is open in X.

For a T_1-space X the map $\eta X \colon X \to G(X)$, $\eta X(x) = \{F \underset{cl}{\subset} X \colon x \in F\}$, is an embedding (see [4]), so we can identify principal inclusion hyperspaces with elements of the space X.

For a T_1-space X the space $G(X)$ is Hausdorff if and only if the space X is normal, see [4], [9]. In the latter case the map

$$h\colon\ G(X) \to G(\beta X), \quad h(\mathcal{F}) = \mathrm{cl}_{\exp(\beta X)}\{\mathrm{cl}_{\beta X} F \mid F \in \mathcal{F}\},$$

is a homeomorphism, so we can identify the space $G(X)$ with the space $G(\beta X)$ of inclusion hyperspaces over the Stone-Čech compactification βX of the normal space X, see [9]. Thus we reduce the study of inclusion hyperspaces over normal topological spaces to the compact case where this construction is well-studied.

For a (discrete) T_1-space X the space $G(X)$ contains a (discrete and) dense subspace $G^\bullet(X)$ consisting of inclusion hyperspaces with finite support. An inclusion hyperspace $\mathcal{A} \in G(X)$ is defined to have *finite support in* X if $\mathcal{A} = \langle \mathcal{F} \rangle$ for some finite family $\mathcal{F}$ of finite subsets of X.

[1]In [4] the inclusion hyperspace $\langle \mathcal{B} \rangle$ generated by a base $\mathcal{B}$ is denoted by $\overline{\uparrow\mathcal{B}}$.

An inclusion hyperspace $\mathcal{F} \in G(X)$ on a non-compact space X is called *free* if for each compact subset $K \subset X$ and any element $F \in \mathcal{F}$ there is another element $E \in \mathcal{F}$ such that $E \subset F \setminus K$. By $G^\circ(X)$ we shall denote the subset of $G(X)$ consisting of free inclusion hyperspaces. By [4], for a normal locally compact space X the subset $G^\circ(X)$ is closed in $G(X)$. In the simplest case of a countable discrete space $X = \mathbb{N}$ free inclusion hyperspaces (called semifilters) on $X = \mathbb{N}$ have been introduced and intensively studied in [1].

1.2. Inclusion hyperspaces in the category of compacta. The construction of the space of inclusion hyperspaces is functorial and monadic in the category $\mathcal{C}omp$ of compact Hausdorff spaces and their continuous map, see [13]. To complete G to a functor on $\mathcal{C}omp$ observe that each continuous map $f\colon X \to Y$ between compact Hausdorff spaces induces a continuous map $Gf\colon G(X) \to G(Y)$ defined by

$$Gf(\mathcal{A}) = \langle f(\mathcal{A})\rangle = \{B \underset{cl}{\subset} Y\colon B \supset f(A) \text{ for some } A \in \mathcal{A}\}$$

for $\mathcal{A} \in G(X)$. The map Gf is well-defined and continuous, and G is a functor in the category $\mathcal{C}omp$ of compact Hausdorff spaces and their continuous maps, see [13]. By Proposition 2.3.2 [13], this functor is weakly normal in the sense that it is continuous, monomorphic, epimorphic and preserves intersections, singletons, the empty set and weight of infinite compacta.

Since the functor G preserves monomorphisms, for each closed subspace A of a compact Hausdorff space X the inclusion map $i\colon A \to X$ induces a topological embedding $Gi\colon G(A) \to G(X)$. So we can identify $G(A)$ with a subspace of $G(X)$. Now for each inclusion hyperspace $\mathcal{A} \in G(X)$ we can consider the support of $\mathcal{A}$

$$\operatorname{supp} \mathcal{A} = \bigcap\{A \underset{cl}{\subset} X\colon \mathcal{A} \in G(A)\}$$

and conclude that $\mathcal{A} \in G(\operatorname{supp} \mathcal{A})$ because G preserves intersections, see [13, §2.4].

Next, we consider the monadic properties of the functor G. We recall that a functor $T\colon \mathcal{C}omp \to \mathcal{C}omp$ is *monadic* if it can be completed to a monad $\mathbb{T} = (T, \eta, \mu)$ where $\eta\colon \mathrm{Id} \to T$ and $\mu\colon T^2 \to T$ are natural transformations (called the unit and multiplication) such that $\mu \circ T(\mu_X) = \mu \circ \mu_{TX}\colon T^3X \to TX$ and $\mu \circ \eta_{TX} = \mu \circ T(\eta_X) = \mathrm{Id}_{TX}$ for each compact Hausdorff space X, see [13].

For the functor G the unit $\eta\colon \mathrm{Id} \to G$ has been defined above while the multiplication $\mu = \{\mu_X\colon G^2X \to G(X)\}$ is defined by the formula

$$\mu_X(\Theta) = \bigcup\{\bigcap \mathcal{M} \mid \mathcal{M} \in \Theta\},\ \Theta \in G^2X.$$

By Proposition 3.2.9 of [13], the triple $\mathbb{G} = (G, \eta, \mu)$ is a monad in $\mathcal{C}omp$.

1.3. Some important subspaces of $\mathbf{G}(\mathbf{X})$. The space $G(X)$ of inclusion hyperspaces contains many interesting subspaces. Let X be a topological space and $k \geq 2$ be a natural number. An inclusion hyperspace $\mathcal{A} \in G(X)$ is defined to be

- *k-linked* if $\cap\mathcal{F} \neq \varnothing$ for any subfamily $\mathcal{F} \subset \mathcal{A}$ with $|\mathcal{F}| \leq k$;
- *centered* if $\cap\mathcal{F} \neq \varnothing$ for any finite subfamily $\mathcal{F} \subset \mathcal{A}$;
- *a filter* if $A_1 \cap A_2 \in \mathcal{A}$ for all sets $A_1, A_2 \in \mathcal{A}$;
- *an ultrafilter* if $\mathcal{A} = \mathcal{A}'$ for any filter $\mathcal{A}' \in G(X)$ containing $\mathcal{A}$;
- *maximal k-linked* if $\mathcal{A} = \mathcal{A}'$ for any k-linked inclusion hyperspace $\mathcal{A}' \in G(X)$ containing $\mathcal{A}$.

By $N_k(X)$, $N_{<\omega}(X)$, and $\mathrm{Fil}(X)$ we denote the subsets of $G(X)$ consisting of k-linked, centered, and filter inclusion hyperspaces, respectively. Also by $\beta(X)$ and $\lambda_k(X)$ we denote the subsets of $G(X)$ consisting of ultrafilters and maximal k-linked inclusion hyperspaces, respectively. The space $\lambda(X) = \lambda_2(X)$ is called the *superextension* of X.

The following diagram describes the inclusion relations between the subspaces N_kX, $N_{<\omega}X$, $\mathrm{Fil}(X)$, λX and βX of $G(X)$ (an arrow $A \to B$ means that A is a subset of B).

$$\begin{array}{ccccccccc}
\mathrm{Fil}(X) & \to & N_{<\omega}X & \to & N_kX & \to & N_2X & \to & G(X) \\
\uparrow & & & & & & \uparrow & & \\
\beta X & \longrightarrow & & & & \longrightarrow & \lambda X & &
\end{array}$$

For a normal space X all the subspaces from this diagram are closed in $G(X)$, see [4].

For a non-compact space X we can also consider the intersections

$$\begin{aligned}
\mathrm{Fil}^\circ(X) =& \mathrm{Fil}(X) \cap G^\circ(X), \quad N^\circ_{<\omega}(X) = N_{<\omega}(X) \cap G^\circ(X), \\
N^\circ_k(X) =& N_k(X) \cap G^\circ(X), \quad \lambda^\circ_k(X) = \lambda_k(X) \cap G^\circ(X), \text{ and} \\
\beta^\circ(X) =& \beta X \cap G^\circ(X) = \beta X \setminus X.
\end{aligned}$$

Elements of those sets will be called free filters, free centered inclusion hyperspaces, free k-linked inclusion hyperspaces, etc. For a normal locally compact space X the subsets $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, $N^\circ_k(X)$, $\lambda^\circ(X) = \lambda^\circ_2(X)$, and $\beta^\circ(X)$ are closed in $G(X)$, see [4]. In contrast, $\lambda^\circ_k(\mathbb{N})$ is not closed in $G(\mathbb{N})$ for $k \geq 3$, see [5].

1.4. The inner algebraic structure of $\mathbf{G(X)}$. In this subsection we discuss the algebraic structure of the space of inclusion hyperspaces $G(X)$ over a topological space X. The space of inclusion hyperspaces $G(X)$ possesses two binary operations $\cup$, $\cap$, and one unary operation

$$\perp\colon G(X) \to G(X), \ \perp\colon \mathcal{F} \mapsto \mathcal{F}^\perp = \{E \underset{cl}{\subset} X : \forall F \in \mathcal{F} \ \ E \cap F \neq \varnothing\}$$

called the transversality map. These three operations are continuous and turn $G(X)$ into a symmetric lattice, see [4].

Definition 1. A *symmetric lattice* is a complete distributive lattice $(L, \vee, \wedge)$ endowed with an additional unary operation $\perp\colon L \to L$, $\perp\colon x \mapsto x^\perp$, that is an involutive anti-isomorphism in the sense that: (i) $x^{\perp\perp} = x$ for all $x \in L$; (ii) $(x \vee y)^\perp = x^\perp \wedge y^\perp$; (iii) $(x \wedge y)^\perp = x^\perp \vee y^\perp$.

The smallest element of the lattice $G(X)$ is the inclusion hyperspace $\{X\}$ while the largest is $\exp(X)$.

For a discrete space X the set $G(X)$ of all inclusion hyperspaces on X is a subset of the double power-set $\mathcal{P}(\mathcal{P}(X))$ (which is a complete distributive lattice) and is closed under the operations of union and intersection (of arbitrary families of inclusion hyperspaces).

Since each inclusion hyperspace is a union of filters and each filter is an intersection of ultrafilters, we obtain the following proposition showing that the lattice $G(X)$ is a rather natural object.

Proposition 1. *For a discrete space X the lattice $G(X)$ coincides with the smallest complete sublattice of $\mathcal{P}(\mathcal{P}(X))$ containing all ultrafilters.*

2. Extending algebraic operations to inclusion hyperspaces. In this section, given a binary (associative) operation $*\colon X \times X \to X$ on a discrete space X we extend this operation to a right-topological (associative) operation on $G(X)$. This can be done in two steps by analogy with the extension of the operation to the Stone-Čech compactification βX of X.

First, for each element $a \in X$ consider the left shift $L_a\colon X \to X$, $L_a(x) = a*x$ and extend it to a continuous map $\overline{L}_a\colon \beta X \to \beta X$ between the Stone-Čech compactifications of X. Next, apply to this extension the functor G to obtain the continuous map $G\overline{L}_a\colon G(\beta X) \to G(\beta X)$. Clearly, for every inclusion hyperspace $\mathcal{F} \in G(\beta X)$ the inclusion hyperspace $G\overline{L}_a(\mathcal{F})$ has a base $\{a * F \mid F \in \mathcal{F}\}$. Thus, we have defined the product $a * \mathcal{F} = G\overline{L}_a(\mathcal{F})$ of the element $a \in X$ and the inclusion hyperspace $\mathcal{F}$.

Further, for each inclusion hyperspace $\mathcal{F} \in G(\beta X) = G(X)$ we can consider the map $R_{\mathcal{F}}\colon X \to G(\beta X)$ defined by the formula $R_{\mathcal{F}}(x) = x * \mathcal{F}$ for every $x \in X$. Extend the map $R_{\mathcal{F}}$ to a continuous map $\overline{R}_{\mathcal{F}}\colon \beta X \to G(\beta X)$ and apply to this extension the functor G to obtain a map $G\overline{R}_{\mathcal{F}}\colon G(\beta X) \to G^2(\beta X)$. Finally, compose the map $G\overline{R}_{\mathcal{F}}$ with the multiplication $\mu X = \mu_G X\colon G^2X \to G(X)$ of the monad $\mathbb{G} = (G, \eta, \mu)$ and obtain a map $\mu_X \circ G\overline{R}_{\mathcal{F}}\colon G(\beta X) \to G(\beta X)$. For an inclusion hyperspace $\mathcal{U} \in G(\beta X)$, the image $\mu_G X \circ G\overline{R}_{\mathcal{F}}(\mathcal{U})$ is called the product of the inclusion hyperspaces $\mathcal{U}$ and $\mathcal{F}$ and is denoted by $\mathcal{U} \circ \mathcal{F}$.

It follows from the continuity of the maps $G\overline{R}_{\mathcal{F}}$ that the extended binary operation on $G(X)$ is continuous with respect to the first argument with the second argument fixed. We are going to show that the operation $\circ$ on $G(X)$ nicely agrees with the lattice structure of $G(X)$ and is associative if so is the operation $*$. Also we shall establish an easy formula

$$\mathcal{U} \circ \mathcal{F} = \Big\langle \bigcup_{x\in U} x * F_x \colon U \in \mathcal{U},\ \{F_x\}_{x\in U} \subset \mathcal{F} \Big\rangle$$

for calculating the product $\mathcal{U} \circ \mathcal{F}$ of two inclusion hyperspaces $\mathcal{U}, \mathcal{F}$. We start with necessary definitions.

Definition 2. Let $\star\colon G(X) \times G(X) \to G(X)$ be a binary operation on $G(X)$. We shall say that $\star$ *respects* the lattice structure of $G(X)$ if for any $\mathcal{U}, \mathcal{V}, \mathcal{W} \in G(X)$ and $a \in X$

1. $(\mathcal{U} \cup \mathcal{V}) \star \mathcal{W} = (\mathcal{U} \star \mathcal{W}) \cup (\mathcal{V} \star \mathcal{W})$; 2. $(\mathcal{U} \cap \mathcal{V}) \star \mathcal{W} = (\mathcal{U} \star \mathcal{W}) \cap (\mathcal{V} \star \mathcal{W})$;
3. $a \star (\mathcal{V} \cup \mathcal{W}) = (a \star \mathcal{V}) \cup (a \star \mathcal{W})$; 4. $a \star (\mathcal{V} \cap \mathcal{W}) = (a \star \mathcal{V}) \cap (a \star \mathcal{W})$.

Definition 3. We will say that a binary operation $\star\colon G(X) \times G(X) \to G(X)$ is *right-topological* if

- for any $\mathcal{U} \in G(X)$ the right shift $R_{\mathcal{U}}\colon G(X) \to G(X)$, $R_{\mathcal{U}}\colon \mathcal{F} \mapsto \mathcal{F} \star \mathcal{U}$, is continuous;
- for any $a \in X$ the left shift $L_a\colon G(X) \to G(X)$, $L_a\colon \mathcal{F} \mapsto a \star \mathcal{F}$, is continuous.

The following uniqueness theorem will be used to find an equivalent description of the induced operation on $G(X)$.

Theorem 1. *Let $\star, \circ\colon G(X) \times G(X) \to G(X)$ be two right-topological binary operations that respect the lattice structure of $G(X)$. These operations coincide if and only if they coincide on the product $X \times X \subset G(X) \times G(X)$.*

Proof. It is clear that if these operations coincide on $G(X) \times G(X)$, then they coincide on the product $X \times X$ identified with a subset of $G(X) \times G(X)$. We recall that each point $x \in X$ is identified with the ultrafilter $\langle x \rangle$ generated by x.

Now assume conversely that $x \star y = x \circ y$ for any two points $x, y \in X \subset G(X)$. First we check that $a \star \mathcal{F} = a \circ \mathcal{F}$ for any $a \in X$ and $\mathcal{F} \in G(X)$. Since the left shifts $\mathcal{F} \mapsto a \star \mathcal{F}$ and $\mathcal{F} \mapsto a \circ \mathcal{F}$ are continuous, it suffices to establish the equality $a \star \mathcal{F} = a \circ \mathcal{F}$ for inclusion hyperspaces $\mathcal{F}$ having finite support in X (because the set $G^\bullet(X)$ of all such inclusion hyperspaces is dense in $G(X)$, see [4]). Any such a hyperspace $\mathcal{F}$ is generated by a finite family of finite subsets of X.

If $\mathcal{F}=\langle F\rangle$ is generated by a single finite subset $F=\{a_1,\dots,a_n\}\subset X$, then $\mathcal{F}=\bigcap_{i=1}^n\langle a_i\rangle$ is the finite intersection of principal ultrafilters, and hence

$$\langle a\rangle\star\mathcal{F}=\langle a\rangle\star\bigcap_{i=1}^n\langle a_i\rangle=\bigcap_{i=1}^n\langle a\rangle\star\langle a_i\rangle=\bigcap_{i=1}^n\langle a\rangle\circ\langle a_i\rangle=\langle a\rangle\circ\bigcap_{i=1}^n\langle a_i\rangle=\langle a\rangle\circ\mathcal{F}.$$

If $\mathcal{F}=\langle F_1,\dots,F_n\rangle$ is generated by finite family of finite sets, then $\mathcal{F}=\bigcup_{i=1}^n\langle F_i\rangle$ and we can use the preceding case to prove that

$$\langle a\rangle\star\mathcal{F}=\langle a\rangle\star\bigcup_{i=1}^n\langle F_i\rangle=\bigcup_{i=1}^n\langle a\rangle\star\langle F_i\rangle=\bigcup_{i=1}^n\langle a\rangle\circ\langle F_i\rangle=\langle a\rangle\circ\bigcup_{i=1}^n\langle F_i\rangle=\langle a\rangle\circ\mathcal{F}.$$

Now fixing any inclusion hyperspace $\mathcal{U}\in G(X)$ by a similar argument one can prove the equality $\mathcal{F}\star\mathcal{U}=\mathcal{F}\circ\mathcal{U}$ for all inclusion hyperspaces $\mathcal{F}\in G^\bullet(X)$ having finite support in X. Finally, using the density of $G^\bullet(X)$ in $G(X)$ and the continuity of right shifts $\mathcal{F}\mapsto\mathcal{F}\circ\mathcal{U}$ and $\mathcal{F}\mapsto\mathcal{F}\star\mathcal{U}$ one can establish the equality $\mathcal{F}\star\mathcal{U}=\mathcal{F}\circ\mathcal{U}$ for all inclusion hyperspaces $\mathcal{F}\in G(X)$. □

The above theorem will be applied to show that the operation $\circ\colon G(X)\times G(X)\to G(X)$ induced by the operation $*\colon X\times X\to X$ coincides with the operation $\star\colon G(X)\times G(X)\to G(X)$ defined by the formula

$$\mathcal{U}\star\mathcal{V}=\langle\bigcup_{x\in U}x*V_x\colon U\in\mathcal{U},\ \{V_x\}_{x\in U}\subset\mathcal{V}\rangle$$

for $\mathcal{U},\mathcal{V}\in G(X)$.

First we establish some properties of the operation $\star$.

Proposition 2. *The operation $\star$ commutes with the transversality operation in the sense that $(\mathcal{U}\star\mathcal{V})^\perp=\mathcal{U}^\perp\star\mathcal{V}^\perp$ for any $\mathcal{U},\mathcal{V}\in G(X)$.*

Proof. To prove that $\mathcal{U}^\perp\star\mathcal{V}^\perp\subset(\mathcal{U}\star\mathcal{V})^\perp$, take any element $A\in\mathcal{U}^\perp\star\mathcal{V}^\perp$. We should check that A intersects each set $B\in\mathcal{U}\star\mathcal{V}$. Without loss of generality, the sets A and B are of the basic form:

$$A=\bigcup_{x\in F}x*G_x\quad\text{for some sets } F\in\mathcal{U}^\perp \text{ and } \{G_x\}_{x\in F}\subset\mathcal{V}^\perp$$

and

$$B=\bigcup_{x\in U}x*V_x\quad\text{for some sets } U\in\mathcal{U} \text{ and } \{V_x\}_{x\in U}\subset\mathcal{V}.$$

Since $U\in\mathcal{U}$ and $F\in\mathcal{U}^\perp$, the intersection $F\cap U$ contains some point x. For this point x the sets $V_x\in\mathcal{V}$ and $G_x\in\mathcal{V}^\perp$ are well-defined and their intersection $V_x\cap G_x$ contains some point y. Then the intersection $A\cap B$ contains the point $x*y$ and hence is not empty, which proves that $A\in(\mathcal{U}\star\mathcal{V})^\perp$.

To prove that $(\mathcal{U}\star\mathcal{V})^\perp\subset\mathcal{U}^\perp\star\mathcal{V}^\perp$, fix a set $A\in(\mathcal{U}\star\mathcal{V})^\perp$. We claim that the set

$$F=\{x\in X\colon x^{-1}A\in\mathcal{V}^\perp\}$$

belongs to $\mathcal{U}^\perp$ (here $x^{-1}A=\{y\in X\colon x*y\in A\}$). Assuming conversely that $F\notin\mathcal{U}^\perp$, we would find a set $U\in\mathcal{U}$ with $F\cap U=\varnothing$. By the definition of F, for each $x\in U$ the set $x^{-1}A\notin\mathcal{V}^\perp$ and thus we can find a set $V_x\in\mathcal{V}$ with empty intersection $V_x\cap x^{-1}A$. By the definition of the product $\mathcal{U}\star\mathcal{V}$, the set $B=\bigcup_{x\in U}x*V_x$ belongs to $\mathcal{U}\star\mathcal{V}$ and hence intersects the set A. Consequently, $x*y\in A$ for some $x\in U$ and $y\in V_x$. The inclusion $x*y\in A$ implies that $y\in x^{-1}A\subset X\setminus V_x$, which is a contradiction proving that $F\in\mathcal{U}^\perp$. Then the sets $A\supset\bigcup_{x\in F}x*x^{-1}A$ belong to $\mathcal{U}^\perp\star\mathcal{V}^\perp$. □

Proposition 3. *The equality $(\mathcal{U}\cap\mathcal{V})\star\mathcal{W}=(\mathcal{U}\star\mathcal{W})\cap(\mathcal{V}\star\mathcal{W})$ holds for any $\mathcal{U},\mathcal{V},\mathcal{W}\in G(X)$.*

Proof. It is easy to show that $(\mathcal{U}\cap\mathcal{V})\star\mathcal{W}\subset(\mathcal{U}\star\mathcal{W})\cap(\mathcal{V}\star\mathcal{W})$.

To prove the reverse inclusion, fix a set $F \in (\mathcal{U} \star \mathcal{W}) \cap (\mathcal{V} \star \mathcal{W})$. Then

$$F \supset \bigcup_{x \in U} x * W'_x \text{ and } F \supset \bigcup_{y \in V} y * W''_y$$

for some $U \in \mathcal{U}$, $\{W'_x\}_{x \in U} \subset \mathcal{W}$, and $V \in \mathcal{V}$, $\{W''_y\}_{y \in V} \subset \mathcal{W}$. Since $\mathcal{U}, \mathcal{V}$ are inclusion hyperspaces, $U \cup V \in \mathcal{U} \cap \mathcal{V}$. For each $z \in U \cup V$ let $W_z = W'_z$ if $z \in U$ and $W_z = W''_z$ if $z \notin U$. It follows that $F \supset \bigcup_{z \in U \cup V} z * W_z$ and hence $F \in (\mathcal{U} \cap \mathcal{V}) \star \mathcal{W}$. □

By analogy one can prove

Proposition 4. *For any $\mathcal{U}, \mathcal{V}, \mathcal{W} \in G(X)$ and $a \in X$*

$$a \star (\mathcal{V} \cup \mathcal{W}) = (a \star \mathcal{V}) \cup (a \star \mathcal{W}) \ \text{ and } \ a \star (\mathcal{V} \cap \mathcal{W}) = (a \star \mathcal{V}) \cap (a \star \mathcal{W}).$$

Combining Propositions 2 and 3 we get

Corollary 1. *For any $\mathcal{U}, \mathcal{V}, \mathcal{W} \in G(X)$ we get $(\mathcal{U} \cup \mathcal{V}) \star \mathcal{W} = (\mathcal{U} \star \mathcal{W}) \cup (\mathcal{V} \star \mathcal{W})$.*

Proof. Indeed, $(\mathcal{U} \cup \mathcal{V}) \star \mathcal{W} = \big(((\mathcal{U} \cup \mathcal{V}) \star \mathcal{W})^\perp\big)^\perp = ((\mathcal{U} \cup \mathcal{V})^\perp \star \mathcal{W}^\perp)^\perp = ((\mathcal{U}^\perp \cap \mathcal{V}^\perp) \star \mathcal{W}^\perp)^\perp =$
$= ((\mathcal{U}^\perp \star \mathcal{W}^\perp) \cap (\mathcal{V}^\perp \star \mathcal{W}^\perp))^\perp = (\mathcal{U}^\perp \star \mathcal{W}^\perp)^\perp \cup (\mathcal{V}^\perp \star \mathcal{W}^\perp)^\perp = (\mathcal{U} \star \mathcal{W}) \cup (\mathcal{V} \star \mathcal{W})$. □

Proposition 5. *The operation $\star\colon G(X) \times G(X) \to G(X)$, $\mathcal{U} \star \mathcal{V} = \langle \bigcup_{x \in U} x * V_x \colon U \in \mathcal{U},\ \{V_x\}_{x \in U} \subset \mathcal{V} \rangle$, respects the lattice structure of $G(X)$ and is right-topological.*

Proof. Propositions 3, 4 and Corollary 1 imply that the operation $\star$ respects the lattice structure of $G(X)$.

So it remains to check that the operation $\star$ is right-topological. First we check that for any $\mathcal{U} \in G(X)$ the right shift $R_\mathcal{U}\colon G(X) \to G(X)$, $R_\mathcal{U}\colon \mathcal{F} \mapsto \mathcal{F} \star \mathcal{U}$, is continuous.

Fix any inclusion hyperspaces $\mathcal{F}, \mathcal{U} \in G(X)$ and let W^+ be a sub-basic neighborhood of their product $\mathcal{F} \star \mathcal{U}$. Find sets $F \in \mathcal{F}$ and $\{U_x\}_{x \in F} \subset \mathcal{U}$ such that $\bigcup_{x \in F} x * U_x \subset W$. Then F^+ is a neighborhood of $\mathcal{F}$ with $F^+ \star \mathcal{U} \subset W^+$.

Now assume that $\mathcal{F} \star \mathcal{U} \in W^-$ for some $W \subset X$. Observe that for any inclusion hyperspace $\mathcal{V} \in G(X)$ we get the equivalences $\mathcal{V} \in W^- \Leftrightarrow W \in \mathcal{V}^\perp \Leftrightarrow \mathcal{V}^\perp \in W^+$. Consequently, $\mathcal{F} \star \mathcal{U} \in W^-$ is equivalent to $\mathcal{F}^\perp \star \mathcal{U}^\perp = (\mathcal{F} \star \mathcal{U})^\perp \in W^+$. The preceding case yields a neighborhood $O(\mathcal{F}^\perp)$ such that $O(\mathcal{F}^\perp) \star \mathcal{U}^\perp \in W^+$. Now the continuity of the transversality operation implies that $O(\mathcal{F}^\perp)^\perp$ is a neighborhood of $\mathcal{F}$ with $O(\mathcal{F}^\perp)^\perp \star \mathcal{U} \in W^-$.

Finally, we prove that for every $a \in X$ the left shift $L_a\colon G(X) \to G(X)$, $L_a\colon \mathcal{F} \mapsto a \star \mathcal{F}$, is continuous. Given a sub-basic open set $W^+ \subset G(X)$ note that $L_a^{-1}(W^+)$ is open because $L_a^{-1}(W^+) = (a^{-1}W)^+$ where $a^{-1}W = \{x \in X \colon a * x \in W\}$. On the other hand, $a \star \mathcal{F} \in W^-$ is equivalent to $a \star \mathcal{F}^\perp = (a \star \mathcal{F})^\perp \in (W^-)^\perp = W^+$ which implies that the preimage $L_a^{-1}(W^-) = (L_a(W^+))^\perp$ is also open. □

The operation $\circ$ has the same properties.

Proposition 6. *The operation $\circ\colon G(X) \times G(X) \to G(X)$, $\mathcal{U} \circ \mathcal{V} = \mu_G X \circ G\overline{R}_\mathcal{F}(\mathcal{U})$ respects the lattice structure of $G(X)$ and is right-topological.*

Proof. For any $\mathcal{U} \in G(X)$ the right shift $R_\mathcal{U} = \mu_{G(X)} \circ G\overline{R}_\mathcal{U}\colon G(X) \to G(X)$, $R_\mathcal{U}\colon \mathcal{F} \mapsto \mathcal{F} \circ \mathcal{U}$ is continuous being the composition of continuous maps. Next for any $a \in X$ and $\mathcal{F} \in G(X)$ we have $L_a(\mathcal{F}) = a \circ \mathcal{F} = \mu_G X(\langle a \rangle * \mathcal{F}) = \mu_G X(\langle a * \mathcal{F} \rangle) = a * \mathcal{F} = G\overline{L}_a(\mathcal{F})$ and the map $L_a \equiv G\overline{L}_a$ is continuous.

It is known (and easy to verify) that the multiplication $\mu_{G(X)}\colon G^2(X) \to G(X)$ is a lattice homomorphism in the sense that $\mu_{G(X)}(\mathcal{U} \cup \mathcal{V}) = \mu_{G(X)}(\mathcal{U}) \cup \mu_G X(\mathcal{V})$ and $\mu_{G(X)}(\mathcal{U} \cap \mathcal{V}) =$

$= \mu_{G(X)}(\mathcal{U}) \cap \mu_{G(X)}(\mathcal{V})$ for any $\mathcal{U},\mathcal{V} \in G(X)$. Then for any $\mathcal{U},\mathcal{V},\mathcal{W} \in G(X)$ and $a \in X$ we get $(\mathcal{U}\cup\mathcal{V})\circ\mathcal{W} = \mu_{G(X)}\circ G\overline{R}_{\mathcal{W}}(\mathcal{U}\cup\mathcal{V}) = \mu_{G(X)}(G\overline{R}_{\mathcal{W}}(\mathcal{U})\cup G\overline{R}_{\mathcal{W}}(\mathcal{V})) = \mu_{G(X)}\circ G\overline{R}_{\mathcal{W}}(\mathcal{U})\cup$ $\cup\mu_{G(X)}\circ G\overline{R}_{\mathcal{W}}(\mathcal{V}) = (\mathcal{U}\circ\mathcal{W})\cup(\mathcal{U}\circ\mathcal{W})$ and similarly $(\mathcal{U}\cap\mathcal{V})\circ\mathcal{W} = (\mathcal{U}\circ\mathcal{W})\cap(\mathcal{U}\circ\mathcal{W})$.

Note that $a\circ\mathcal{W} = \mu_{G(X)}(G\overline{R}_{\mathcal{W}}(\langle a\rangle) = \langle\overline{R}_{\mathcal{W}}(\{a\})\rangle = \langle\overline{R}_{\mathcal{W}}(a)\rangle = a*\mathcal{W}$ for any $a\in X$. Consequently, $a\circ(\mathcal{V}\cup\mathcal{W}) = a*(\mathcal{V}\cup\mathcal{W}) = (a*\mathcal{V})\cup(a*\mathcal{W}) = (a\circ\mathcal{V})\cup(a\circ\mathcal{W})$ and similarly $a\circ(\mathcal{V}\cap\mathcal{W}) = (a\circ\mathcal{V})\cap(a\circ\mathcal{W})$. □

Since both operations $\circ$ and $\star$ are right-topological and respect the lattice structure of $G(X)$ we may apply Theorem 1 to get

Corollary 2. *For any binary operation $*\colon X\times X\to X$ the operations $\circ$ and $\star$ on $G(X)$ coincide. Consequently, for any inclusion hyperspaces $\mathcal{U},\mathcal{V}\in G(X)$ their product $\mathcal{U}\circ\mathcal{V}$ is the inclusion hyperspace*

$$\langle\bigcup_{x\in U} x*V_x\colon U\in\mathcal{U},\ \{V_x\}_{x\in U}\subset\mathcal{V}\rangle = \{A\subset X\colon \{x\in X\colon x^{-1}A\in\mathcal{V}\}\in\mathcal{U}\}.$$

Having the apparent description of the operation $\circ$ we can establish its associativity.

Proposition 7. *If the operation $*$ on X is associative, then so is the induced operation $\circ$ on $G(X)$.*

Proof. It is necessary to show that $(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W} = \mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$ for any inclusion hyperspaces $\mathcal{U},\mathcal{V},\mathcal{W}$. Take any subset $A\in(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W}$ and choose a set $B\in\mathcal{U}\circ\mathcal{V}$ such that $A\supset\bigcup_{z\in B} z*W_z$ for some family $\{W_z\}_{z\in B}\subset\mathcal{W}$. Next, for the set $B\in\mathcal{U}\circ\mathcal{V}$ choose a set $U\in\mathcal{U}$ such that $B\supset\bigcup_{x\in U}x*V_x$ for some family $\{V_x\}_{x\in U}\subset\mathcal{V}$. It is clear that for each $x\in U$ and $y\in V_x$ the product $x*y$ is in B and hence W_{x*y} is defined. Consequently, $\bigcup_{y\in V_x} y*W_{x*y}\in\mathcal{V}\circ\mathcal{W}$ for all $x\in U$ and hence $\bigcup_{x\in U}x*(\bigcup_{y\in V_x}y*W_{x*y})\in\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$. Since $\bigcup_{x\in U}\bigcup_{y\in V_x}x*y*W_{x*y}\subset A$, we get $A\in\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$. This proves the inclusion $(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W}\subset\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$.

To prove the reverse inclusion, fix a set $A\in\mathcal{U}\circ(\mathcal{V}\circ\mathcal{W})$ and choose a set $U\in\mathcal{U}$ such that $A\supset\bigcup_{x\in U}x*B_x$ for some family $\{B_x\}_{x\in U}\subset\mathcal{V}\circ\mathcal{W}$. Next, for each $x\in U$ find a set $V_x\in\mathcal{V}$ such that $B_x\supset\bigcup_{y\in V_x}y*W_{x,y}$ for some family $\{W_{x,y}\}_{y\in V_x}\subset\mathcal{W}$. Let $Z=\bigcup_{x\in U}x*V_x$. For each $z\in Z$ we can find $x\in U$ and $y\in V_x$ such that $z=x*y$ and put $W_z=W_{x,y}$. Then $Z\in\mathcal{U}\circ\mathcal{V}$ and $\bigcup_{z\in Z}z*W_z\in(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W}$. Taking into account $\bigcup_{z\in Z}z*W_z\subset\bigcup_{x\in U}\bigcup_{y\in V_x}x*y*W_{x,y}\subset A$, we conclude $A\in(\mathcal{U}\circ\mathcal{V})\circ\mathcal{W}$. □

3. Homomorphisms of semigroups of inclusion hyperspaces. Let us observe that our construction of extension of a binary operation for X to $G(X)$ works well both for associative and non-associative operations. Let us recall that a set S endowed with a binary operation $*\colon X\times X\to X$ is called a *groupoid*. If the operation is associative, then X is called a *semigroup*. In the preceding section we have shown that for each groupoid (semigroup) X the space $G(X)$ is a groupoid (semigroup) with respect to the extended operation.
A map $h\colon X_1\to X_2$ between two groupoids $(X_1,*_1)$ and $(X_2,*_2)$ is called a *homomorphism* if $h(x*_1 y) = h(x)*_2 h(y)$ for all $x,y\in X_1$.

Proposition 8. *For any homomorphism $h\colon X_1\to X_2$ between groupoids $(X_1,*_1)$ and $(X_2,*_2)$ the induced map $Gh\colon G(X_1)\to G(X_2)$ is a homomorphism of the groupoids $G(X_1)$, $G(X_2)$.*

Proof. Given two inclusion hyperspaces $\mathcal{U},\mathcal{V}\in G(X_1)$ observe that

$$Gh(\mathcal{U}\circ_1\mathcal{V})=Gh(\langle\bigcup_{x\in U}x*_1V_x\colon U\in\mathcal{U},\ \{V_x\}_{x\in U}\subset\mathcal{V}\rangle)=$$
$$=\langle h(\bigcup_{x\in U}x*_1V_x)\colon U\in\mathcal{U},\ \{V_x\}_{x\in U}\subset\mathcal{V}\rangle)=\langle\bigcup_{x\in U}h(x)*_2h(V_x)\colon U\in\mathcal{U},\ \{V_x\}_{x\in U}\subset\mathcal{V}\rangle=$$
$$\langle\bigcup_{x\in h(U)}x*_2h(V_x)\colon U\in\mathcal{U},\ \{h(V_x)\}_{x\in U}\subset Gh(\mathcal{V})\rangle=$$
$$=\langle h(U)\colon U\in\mathcal{U}\rangle\circ_2\langle h(V)\colon V\in\mathcal{V}\rangle=Gh(\mathcal{U})\circ_2Gh(\mathcal{V}).$$

□

Reformulating Proposition 2 in terms of homomorphisms, we obtain

Proposition 9. *For any groupoid X the transversality map $\perp\colon G(X)\to G(X)$ is a homomorphism of the groupoid $G(X)$.*

4. Subgroupoids of G(X). In this section we shall show that for a groupoid X endowed with the discrete topology all (topologically) closed subspaces of $G(X)$ introduced in Section 1.3 are subgroupoids of $G(X)$. A subset A of a groupoid $(X,*)$ is called a *subgroupoid* of X if $A*A\subset A$, where $A*A=\{a*b\colon a,b\in A\}$.

We assume that $*\colon X\times X\to X$ is a binary operation on a discrete space X and $\circ\colon G(X)\times G(X)\to G(X)$ is the extension of $*$ to $G(X)$. Applying Proposition 9 we obtain

Proposition 10. *If S is a subgroupoid of $G(X)$, then $S^\perp$ is a subgroupoid of $G(X)$ too.*

Our next propositions can be easily derived from Corollary 2.

Proposition 11. *The sets $\mathrm{Fil}(X)$, $N_{<\omega}(X)$ and $N_k(X)$, $k\ge2$, are subgroupoids in $G(X)$.*

Proposition 12. *The Stone-Čech extension βX and the superextension λX both are closed subgroupoids in $G(X)$.*

Proof. The superextension λX is a subgroupoid of $G(X)$ being the intersection $\lambda(X)=N_2(X)\cap(N_2(X))^\perp$ of two subgroupoids of $G(X)$. By analogy, $\beta X=\mathrm{Fil}(X)\cap\lambda(X)$ is a subgroupoid of $G(X)$. □

Remark 1. In contrast to λX for $k\ge3$ the subset $\lambda_k(X)$ need not be a subgroupoid of $G(X)$. For example, for the cyclic group $\mathbb{Z}_5=\{0,1,2,3,4\}$ the subset $\lambda_3(\mathbb{Z}_5)$ of $G(\mathbb{Z}_5)$ contains a maximal 3-linked system $\mathcal{L}=\langle\{0,1,2\},\{0,1,4\},\{0,2,4\},\{1,2,4\}\rangle$ whose square $\mathcal{L}*\mathcal{L}=\langle\{1,2,4,5\},\{0,2,3,4\},\{0,1,3,4\},\{0,1,2,4\},\{0,1,2,3\}\rangle$ is not maximal 3-linked.

By a direct application of Corollary 2 we can also prove

Proposition 13. *The set $G^\bullet(X)$ of all inclusion hyperspaces with finite support is a subgroupoid in $G(X)$.*

Finally we find conditions on the operation $*$ guaranteeing that the subset $G^\circ(X)$ of free inclusion hyperspaces is a subgroupoid of $G(X)$.

Proposition 14. *Assume that for each $b\in X$ there is a finite subset $F\subset X$ such that for each $a\in X\setminus F$ the set $a^{-1}b=\{x\in X\colon a*x=b\}$ is finite. Then the set $G^\circ(X)$ is a closed subgroupoid in $G(X)$ and consequently, $\mathrm{Fil}^\circ(X)$, $\lambda^\circ(X)$, $\beta^\circ(X)$ all are closed subgroupoids in $G(X)$.*

Proof. Take two free inclusion hyperspaces $\mathcal{A}, \mathcal{B} \in G(X)$ and a subset $C \in \mathcal{A} \circ \mathcal{B}$. We should prove that $C \setminus K \in \mathcal{A} \circ \mathcal{B}$ for each compact subset $K \subset X$. Without loss of generality, the set C is of basic form: $C = \bigcup_{a \in A} a * B_a$ for some set $A \in \mathcal{A}$ and some family $\{B_a\}_{a \in A} \subset \mathcal{B}$.

Since X is discrete, the set K is finite. It follows from our assumption that there is a finite set $F \subset X$ such that for every $a \in X \setminus F$ the set $a^{-1}K = \{x \in X : a * x \in K\}$ is finite. The hyperspace $\mathcal{A}$, being free, contains the set $A' = A \setminus F$. By the same reason, for each $a \in A'$ the hyperspace $\mathcal{B}$ contains the set $B_a' = B_a \setminus a^{-1}K$. Since $C \setminus K \supset \bigcup_{a \in A'} a * B_a' \in \mathcal{A} \circ \mathcal{B}$, we conclude that $C \setminus K \in \mathcal{A} \circ \mathcal{B}$. □

Remark 2. If X is a semigroup, then $G(X)$ is a semigroup and all the subgroupoids considered above are closed subsemigroups in $G(X)$. Some of them are well-known in Semigroup Theory. In particular, so is the semigroup βX of ultrafilter and $\beta^{\circ}(X) = \beta X \setminus X$ of free ultrafilters. The semigroup $\mathrm{Fil}(X)$ contains an isomorphic copy of the global semigroup of X, which is the hyperspace $\exp(X)$ endowed with the semigroup operation $A * B = \{a * b : a \in A,\ b \in B\}$.

5. Ideals and zeros in G(X). A non-empty subset I of a groupoid $(X, *)$ is called an *ideal* (resp. *right ideal*, *left ideal*) if $I * X \cup X * I \subset I$ (resp. $I * X \subset I$, $X * I \subset I$). An element O of a groupoid $(X, *)$ is called a *zero* (resp. *left zero*, *right zero*) in X if $\{O\}$ is an ideal (resp. right ideal, left ideal) in X. Each right or left zero $z \in X$ is an *idempotent* in the sense that $z * z = z$.

For a groupoid $(X, *)$ right zeros in $G(X)$ admit a simple description. We define an inclusion hyperspace $\mathcal{A} \in G(X)$ to be *shift-invariant* if for every $A \in \mathcal{A}$ and $x \in X$ the sets $x * A$ and $x^{-1}A = \{y \in X : x * y \in A\}$ belong to $\mathcal{A}$.

Proposition 15. *An inclusion hyperspace $\mathcal{A} \in G(X)$ is a right zero in $G(X)$ if and only if $\mathcal{A}$ is shift-invariant.*

Proof. Assuming that an inclusion hyperspace $\mathcal{A} \in G(X)$ is shift-invariant, we shall show that $\mathcal{B} \circ \mathcal{A} = \mathcal{A}$ for every $\mathcal{B} \in G(X)$. Take any set $F \in \mathcal{B} \circ \mathcal{A}$ and find a set $B \in \mathcal{B}$ and a family $\{A_x\}_{x \in B} \subset \mathcal{A}$ such that $\bigcup_{x \in B} x * A_x \subset F$. Since $\mathcal{A} \in G(X)$ is shift-invariant, $\bigcup_{x \in B} x * A_x \in \mathcal{A}$ and thus $F \in \mathcal{A}$. This proves the inclusion $\mathcal{B} \circ \mathcal{A} \subset \mathcal{A}$. On the other hand, for every $F \in \mathcal{A}$ and every $x \in X$ we get $x^{-1}F \in \mathcal{A}$ and thus $F \supset \bigcup_{x \in X} x * x^{-1}F \in \mathcal{B} \circ \mathcal{A}$. This shows that $\mathcal{A}$ is a right zero of the semigroup $G(X)$.

Now assume that $\mathcal{A}$ is a right zero of $G(X)$. Observe that for every $x \in X$ the equality $\langle x \rangle \circ \mathcal{A} = \mathcal{A}$ implies $x * A \in \mathcal{A}$ for every $A \in A$.

One the other hand, the equality $\{X\} \circ \mathcal{A} = \mathcal{A}$ implies that for every $A \in \mathcal{A}$ there is a family $\{A_x\}_{x \in X} \subset \mathcal{A}$ such that $\bigcup_{x \in X} x * A_x \subset A$. Then for every $x \in X$ the set $x^{-1}A = \{z \in X : x * z \in A\} \supset A_x \in \mathcal{A}$ belongs to $\mathcal{A}$ witnessing that $\mathcal{A}$ is shift-invariant. □

By $\overset{\leftrightarrow}{G}(X)$ we *denote* the set of shift-invariant inclusion hyperspaces in $G(X)$. Proposition 15 implies that $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \overset{\leftrightarrow}{G}(X)$. This means that $\overset{\leftrightarrow}{G}(X)$ is a rectangular semigroup.

We recall that a semigroup $(S, *)$ is called *rectangular* (or else a *semigroup of right zeros*) if $x * y = y$ for all $x, y \in S$.

Proposition 16. *The set $\overset{\leftrightarrow}{G}(X)$ is closed in $G(X)$, is a rectangular subsemigroup of the groupoid $G(X)$ and is closed complete sublattice of the lattice $G(X)$ invariant under the* *transversality map. Moreover, if $\overset{\leftrightarrow}{G}(X)$ is non-empty, then it is a left ideal that lies in each right ideal of $G(X)$.*

Proof. If $\mathcal{A} \in G(X) \setminus \overset{\leftrightarrow}{G}(X)$, then there exists $x \in X$ and $A \in \mathcal{A}$ such that $x * A \notin \mathcal{A}$ or $x^{-1}A \notin \mathcal{A}$. Then

$$O(\mathcal{A}) = \{\mathcal{A}' \in G(X)\colon A \in \mathcal{A}' \text{ and } (x * A \notin \mathcal{A}' \text{ or } x^{-1}A \notin \mathcal{A})\}$$

is an open neighborhood of $\mathcal{A}$ missing the set $\overset{\leftrightarrow}{G}(X)$ and witnessing that the set $\overset{\leftrightarrow}{G}(X)$ is closed in $G(X)$.

Since $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \overset{\leftrightarrow}{G}(X)$, the set $\overset{\leftrightarrow}{G}(X)$ is a rectangular subsemigroup of the groupoid $G(X)$.

To show that $\overset{\leftrightarrow}{G}(X)$ is invariant under the transversality operation, note that for every $\mathcal{A} \in G(X)$ and $\mathcal{Z} \in \overset{\leftrightarrow}{G}(X)$ we get $\mathcal{A} \circ \mathcal{Z}^\perp = (\mathcal{A}^\perp \circ \mathcal{Z})^\perp = \mathcal{Z}^\perp$ which means that $\mathcal{Z}^\perp$ is a right zero in $G(X)$ and thus belongs to $\overset{\leftrightarrow}{G}(X)$ according to Proposition 15.

To show that $\overset{\leftrightarrow}{G}(X)$ is a complete sublattice of $G(X)$ it is necessary to check that $\overset{\leftrightarrow}{G}(X)$ is closed under arbitrary unions and intersections. It is trivial to check that arbitrary union of shift-invariant inclusion hyperspaces is shift-invariant, which means that $\bigcup_{\alpha\in A} \mathcal{Z}_\alpha \in \overset{\leftrightarrow}{G}(X)$ for any family $\{\mathcal{Z}_\alpha\}_{\alpha\in A} \subset \overset{\leftrightarrow}{G}(X)$. Since $\overset{\leftrightarrow}{G}(X)$ is closed under the transversality operation we also get

$$\bigcap_{\alpha\in A} \mathcal{Z}_\alpha = \big(\bigcup_{\alpha\in A} \mathcal{Z}_\alpha^\perp\big)^\perp \in \overset{\leftrightarrow}{G}(X)^\perp = \overset{\leftrightarrow}{G}(X).$$

If $\overset{\leftrightarrow}{G}(X)$ is not empty, then it is a left ideal in $G(X)$ because it consists of right zeros. Now take any right ideal I in $G(X)$ and fix any element $\mathcal{R} \in I$. Then for every $\mathcal{Z} \in \overset{\leftrightarrow}{G}(X)$ we get $\mathcal{Z} = \mathcal{R} \circ \mathcal{Z} \in I$ which yields $\overset{\leftrightarrow}{G}(X) \subset I$. □

Proposition 17. *If X is a semigroup and $\overset{\leftrightarrow}{G}(X)$ is not empty, then $\overset{\leftrightarrow}{G}(X)$ is the minimal ideal of $G(X)$.*

Proof. In light of the preceding proposition, it suffices to check that $\overset{\leftrightarrow}{G}(X)$ is a right ideal. Take any inclusion hyperspaces $\mathcal{A} \in \overset{\leftrightarrow}{G}(X)$ and $\mathcal{B} \in G(X)$ and take any set $F \in \mathcal{A} \circ \mathcal{B}$. We need to show that the sets $x * F$ and $x^{-1}F$ belong to $\mathcal{A} \circ \mathcal{B}$. Without loss of generality, F is of the basic form:

$$F = \bigcup_{a\in A} a * B_a$$

for some set $A \in \mathcal{A}$ and some family $\{B_a\}_{a\in A} \subset \mathcal{B}$. The associativity of the semigroup operation on S implies that

$$x * F = \bigcup_{a\in A} x * a * B_a = \bigcup_{z\in x*A} z * B_{a(z)} \in \mathcal{A} \circ \mathcal{B}$$

where $a(z) \in \{a \in A\colon x * a = z\}$ for $z \in x * A$. To see that $x^{-1}F \in \mathcal{A}$ observe that the set $A' = \bigcup_{z\in x^{-1}A} z * B_{xz}$ belongs to $\mathcal{A}$ and each point $a' \in A'$ belongs to the set $z * B_{xz}$ for some $z \in x^{-1}A$. Then $x * a' \in x * z * B_{xz} \subset F$ and hence $\mathcal{A} \ni A' \subset x^{-1}F$, which yields the desired inclusion $x^{-1}F \in \mathcal{A}$. □

Now we find conditions on the binary operation $*\colon X \times X \to X$ guaranteeing that the set $\overset{\leftrightarrow}{G}(X)$ is not empty. By $\min GX = \{X\}$ and $\max GX = \{A \subset X\colon A \neq \varnothing\}$ we denote the minimal and maximal elements of the lattice $G(X)$.

Proposition 18. *For a groupoid $(X,*)$ the following conditions are equivalent:*
1) $\min GX \in \overset{\leftrightarrow}{G}(X)$; 2) $\max GX \in \overset{\leftrightarrow}{G}(X)$; 3) for each $a,b \in X$ the equation $a*x=b$ has a solution $x\in X$.

Proof. (1) $\Rightarrow$ (3) Assuming that $\min GX \in \overset{\leftrightarrow}{G}(X)$ and applying Proposition 15 observe that for every $a\in X$ the equation $\langle a\rangle \circ \{X\} = \{X\}$ implies that for every $b\in X$ the equation $a*x=b$ has a solution.

(3) $\Rightarrow$ (1) If for every $a,b\in X$ the equation $a*x=b$ has a solution, then $a*X=X$ and hence $\mathcal{F}\circ\{X\}=\{X\}$ for all $\mathcal{F}\in G(X)$. This means that $\{X\}=\min G(X)$ is a right zero in $G(X)$ and hence belongs to $\overset{\leftrightarrow}{G}(X)$ according to Proposition 15.

(2) $\Rightarrow$ (3) Assume that $\max G(X) \in \overset{\leftrightarrow}{G}(X)$ and take any points $a,b\in X$. Since $\langle a\rangle\circ \max G(X) = \max G(X) \ni \{b\}$, there is a non-empty set $X_a\in\max G(X)$ with $a*X_a\subset\{b\}$. Then any $x\in X_a$ is a solution of $a*x=b$.

(3) $\Rightarrow$ (2) Assume that for every $a,b\in X$ the equation $a*x=b$ has a solution. To show that $\mathcal{F}\circ\max G(X)=\max G(X)$ it suffices to check that $\max G(X)\subset\mathcal{F}\circ\max G(X)$. Take any set $B\in\max G(X)$ and any set $F\in\mathcal{F}$. For every $a\in F$ find a point $x_a\in X$ with $a*x_a\in B$. Then the sets $\bigcup_{a\in F} a*\{x_a\}\subset B$ belong to $\mathcal{F}\circ\max G(X)$, which yields the desired inclusion $\max G(X)\subset\mathcal{F}\circ\max G(X)$. □

By analogy we can establish a similar description of zeros and the minimal ideal in the semigroup $G^\circ(X)$ of free inclusion hyperspaces.

Proposition 19. *Assume that $(X,*)$ is an infinite groupoid such that for each $b\in X$ there is a finite subset $F\subset X$ such that for each $a\in X\setminus F$ the set $a^{-1}b=\{x\in X: a*x=b\}$ is finite and not empty. Then: 1) $G^\circ(X)$ is a closed subgroupoid of $G(X)$; 2) $G^\circ(X)$ is a left ideal in $G(X)$ provided if for each $a,b\in X$ the set $a^{-1}b$ is finite; 3) the set $\overset{\leftrightarrow}{G}{}^\circ(X)=\overset{\leftrightarrow}{G}(X)\cap G^\circ(X)$ of shift-invariant free inclusion hyperspaces is the minimal ideal in $G^\circ(X)$; 4) the set $\overset{\leftrightarrow}{G}{}^\circ(X)$ is a rectangular subsemigroup of the groupoid $G(X)$ and is closed complete sublattice of the lattice $G(X)$ invariant under the transversality map.*

Remark 3. It follows from Propositions 16 and 19 that the minimal ideals of the semigroups $G(\mathbb{Z})$ and $G^\circ(X)$ are closed. In contrast, the minimal ideals of the semigroups $\beta\mathbb{Z}$ and $\beta^\circ\mathbb{Z}=\beta\mathbb{Z}\setminus\mathbb{Z}$ are not closed, see [8, §4.4].

Minimal left ideals of the semigroup $\beta^\circ(\mathbb{Z})$ play an important role in Combinatorics of Numbers, see [8]. We believe that the same will happen for the semigroup $\lambda^\circ(\mathbb{Z})$. The following proposition implies that minimal left ideals of $\lambda^\circ(\mathbb{Z})$ contain no ultrafilter!

Proposition 20. *If a groupoid X admits a homomorphism $h\colon X\to\mathbb{Z}_3$ such that for every $y\in\mathbb{Z}_3$ the preimage $h^{-1}(y)$ is not empty (is infinite) then each minimal left ideal I of $\lambda(X)$ (of $\lambda^\circ(X)$) is disjoint from $\beta(X)$.*

Proof. It follows that the induced map $\lambda h\colon \lambda(X)\to\lambda(\mathbb{Z}_3)$ is a surjective homomorphism. Consequently, $\lambda h(I)$ is a minimal left ideal in $\lambda(\mathbb{Z}_3)$. Now observe that $\lambda(\mathbb{Z}_3)$ consists of four maximal linked inclusion hyperspaces. Besides three ultrafilters there is a maximal linked inclusion hyperspace $\mathcal{L}_\Delta=\langle\{0,1\},\{0,2\},\{1,2\}\rangle$ where $\mathbb{Z}_3=\{0,1,2\}$. One can check that $\{\mathcal{L}_\Delta\}$ is a zero of the semigroup $\lambda(\mathbb{Z}_3)$. Consequently, $\lambda(h)(I)=\{\mathcal{L}_\Delta\}$, which implies that $I\cap\beta(X)=\varnothing$.

Now assume that for every $y \in \mathbb{Z}_3$ the preimage $h^{-1}(y)$ is infinite. We claim that $\lambda h(\lambda^\circ(X)) = \lambda(\mathbb{Z}_3)$. Take any maximal linked inclusion hyperspace $\mathcal{L} \in \lambda(\mathbb{Z}_3)$. If $\mathcal{L}$ is an ultrafilter supported by a point $y \in \mathbb{Z}_3$, then we can take any free ultrafilter $\mathcal{U}$ on X containing the infinite set $h^{-1}(y)$ and observe that $\lambda h(\mathcal{U}) = \mathcal{L}$. It remains to consider the case $\mathcal{L} = \mathcal{L}_\triangle$. Fix free ultrafilters $\mathcal{U}_0, \mathcal{U}_1, \mathcal{U}_2$ on X containing the sets $h^{-1}(0)$, $h^{-1}(1)$, $h^{-1}(2)$, respectively. Then $\mathcal{L} = (\mathcal{U}_0 \cap \mathcal{U}_1) \cup (\mathcal{U}_0 \cap \mathcal{U}_2) \cup (\mathcal{U}_1 \cap \mathcal{U}_2)$ is a free maximal linked inclusion hyperspace whose image $\lambda h(\mathcal{L}_X) = \mathcal{L}_\triangle$.

Given any minimal left ideal $I \subset \lambda^\circ(X)$ we obtain that the image $\lambda h(I)$, being a minimal left ideal of $\lambda(\mathbb{Z}_3)$ coincides with $\{\mathcal{L}_\triangle\}$ and is disjoint from $\beta(\mathbb{Z}_3)$. Consequently, I is disjoint from $\beta(X)$. □

6. The center of G(X). In this section we describe the structure of the center of the groupoid $G(X)$ for each (quasi)group X. By definition, the *center* of a groupoid X is the set

$$C = \{x \in X\colon \forall y \in X \;\; xy = yx\}.$$

A groupoid X is called a *quasigroup* if for every $a, b \in X$ the system of equations $a * x = b$ and $y * a = b$ has a unique solution $(x, y) \in X \times X$. It is clear that each group is a quasigroup. On the other hand, there are many examples of quasigroups, not isomorphic to groups, see [10], [3].

Theorem 2. *Let X be a quasigroup. If an inclusion hyperspace $\mathcal{C} \in G(X)$ commutes with the extremal elements $\max G(X)$ and $\min G(X)$ of $G(X)$, then $\mathcal{C}$ is a principal ultrafilter.*

Proof. By Proposition 18, the inclusion hyperspaces $\max G(X)$ and $\min G(X)$ are right zeros in $G(X)$ and thus $\max G(X) \circ \mathcal{C} = \mathcal{C} \circ \max G(X) = \max G(X)$ and $\min G(X) \circ \mathcal{C} = \mathcal{C} \circ \min G(X) = \min G(X)$. It follows that for every $b \in X$ we get $\{b\} \in \max G(X) = \max G(X) \circ \mathcal{C}$, which means that $a * C \subset \{b\}$ for some $C \in \mathcal{C}$ and some $a \in X$. Since the equation $a * y = b$ has a unique solution $y \in X$, the set C is a singleton, say $C = \{c\}$. It remains to prove that $\mathcal{C}$ coincides with the principal ultrafilter $\langle c \rangle$ generated by c. Assuming the converse, we would conclude that $X \setminus \{c\} \in \mathcal{C}$. By our hypothesis, the equation $y * c = c$ has a unique solution $y_0 \in X$. Since the equation $y_0 * x = c$ has a unique solution $x = c$, $y_0 * (X \setminus \{c\}) \subset X \setminus \{c\}$. Letting $C_x = \{c\}$ for all $x \in X \setminus \{y_0\}$ and $C_x = X \setminus \{c\}$ for $x = y_0$, we conclude that $X \setminus \{c\} \supset \bigcup_{x \in X} x * C_x \in \min G(X) \circ \mathcal{C} = \mathcal{C} \circ \min G(X) = \min G(X)$, which is not possible. □

Corollary 3. *For any quasigroup X the center of the groupoid $G(X)$ coincides with the center of X.*

Proof. If an inclusion hyperspace $\mathcal{C}$ belongs to the center of the groupoid $G(X)$, then $\mathcal{C}$ is a principal ultrafilter generated by some point $c \in X$. Since $\mathcal{C}$ commutes with all the principal ultrafilters, c commutes with all elements of X and thus c belongs to the center of X.

Conversely, if $c \in X$ belongs to the center of X, then for every inclusion hyperspace $\mathcal{F} \in G(X)$ we get

$$c \circ \mathcal{F} = \{c * F\colon F \in \mathcal{F}\} = \{F * c\colon F \in \mathcal{F}\} = \mathcal{F} \circ c,$$

which means that (the principal ultrafilter generated by) c belongs to the center of the groupoid $G(X)$. □

Remark 4. It is interesting to note that for any group X the center of the semigroup βX also coincides with the center of the group X, see Theorem 6.54 of [8].

Problem 1. *Given a group X describe the centers of the subsemigroups $\lambda(X)$, $\mathrm{Fil}(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \ge 2$ of the semigroup $G(X)$.*

Problem 2. *Given an infinite group X describe the centers of the semigroups $G^\circ(X)$, $\lambda^\circ(X)$, $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, and $N^\circ_k(X)$, $k \ge 2$.* (By Theorem 6.54 of [8], the center of the semigroup of free ultrafilters $\beta^\circ(X)$ is empty).

7. The topological center of G(X). In this section we describe the topological center of $G(X)$. By the *topological center* of a groupoid X endowed with a topology we understand the set $\Lambda(X)$ consisting of all points $x \in X$ such that the left and right shifts

$$l_x\colon X \to X,\ \ l_x\colon z \mapsto xz, \quad \text{and} \quad r_x\colon X \to X,\ \ r_x\colon z \mapsto zx$$

both are continuous.

Since all right shifts on $G(X)$ are continuous, the topological center of the groupoid $G(X)$ consists of all inclusion hyperspaces $\mathcal{F}$ with continuous left shifts $l_{\mathcal{F}}$.

We recall that $G^\bullet(X)$ stands for the set of inclusion hyperpsaces with finite support.

Theorem 3. *For a quasigroup X the topological center of the groupoid $G(X)$ coincides with $G^\bullet(X)$.*

Proof. By Proposition 5, the topological center $\Lambda(GX)$ of $G(X)$ contains all principal ultrafilters and is a sublattice of $G(X)$. Consequently, $\Lambda(GX)$ contains the sublatttice $G^\bullet(X)$ of $G(X)$ generated by X.

Next, we show that each inclusion hyperspace $\mathcal{F} \in \Lambda(GX)$ has finite support and hence belongs to $G^\bullet(X)$. By Theorem 9.1 of [4], this will follow as soon as we check that both $\mathcal{F}$ and $\mathcal{F}^\perp$ have bases consisting of finite sets.

Take any set $F \in \mathcal{F}$, choose any point $e \in X$, and consider the inclusion hyperspace $\mathcal{U} = \{U \subset X\colon e \in F * U\}$. Since for every $f \in F$ the equation $f * u = e$ has a solution in X, we conclude that $\{e\} \in \mathcal{F} \circ \mathcal{U}$ and by the continuity of the left shift $l_{\mathcal{F}}$, there is an open neighborhood $\mathcal{O}(\mathcal{U})$ of $\mathcal{U}$ such that $\{e\} \in \mathcal{F} \circ \mathcal{A}$ for all $\mathcal{A} \in \mathcal{O}(\mathcal{U})$. Without loss of generality, the neighborhood $\mathcal{O}(\mathcal{U})$ is of basic form

$$\mathcal{O}(\mathcal{U}) = U_1^+ \cap \dots \cap U_n^+ \cap V_1^- \cap \dots \cap V_m^-$$

for some sets $U_1, \dots, U_n \in \mathcal{U}$ and $V_1, \dots, V_m \in \mathcal{U}^\perp$. Take any finite set $A \subset F^{-1}e = \{x \in X\colon e \in F * x\}$ intersecting each set U_i, $i \le n$, and consider the inclusion hyperspace $\mathcal{A} = \langle A\rangle^\perp$. It is clear that $\mathcal{A} \subset U_1^+ \cap \dots \cap U_n^+$. Since each set V_j, $j \le m$, contains the set $F^{-1}e \supset A$, we get also that $\mathcal{A} \in V_1^- \cap \dots \cap V_m^-$. Then $\mathcal{F} \circ \mathcal{A} \ni \{e\}$ and hence there is a set $E \in \mathcal{F}$ and a family $\{A_x\}_{x \in E} \subset \mathcal{A}$ with $\bigcup_{x \in E} x * A_x \subset \{e\}$. It follows that the set $E \subset eA^{-1} = \{x \in X\colon \exists a \in A \text{ with } xa = e\}$ is finite. We claim that $E \subset F$. Indeed, take any point $x \in E$ and find a point $a \in A$ with $x * a = e$. Since $A \subset F^{-1}e$, there is a point $y \in F$ with $e = y * a$. Hence $xa = ya$ and the right cancellativity of X yields $x = y \in F$. Therefore, using the continuity of the left shift $l_{\mathcal{F}}$, for every $F \in \mathcal{F}$ we have found a finite subset $E \in \mathcal{F}$ with $E \subset F$. This means that $\mathcal{F}$ has a base of finite sets.

The continuity of the left shift $l_{\mathcal{F}}$ and Proposition 2 imply the continuity of the left shift $l_{\mathcal{F}^\perp}$. Repeating the preceding argument, we can prove that the inclusion hyperspace $\mathcal{F}^\perp$ has a base of finite sets too. Finally, applying Theorem 9.1 of [4], we conclude that $\mathcal{F} \in G^\bullet(X)$. □

Problem 3. *Given an infinite group X describe the topological center of the subsemigroups $\lambda(X)$, $\mathrm{Fil}(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \ge 2$, of the semigroup $G(X)$. Is it true that the topological center of any subsemigroup $S \subset G(X)$ containing $\beta(X)$ coincides with $S \cap G^\bullet(X)$?* (This is true for the subsemigroups $S = G(X)$ (see Theorem 3) and $S = \beta(X)$, see Theorems 4.24 and 6.54 of [8]).

Problem 4. *Given an infinite group X describe the topological centers of the semigroups $G^\circ(X)$, $\lambda^\circ(X)$, $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, and $N^\circ_k(X)$, $k \geq 2$.* (It should be mentioned that the topological center of the semigroup $\beta^\circ(X)$ of free ultrafilters is empty [12]).

8. Left cancelable elements of $\mathbf{G(X)}$. An element a of a groupoid S is called *left cancelable* (resp. *right cancelable*) if for any points $x, y \in S$ the equation $ax = ay$ (resp. $xa = ya$) implies $x = y$. In this section we characterize left cancelable elements of the groupoid $G(X)$ over a quasigroup X.

Theorem 4. *Let X be a quasigroup. An inclusion hyperspace $\mathcal{F} \in G(X)$ is left cancelable in the groupoid $G(X)$ if and only if $\mathcal{F}$ is a principal ultrafilter.*

Proof. Assume that $\mathcal{F}$ is left cancelable in $G(X)$. First we show that $\mathcal{F}$ contains some singleton. Assuming the converse, take any point $x_0 \in X$ and note that $F * (X \setminus \{x_0\}) = X$ for any $F \in \mathcal{F}$. To see that this equality holds, take any point $a \in X$, choose two distinct points $b, c \in F$ and find solutions $x, y \in X$ of the equation $b*x = a$ and $c*y = a$. Since X is right cancellative, $x \neq y$. Consequently, one of the points x or y is distinct from x_0. If $x \neq x_0$, then $a = b*x \in F*(X\setminus\{x_0\})$. If $y \neq x_0$, then $a = c*y \in F*(X\setminus\{x_0\})$. Now for the inclusion hyperspace $\mathcal{U} = \langle X \setminus \{x_0\}\rangle \neq \min G(X)$, we get $\mathcal{F} \circ \mathcal{U} = \min G(X) = \mathcal{F} \circ \min G(X)$, which contradicts the choice of $\mathcal{F}$ as a left cancelable element of $G(X)$.

Thus $\mathcal{F}$ contains some singleton $\{c\}$. We claim that $\mathcal{F}$ coincides with the principal ultrafilter generated by c. Assuming the converse, we would conclude that $X \setminus \{c\} \in \mathcal{F}$. Let $\mathcal{A} = \langle X \setminus \{c\}\rangle^\perp$ be the inclusion hyperspace consisting of subsets that meet $X \setminus \{c\}$. It is clear that $\mathcal{A} \neq \max G(X)$. We claim that $\mathcal{F} \circ \mathcal{A} = \max G(X) = \mathcal{F} \circ \max G(X)$ which will contradict the left cancelability of $\mathcal{F}$. Indeed, given any singleton $\{a\} \in \max G(X)$, consider two cases: if $a \neq c*c$, then we can find a unique $x \in X$ with $c*x = a$. Since $x \neq c$, $\{x\} \in \mathcal{A}$ and hence $\{a\} = c * \{x\} \in \mathcal{F} \circ \mathcal{A}$. If $a = c*c$, then for every $y \in X \setminus \{c\}$ we can find $a_y \in X$ with $y * a_y = a$ and use the left cancelativity of X to conclude that $a_y \neq c$ and hence $\{a_y\} \in \mathcal{A}$. Then $\{a\} = \bigcup_{y \in X\setminus\{c\}} y * \{a_y\} \in \mathcal{F} \circ \mathcal{A}$.

Therefore $\mathcal{F} = \langle c\rangle$ is a principal ultrafilter, which proves the "only if" part of the theorem. To prove the "if" part, take any principal ultrafilter $\langle x\rangle$ generated by a point $x \in X$. We claim that two inclusion hyperspaces $\mathcal{F}, \mathcal{U} \in G(X)$ are equal provided $\langle x\rangle \circ \mathcal{F} = \langle x\rangle \circ \mathcal{U}$. Indeed, given any set $F \in \mathcal{F}$ observe that $x * F \in \langle x\rangle \circ \mathcal{F} = \langle x\rangle \circ \mathcal{U}$ and hence $x * F = x * U$ for some $U \in \mathcal{U}$. The left cancelativity of X implies that $F = U \in \mathcal{U}$, which yields $\mathcal{F} \subset \mathcal{U}$. By the same argument we can also check that $\mathcal{U} \subset \mathcal{F}$. □

Problem 5. *Given an (infinite) group X describe left cancelable elements of the subsemigroups $\lambda(X)$, $\mathrm{Fil}(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \geq 2$ (and $G^\circ(X)$, $\lambda^\circ(X)$, $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, $N^\circ_k(X)$, for $k \geq 2$).*

Remark 5. Theorem 4 implies that for a countable Abelian group X the set of left cancelable elements in $G(X)$ coincides with X. On the other hand, the set of (left) cancelable elements of $\beta(X)$ contains an open dense subset of $\beta^\circ(X)$, see Theorem 8.34 of [8].

9. Right cancelable elements of $\mathbf{G(X)}$. As we saw in the preceding section, for any quasigroup X the groupoid $G(X)$ contains only trivial left cancelable elements. For right cancelable elements the situation is much more interesting. First note that the right cancelativity of an inclusion hyperspace $\mathcal{F} \in G(X)$ is equivalent to the injectivity of the map $\mu_X \circ G\overline{R}_{\mathcal{F}}\colon G(X) \to G(X)$ considered at the begining of Section 2. We recall that $\mu_X\colon G^2(X) \to G(X)$ is the multiplication of the monad $\mathbb{G} = (G, \mu, \eta)$ while $\overline{R}_{\mathcal{F}}\colon \beta X \to G(X)$ is the Stone-Čech extension of the right shift $R_{\mathcal{F}}\colon X \to G(X)$, $R_{\mathcal{F}}\colon x \mapsto x * \mathcal{F}$. The map $\overline{R}_{\mathcal{F}}$ certainly is not injective if $R_{\mathcal{F}}$ is not an embedding, which is equivalent to the discreteness of the indexed set $\{x * \mathcal{F}\colon x \in X\}$ in $G(X)$. Therefore we have obtained the following necessary condition for the right cancelability.

Proposition 21. *Let X be a groupoid. If an inclusion hyperspace $\mathcal{F} \in G(X)$ is right cancelable in $G(X)$, then the indexed set $\{x\mathcal{F}\colon x \in X\}$ is discrete in $G(X)$ in the sense that each point $x\mathcal{F}$ has a neighborhood $O(x\mathcal{F})$ containing no other points $y\mathcal{F}$ with $y \in X \setminus \{x\}$.*

Next we give a sufficient condition of the right cancelability.

Proposition 22. *Let X be a groupoid. An inclusion hyperspace $\mathcal{F} \in G(X)$ is right cancelable in $G(X)$ provided there is a family of sets $\{S_x\}_{x\in X} \subset \mathcal{F} \cap \mathcal{F}^\perp$ such that $xS_x \cap yS_y = \varnothing$ for any distinct $x, y \in X$.*

Proof. Assume that $\mathcal{A} \circ \mathcal{F} = \mathcal{B} \circ \mathcal{F}$ for two inclusion hyperspaces $\mathcal{A}, \mathcal{B} \in G(X)$. First we show that $\mathcal{A} \subset \mathcal{B}$. Take any set $A \in \mathcal{A}$ and observe that the set $\bigcup_{a\in A} aS_a$ belongs to $\mathcal{A} \circ \mathcal{F} = \mathcal{B} \circ \mathcal{F}$. Consequently, there is a set $B \in \mathcal{B}$ and a family of sets $\{F_b\}_{b\in B} \subset \mathcal{F}$ such that $\bigcup_{b\in B} bF_b \subset \bigcup_{a\in A} aS_a$. It follows from $S_b \in \mathcal{F}^\perp$ that $F_b \cap S_b$ is not empty for every $b \in B$. Since the sets aS_a and bS_b are disjoint for different $a, b \in X$, the inclusion $\bigcup_{b\in B} b(F_b \cap S_b) \subset \bigcup_{b\in B} bF_b \subset \bigcup_{a\in A} aS_a$ implies $B \subset A$ and hence $A \in \mathcal{B}$.

By analogy we can prove that $\mathcal{B} \subset \mathcal{A}$. □

Propositions 21 and 22 imply the following characterization of right cancelable ultrafilters in $G(X)$ generalizing a known characterization of right cancelable elements of the semigroups βX, see [8, 8.11].

Corollary 4. *Let X be a countable groupoid. For an ultrafilter $\mathcal{U}$ on X the following conditions are equivalent: 1) $\mathcal{U}$ is right cancelable in $G(X)$; 2) $\mathcal{U}$ is right cancelable in βX; 3) the indexed set $\{x\mathcal{U}\colon x \in X\}$ is discrete in βX; 4) there is an indexed family of sets $\{U_x\}_{x\in X} \subset \mathcal{U}$ such that for any distinct $x, y \in X$ the shifts $x\,U_x$ and $y\,U_y$ are disjoint.*

This characterization can be used to show that for any countable group X the semigroup $\beta^\circ(X)$ of free ultrafilters contains an open dense subset of right cancelable ultrafilters, see [8, 8.10]. It turns out that a similar result can be proved for the semigroup $G^\circ(X)$.

Proposition 23. *For any countable quasigroup, the groupoid $G^\circ(X)$ contains an open dense subset of right cancelable free inclusion hyperspaces.*

Proof. Let $X = \{x_n\colon n \in \omega\}$ be an injective enumeration of the countable quasigroup X. Given a free inclusion hyperspace $\mathcal{F} \in G^\circ(X)$ and a neighborhood $O(\mathcal{F})$ of $\mathcal{F}$ in $G^\circ(X)$, we should find a non-empty open subset in $O(\mathcal{F})$. Without loss of generality, the neighborhood $O(\mathcal{F})$ is of basic form $O(\mathcal{F}) = G^\circ(X) \cap U_0^+ \cap \dots \cap U_n^+ \cap U_{n+1}^- \cap \dots \cap U_{m-1}^-$ for some sets $U_1, \dots, U_{m-1}$ of X. Those sets are infinite because $\mathcal{F}$ is free. We are going to construct an infinite set $C = \{c_n\colon n \in \omega\} \subset X$ that has infinite intersection with the sets U_i, $i < m$, and such that for any distinct $x, y \in X$ the intersection $xC \cap yC$ is finite. The points c_k,

$k \in \omega$, composing the set C will be chosen by induction to satisfy the following conditions:
- $c_k \in U_j$ where $j = k \mod m$;
- c_k does not belong to the finite set $F_k = \{z \in X : \exists i, j \le k\ \exists l < k\ \ (x_i z = x_j c_l)\}$.

It is clear that the so-constructed set $C = \{c_k : k \in \omega\}$ has infinite intersection with each set U_i, $i < m$. Since X is right cancellative, for any $i < j$ the set $Z_{i,j} = \{z \in X : x_i z = x_j z\}$ is finite. Now the choice of the points c_k for $k > j$ implies that $x_i C \cap x_j C \subset x_i(Z_{i,j} \cup \{c_l : l \le j\})$ is finite.

Now let $\mathcal{C}$ be the free inclusion hyperspace on X generated by the sets C and $U_0, \dots, U_n$. It is clear that $\mathcal{C} \in O(\mathcal{F})$ and $C \in \mathcal{C} \cap \mathcal{C}^\perp$. Consider the open neighborhood of $\mathcal{C}$ in $G^\circ(X)$

$$O(\mathcal{C}) = O(\mathcal{F}) \cap C^+ \cap (C^+)^\perp.$$

We claim that each inclusion hyperspace $\mathcal{A} \in O(\mathcal{C})$ is right cancelable in $G(X)$. This will follow from Proposition 22 as soon as we construct a family of sets $\{A_i\}_{i\in\omega} \in \mathcal{A} \cap \mathcal{A}^\perp$ such that $x_i A_i \cap x_j A_j = \varnothing$ for any numbers $i < j$. The sets A_i, $i \in \omega$, can be defined by the formula $A_k = C \setminus F_k$ where $F_k = \{c \in C : \exists i < k \text{ with } x_k c = x_i C\}$ is finite by the choice of the set C. □

Problem 6. *Given an (infinite) group X describe right cancelable elements of the subsemigroups $\lambda(X)$, $\mathrm{Fil}(X)$, $N_{<\omega}(X)$, $N_k(X)$, $k \ge 2$ ($\lambda^\circ(X)$, $\mathrm{Fil}^\circ(X)$, $N^\circ_{<\omega}(X)$, $N^\circ_k(X)$, for $k \ge 2$).*

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