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On King type modification of (p,q)-Lupaş Bernstein operators with improved estimates

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This paper aims to modify the (p,q)-Lupaş Bernstein operators using King's technique and to establish convergence results of these operators by using of modulus of continuity and Lipschitz class functions. Some approximation results for this new sequence of operators are obtained. It has been shown that the convergence rate of King type modification is better than the (p,q)-Lupaş Bernstein operators. King type modification of operators also provide better error estimation within some subinterval of $[0,1]$ in comparison to (p,q)-Lupaş Bernstein operators. In the last section, some graphs and tables provided for simulation purposes using MATLAB (R2015a).

Key words and phrases: post-quantum calculus, (p,q)-Lupaş Bernstein operator, modulus of continuity, King type approximation, error estimate.

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1 Introduction

The applications of post quantum calculus ((p,q)-calculus) emerged as a new area in the field of constructive approximation and computer-aided geometric design. The concept of (p,q)-calculus in approximation theory has been introduced by M. Mursaleen et al. in [23], where they constructed post quantum analogue of Bernstein operators. After this, several researchers started working in this area.

K. Khan and D.K. Lobiyal [18] recently defined post quantum analogue of Lupaş Bernstein operators $L^m_{p,q} : C[0,1] \to C[0,1]$ (an extension of q-analogue of Lupaş Bernstein operators [12]) as follows

$$L^m_{p,q}(f;u) = L_m(f,p,q;u) = \sum_{j=0}^{m} \frac{f\left(\frac{p^{m-j}\,[j]_{p,q}}{[m]_{p,q}}\right) \left[\begin{array}{c} m \\ j \end{array} \right]_{p,q} p^{\frac{(m-j)(m-j-1)}{2}} q^{\frac{j(j-1)}{2}} \, u^j \, (1-u)^{m-j}}{\prod\limits_{j=1}^{m} \{p^{j-1}(1-u)+q^{j-1}u\}}, \tag{1}$$

for any $p,q > 0$ and $m \geq 2$.

They used basis functions of this post quantum analogue of Lupaş q-Bernstein operators to construct Bèzier curves and surfaces which add parameter flexibility in approximation and mimics the shape of control polygon. For basic definitions and related results about post quantum calculus we refer the reader to [1,18].

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We recall the results for post quantum analogue of Lupaş q-Bernstein operators [18] for Chebyshev system of test functions.

Lemma 1 ([18])**.** *The following equalities are true:*

(1) $L_m(1,p,q;u) = 1$;

(2) $L_m(t,p,q;u) = u$;

(3) $L_m(t^2,p,q;u) = u^2 + \frac{u(1-u)p^{m-1}}{[m]_{p,q}} - \frac{u^2(p-q)(1-u)}{p(1-u)+qu}\left(1 - \frac{p^{m-1}}{[m]_{p,q}}\right)$.

Here, one can observe that the operator $L_m(t^2;p;q;u)$ does not preserve the quadratic test function. J.P. King [17] established a technique to compute better approximation results for the Bernstein polynomials at the beginning of the 21st century. In this method, these operators approximate every continuous function $f \in C[0,1]$, while preserving the function $e_2(u) = u^2$.

Thus, J.P. King constructed the sequence of (non-trivial) operators preserving the functions e_0 and e_2, where $(e_i = u^i, i = 0,1,2)$ and proved that these operators have a better convergence rate for $0 \le u \le \frac{1}{3}$ than the classical Bernstein polynomials [5].

For classical approximation theory related to positive linear operators we refer the reader to [5] and for quantum calculus to [14,19,20,27]. T. Acar et al. [2–4,7] investigated approximation properties by constructing some operators via post quantum calculus. Q.-B. Cai et al. [8,9] investigated approximation properties by constructing post quantum analogue of Lambda-Bernstein operators. U. Kadak et al. [15] studied (p,q)-Szasz operators involving Brenke type polynomials. A. Wafi and N. Rao [26] studied approximation properties by (p,q)-Bivariate-Bernstein-Chlowdosky operators. M. Mursaleen et al. further analyzed approximation properties by considering operators via post quantum calculus like generalized (p,q)-Bleimann-Butzer-Hahn operators [21], higher order generalization of Bernstein type operators [22] and (p,q)-Lorentz polynomials on a compact disk [24] etc.

In this paper, we have introduced King type modification of post quantum analogue of Lupaş q-Bernstein operators and investigated statistical approximation properties of these operators defined in [12] for $0 < q < p \le 1$. In comparison to King type modification of Phillips type (p,q)-Bernstein operators [13], the calculations involved are more complex and challenging but it has advantage in approximation due to rational basis functions. In the last section, we show that our modification gives us better error estimation than the (p,q)-Lupaş Bernstein operators on some subintervals of $[0,1]$. In case $p = 1$, it reduces to King type modification of q-Lupaş Bernstein operators.

2 Construction of operators

Using King's technique, we will modify (p,q)-Lupaş Bernstein operators (1), which preserve monomials $e_i(u) = u^i$ for $i = 0,2$. For this study, we consider $0 < q < p \le 1$ satisfying the following condition

$$pq([m]_{p,q} - 1) > p^m(p-q) \tag{2}$$

for $m \ge 2$. Here, we define the operators

$$L^{*m}_{p,q}(f;u) = L^*_m(f,p,q;u) = \sum_{j=0}^{m} \frac{f\left(\frac{p^{m-j}\,[j]_{p,q}}{[m]_{p,q}}\right)\left[\begin{array}{c} m \\ j \end{array}\right]_{p,q} p^{\frac{(m-j)(m-j-1)}{2}} q^{\frac{j(j-1)}{2}}\, s_m(u)^j\,(1-s_m(u))^{m-j}}{\prod_{j=1}^{m}\{p^{j-1}(1-s_m(u)) + q^{j-1}s_m(u)\}} \tag{3}$$

for every $f \in C[0,1]$, $u \in [0,1]$ and $m \geq 2$, where $s_m(u) : [0,1] \longrightarrow [0,1]$ are continuous functions. It is easy to observe that these operators $L_m^*(f,p,q;u)$ are positive and linear. One can notice that if we choose $s_m(u) = u$ then it reduces to (p,q)-Lupaş Bernstein operators.

Lemma 2. *$L^*(f,p,q;x)$ satisfies the following properties:*
(1) $L_m^*(e_0,p,q;u) = 1$;
(2) $L_m^*(e_1,p,q;u) = s_m(u)$;
(3) $L_m^*(e_2,p,q;u) = s_m^2(u) + \frac{s_m(x)(1-s_m(u))p^{m-1}}{[m]_{p,q}} - \frac{s_m(u)^2(p-q)(1-s_m(u))}{p(1-s_m(u))+qs_m(u)}\left(1-\frac{p^{m-1}}{[m]_{p,q}}\right)$.

Based on the restriction mentioned in (2), if we take (using King's approach)

$$s_m(u) = \frac{p^m + u^2[m]_{p,q}(p-q)}{2\left(p^{m-1}q - p^m + q^2[m-1]_{p,q}\right)} + \frac{\sqrt{p^{2m} + u^4[m]_{p,q}^2(p-q)^2 + 2u^2[m]_{p,q}\left(2pq([m]_{p,q}-1) - p^m(p-q)\right)}}{2\left(p^{m-1}q - p^m + q^2[m-1]_{p,q}\right)}, \tag{4}$$

then $L_m^*(f,p,q;u)$ preserves monomials, $L_m^*(e_0,p,q;u) = e_0(u) = 1$ and $L_m^*(e_2,p,q;u) = e_2 = u^2$ for $m \geq 2$. Also $0 \leq s_m(u) \leq 1$ for $s_m(u)$ defined in (4).

From the equation (2) we have

$$2pq\left([m]_{p,q}-1\right) - p^m(p-q) > p^m(p-q).$$

Using the above inequality we get

$$p^{2m} + 2u^2[m]_{p,q}p^m(p-q) + u^4[m]_{p,q}^2(p-q)^2 = \left(p^m + u^2[m]_{p,q}(p-q)\right)^2. \tag{5}$$

This imply $s_m(u) \geq 0$ under the condition (2). Since $(1-u)^2 \geq 0$ for $0 \leq u \leq 1$, we have

$$4q^2[m]_{p,q}^2(1-u^2) + 4q[m]_{p,q}(pu^2 - p^m) > 0$$

provided $4q[m]_{p,q}(pu^2 - p^m) > 0$. From the above inequality we get

$$p^2 + 2u^2[m]_{p,q}(2pq([m]_{p,q}-1) - p^m(p-q)) + u^4[m]_{p,q}^2(p-q)^2 \leq \left(2(p^{m-1}q - p^m + q^2[m-1]) + p^m + u^2[m]_{p,q}(p-q)\right)^2. \tag{6}$$

If we use the equation (6) in (4), then we get $s_m(u) \leq 1$.

Remark 1. *If $q \in (0,1)$ and $p \in (q,1]$, one can observe that $\lim_{m\to\infty}[m]_{p,q} = 0$ or $\frac{1}{p-q}$. To get convergence of the operator $L_m^*(f,p,q;u)$, we choose a sequence $q_m \subseteq (0,1)$ and $p_m \subseteq (q_m,1]$ such that $\lim_{m\to\infty} p_m = 1$, $\lim_{m\to\infty} q_m = 1$ and $\lim_{m\to\infty} p_m^m = 1$, $\lim_{n\to\infty} q_m^m = 1$. So we get $\lim_{m\to\infty}[m]_{p_m,q_m} = \infty$.*

Theorem 1. *Let $L_m^*(f,p,q;u)$ be the sequence of operators and the sequences $p = p_m$ and $q = q_m$ satisfy above Remark 1. Then for any function $f \in C[0,1]$ we have*

$$\lim_{n\to\infty}|L_m^*(f,p,q;u_0) - f(u_0)| = 0 \quad \textit{for fixed} \quad u_0 \in [0,1].$$

Proof. We know that $L_m^*(f,p,q;u)$ is a linear positive operator. If we choose the sequences satisfying the Remark 1 together with the Lemma 2, we obtain

$$\lim_{n\to\infty}|L_m^*(e_i,p,q;u_0) - e_i(u_0)| = 0 \quad \text{for} \quad i = 0,1,2.$$

Hence by using Korovkin's theorem the proof is completed. □

3 The convergence rate

Let $f \in C[0,1]$. Then $w(f;\delta)$ is said to be the modulus of continuity of f defined by

$$w(f;\delta) = \sup_{u,t\in[0,1],\ |t-u|<\delta} |f(t)-f(u)|.$$

For all $f \in C[0,1]$, it satisfies the following conditions:

$$\lim_{\delta\to 0} w(f;\delta) = 0,$$

and

$$\big|f(t)-f(u)\big| \leq w(f;\delta)\left(\frac{|t-u|}{\delta}+1\right). \tag{7}$$

It is evident that if we put $\alpha=\beta=0$ in [16], then we get the following convergence rate of the operators (1)

$$\Big|L_m(f,p_m,q_m;u)-f(u)\Big| \leq 2w(f;\delta_m(u))$$

for every $f \in C[0,1]$ and $\delta > 0$, where

$$\delta_m(u) = \sqrt{\left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}}-1\right)u^2 + \frac{p^{m-1}}{[m]_{p,q}}u}. \tag{8}$$

Next, rates of convergence will be computed for the operators $L_m^*(f,p,q;u)$ to $f(u)$ given by (3) by using the modulus of continuity and we further show that error estimation obtained is better than the (p,q)-Lupaş operator given by (1).

Theorem 2. *Let (p_m) and (q_m) be the sequences satisfying Remark 1 for every $m \geq 2$. For fixed $u \in [0,1]$, $f \in C[0,1]$ and $\delta_m > 0$, we have*

$$\Big|L_m^*(f,p_m,q_m;u)-f(u)\Big| \leq 2w^*(f;\delta_m^*(u)),$$

where

$$\left(\delta_m^*(u)\right)^2 = 2u^2 + u\left(\frac{p_m^m+u^2[m]_{p_m,q_m}(p_m-q_m)}{2\left(p_m^{m-1}q_m-p_m^m+q^2[m-1]_{p_m,q_m}\right)} - \frac{\sqrt{p_m^{2m}+u^4[m]_{p_m,q_m}^2(p_m-q_m)^2+2u^2[m]_{p_m,q_m}\left(2p_mq_m([m]_{p_m,q_m}-1)-p_m^m(p_m-q_m)\right)}}{2\left(p_m^{m-1}q_m-p_m^m+q_m^2[m-1]_{p_m,q_m}\right)}\right).$$

Proof. For $f \in C[0,1]$ and by using linearity and positivity of $L_m^*(f,p_m,q_m;u)$, we get

$$\Big|L_m^*(f,p_m,q_m;u)-f(u)\Big| \leq L_m^*\Big(\big|f(t)-f(u)\big|,p_m,q_m;u\Big) \tag{9}$$

for each $m \geq 2$, $m \in \mathbb{N}$ and $u \in [0,1]$. Now using (7) in the inequality (9) we have

$$\Big|L_m^*(f,p_m,q_m;u)-f(u)\Big| \leq w^*\big(f;\delta^*(u)\big)\Big\{1+\frac{1}{\delta^*}\big(L_m^*|t-u|,p_m,q_m:u\big)\Big\} \tag{10}$$

for any $\delta > 0$.

Using the inequality of Cauchy-Schwarz in (10), we get

$$\left|L_m^*(f, p_m, q_m; u) - f(u)\right| \leq w^*\left(f; \delta^*(u)\right)\left\{1 + \frac{1}{\delta^*}\left(L_m^*(t-u)^2, p_m, q_m : u\right)^{\frac{1}{2}}\right\}. \tag{11}$$

On the other hand, we have

$$L_m^*\left((t-u)^2, p_m, q_m; u\right) = 2u^2 + u\left(\frac{p_m^m + u^2[m]_{p_m,q_m}(p_m - q_m)}{2\left(p_m^{m-1}q_m - p_m^m + q_m^2[m-1]_{p_m,q_m}\right)}\right.$$
$$\left. - \frac{\sqrt{p_m^{2m} + u^4[m]_{p_m,q_m}^2(p_m-q_m)^2 + 2u^2[m]_{p_m,q_m}\left(2p_mq_m([m]_{p_m,q_m}-1) - p^m(p_m-q_m)\right)}}{2\left(p_m^{m-1}q_m - p_m^m + q_m^2[m-1]_{p_m,q_m}\right)}\right). \tag{12}$$

For the given sequence satisfying the Remark 1, it is evident that $\lim_{m\to\infty} \frac{1}{[m]_{p,q}} = 0$ and hence

$$\lim_{m\to\infty} \delta_m^*(u) = \lim_{m\to\infty} 2u^2 + u\left(\frac{p_m^m + u^2[m]_{p_m,q_m}(p_m - q_m)}{2\left(p_m^{m-1}q_m - p_m^m + q^2[m-1]_{p_m,q_m}\right)}\right.$$
$$\left. - \frac{\sqrt{p_m^{2m} + u^4[m]_{p_m,q_m}^2(p_m-q_m)^2 + 2u^2[m]_{p_m,q_m}(2p_mq_m([m]_{p_m,q_m}-1) - p_m^m(p_m-q_m))}}{2\left(p_m^{m-1}q_m - p_m^m + q_m^2[m-1]_{p_m,q_m}\right)}\right)$$
$$= \lim_{m\to\infty}\{2u^2 + u(-2u)\} = 0.$$

Thus, using (12) in (11) the proof is completed. Because of $\lim_{n\to\infty} \delta_m^*(u) = 0$, Theorem 2 gives us the pointwise convergence rate of the operators L_m^* to the function $f(u)$. □

If we choose

$$u \leq \frac{p^{m-1}}{2[m]_{p,q}} \quad \text{such that} \quad \delta_m^*(u) \leq \delta_m(u),$$

then the convergence rate given in Theorem 2 is better than the estimate given by (8) for all $m \geq 2$. However, computing exact bound of u from $\delta_m^*(u) \leq \delta_m(u)$, gives us it is complex. Hence we will try to find a bound and take help of MATLAB for simulation purpose to claim better rate of convergence within some subinterval of $\left[0, \frac{p^{m-1}}{2[m]_{p,q}}\right]$.

Indeed, for

$$\delta_m^*(u) \leq \delta_m(u)$$

we have

$$2u^2 + u\left(\frac{p_m + u^2[m]_{p,q}(p-q)}{2\left(p^{m-1}q - p^m + q^2[m-1]_{p,q}\right)}\right.$$
$$\left. - \frac{\sqrt{p^{2m} + u^4[m]_{p,q}^2(p-q)^2 + 2u^2m_{p,q}(2pq(m_{p,q}-1) - p^m(p-q))}}{2\left(p^{m-1}q - p^m + q^2[m-1]_{p,q}\right)}\right)$$
$$\leq \left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}} - 1\right)u^2 + \frac{p^{m-1}}{[m]_{p,q}}u$$

$$\Longrightarrow 2u^2+u\left(\frac{p^m+u^2[m]_{p,q}(p-q)}{2(p^{m-1}q-p^m+q^2[m-1]_{p,q})}-\frac{\sqrt{p^{2m}+u^4[m]_{p,q}^2(p-q)^2+2u^2[m]_{p,q}p^m(p-q)}}{2\left(p^{m-1}q-p^m+q^2[m-1]_{p,q}\right)}\right)$$
$$\leq\left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}}-1\right)u^2+\frac{p^{m-1}}{[m]_{p,q}}u$$
$$\Longrightarrow 2u^2+u(0)\leq\left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}}-1\right)u^2+\frac{p^{m-1}}{[m]_{p,q}}u$$
$$\Longrightarrow 2u^2\leq 0u^2+\frac{p^{m-1}}{[m]_{p,q}}u \quad\Longrightarrow\quad u\leq\frac{p^{m-1}}{2[m]_{p,q}} \quad \text{as} \quad \left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}}-1\right)\leq 0.$$

For all $m\geq 2$ and for u satisfying $\delta_m^*(u)\leq\delta_m(u)$, it provides a bound in which rate of convergence will be better. To verify this, we will do simulation in graphical section.

Now, we can find the convergence rate of our modified operator defined in (3) for the functions belonging to the Lipschitz class.

Let $f\in C[0,1]$ and $0<\rho\leq 1$. We know that $f\in Lip_M(\rho)$ if

$$|f(v)-f(u)|\leq M|v-u|^\rho \qquad \text{for all} \quad u,v\in[0,1]. \tag{13}$$

Theorem 3. *For all $f\in Lip_M(\rho)$ we have*

$$\|L_m(f,p_m,q_m;u)-f(u)\|_{C[0,1]}\;\leq\;M\delta_m^\rho(u),$$

where

$$\delta_m(u)=\sqrt{\left(\frac{q^2}{p(1-u)+qu}\frac{[m-1]_{p,q}}{[m]_{p,q}}-1\right)u^2\;+\;\frac{p^{m-1}}{[m]_{p,q}}u}$$

and M is a positive constant.

Proof. Let $f\in Lip_M(\rho)$ and $0<\rho\leq 1$. By (13) and linearity of $L_n(f,p,q;x)$ we have

$$\left|L_m(f,p,q;u)-f(u)\right|\leq L_m\left(|f(t)-f(u)|,p,q;u\right)\leq ML_m\left(|t-u|^\rho,p,q;u\right).$$

Using the Hölder's inequality with $m=\frac{2}{\rho}$ and $n=\frac{2}{2-\rho}$, we obtain

$$\left|L_m(f,p,q;u)-f(u)\right|\leq M\Big(L_m\left((t-u)^2,p,q;u\right)\Big)^{\frac{\rho}{2}}.$$

If we choose $\delta=\delta_m$, then our proof is completed. □

Theorem 4. *For all $f\in Lip_M(\rho)$, from the condition (2) for $p=p_m$ and $q=q_m$ we obtain*

$$\left\|L_m^*(f,p_m,q_m;u)-f(u)\right\|_{C[0,1]}\;\leq\;M\delta_m^\rho(u),$$

where

$$\delta_m(u)=\sqrt{2u^2+u\left(\frac{p_m^m+u^2m_{p_m,q_m}(p_m-q_m)}{2\left(p_m^{m-1}q_m-p_m^m+q_m^2[m-1]_{p_m,q_m}\right)}-\frac{\sqrt{p_m^{2m}+u^4m_{p_m,q_m}^2(p_m-q_m)^2+2u^2m_{p_m,q_m}\left(p^m(p_m-q_m)\right)}}{2\left(p_m^{m-1}q_m-p_m^m+q_m^2[m-1]_{p_m,q_m}\right)}\right)}$$

and M is a positive constant.

Proof. Let $f \in Lip_M(\rho)$ and $0 < \rho \leq 1$. By (13), we have

$$\left|L_m^*(f,p,q;u) - f(u)\right| \leq L_m^*\left(|f(t)-f(u)|,p,q;u\right) \leq ML_m^*\left(|t-u|^\rho,p,q;u\right).$$

Using the Hölder's inequality with $m = \frac{2}{\rho}$ and $n = \frac{2}{2-\rho}$, we get

$$\left|L_m^*(f,p,q;u) - f(u)\right| \leq M\Big(L_m^*\left((t-u)^2,p,q;u\right)\Big)^{\frac{\rho}{2}}. \tag{14}$$

Combining (12) and (14) we have the desired inequality.

If we choose $u \leq \frac{p^{m-1}}{2[m]_{p,q}}$ and satisfying $\delta_m^*(u) \leq \delta_m(u)$ for all $m \geq 2$, then we get the desired result. □

4 The statistical convergence rate

H. Fast [25] was the first, who introduced this in 1951. In approximation theory, this concept is used first in [11] for linear positive operators. Recently many researchers have investigated the statistical convergence for several operators (see, e.g. [6]). For definition and examples of statistically convergent sequence we refer the reader to [10,16].

Now, let $q = q_m$ and $p = p_m$ be two sequences such that:

$$st\text{-}\lim_m q_m = 1, \qquad st\text{-}\lim_m p_m = 1, \qquad st\text{-}\lim_m q_m^m = 1 \qquad \text{and} \qquad st\text{-}\lim_m p_m^m = 1. \tag{15}$$

Theorem 5. *If the sequences $p = p_m$ and $q = q_m$ satisfy the condition* (15)*, then*

$$\left|L_m^*(f,p_m,q_m;u) - f(u)\right| \leq 2w(f;\sqrt{\delta_{m,u}}) \quad \textit{for all} \quad f \in C[0,1],$$

where $\delta_{m,u} = L_m^*\left((t-u),^2 p_m,q_m;u\right)$.

Proof. We will omit the proof as it is similar to the Theorem 2.

From the conditions (15), one can see that $st\text{-}\lim_m L_m^*\left((t-u)^2,p_m,q_m;u\right) = 0$, which implies $st\text{-}\lim_m w(f;\delta_{m,u}) = 0$ due to the fact that

$$st\text{-}\lim_{m\to\infty} \delta_{m,u} = 0.$$

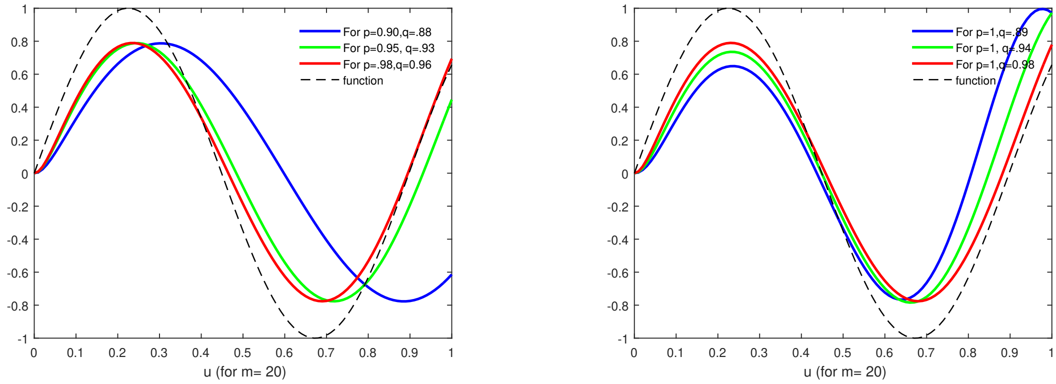
This gives the pointwise statistical convergence rate of the operators $L_m^*(f,p,q;u)$ to the function $f(u)$. □

5 Graphical Analysis

By using MATLAB (2015), we show comparisons between the convergence rate of (p,q)-Lupaş Bernstein operators defined in (1) and their King type modification to the function $f(u) = \sin 7u$ under different parameters.

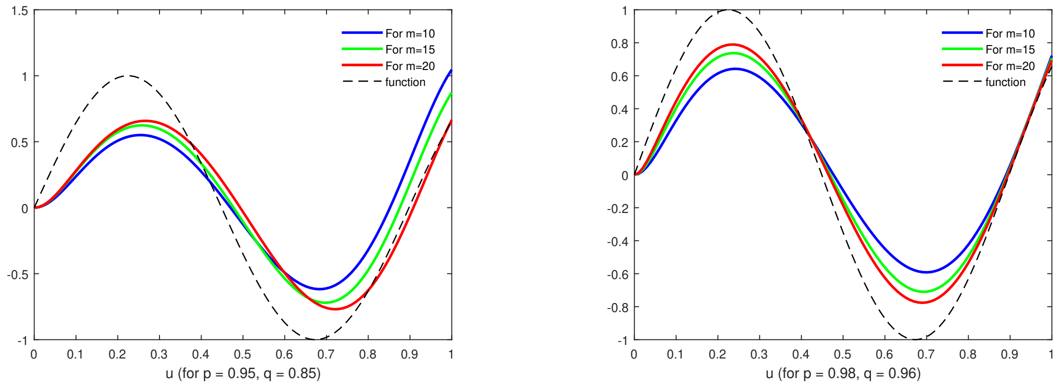
From the Figure 1*(a)*, it can be observed that as the value the p and q approaches towards 1 provided $0 < q < p \leq 1$, the operators given by (3) converges towards the function. If we take $p = 1$, then it reduces to King type modification of q-Lupaş Bernstein operators (see Figure 1*(b)*).

Also, from Figure 2*(a)* and Figure 2*(b)*, one can see the convergence of King type modification of (p,q)-Lupaş Bernstein operator defined in (3) for fixed values of parameters p, q and different values of m.



(a) *(b)*

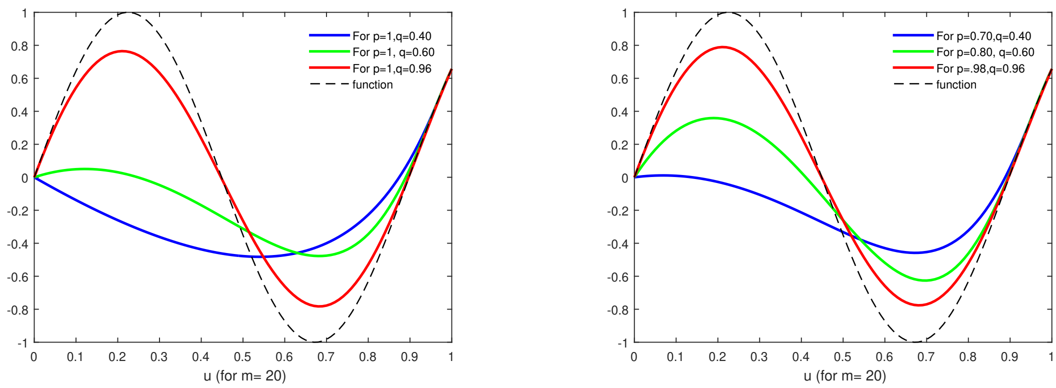
Figure 1. King type modification of (p,q)-Lupaş Bernstein operators



(a) *(b)*

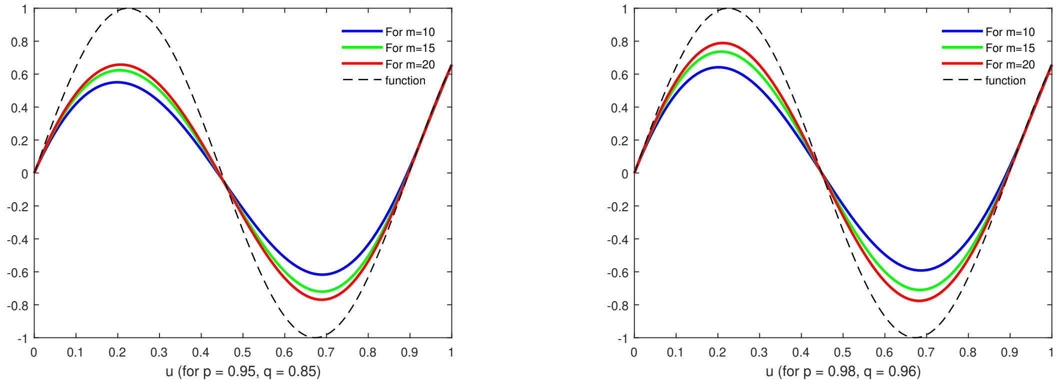
Figure 2. King type modification of (p,q)-Lupaş Bernstein operators

Similarly, one can observe the approximation by (p,q)-Lupaş Bernstein operators defined by (1) for the same function in Figures 3*(a)*, 3*(b)* and Figures 4*(a)*, 4*(b)*.



(a) *(b)*

Figure 3. (p,q)-Lupaş Bernstein operators



(a) (b)

Figure 4. (p,q)-Lupaş Bernstein operators

6 The rates of convergence and error estimate

Here we can compare convergence rate of operators defined in (1) and their modification using King's technique defined in (3) denoted by $\delta_m(u)$ and $\delta_m^*(u)$, respectively. For $f(u) = \sin 7u$ and fixed p and q we will take values of u, that lay in the interval $\left[0, \frac{p^{m-1}}{2[m]_{p,q}}\right]$.

6.1 Error Estimation

In the following tables, one can easily see that our modified operator (3) gives better error estimation than the (p,q)-Lupaş Bernstein operators defined in (1).

m (for $p = 0.95, q = 0.85$)	$\delta_m(u)$ at $u = 0.01$	$\delta_m(u)$ at $u = 0.02$	$\delta_m(u)$ at $u = 0.03$
5	0.0493	0.0691	0.0839
10	0.0392	0.0548	0.0664
15	0.0356	0.0497	0.0601

Table 1. Rate of convergence table for (p,q)-Lupaş Bernstein operator

m (for $p = 0.95, q = 0.85$)	$\delta_m^*(u)$ at $u = 0.01$	$\delta_m^*(u)$ at $u = 0.02$	$\delta_m^*(u)$ at $u = 0.03$
5	0.0136	0.0260	0.0372
10	0.0125	0.0211	0.0247
15	0.0123	0.0205	0.0231

Table 2. Rate of convergence table for King type (p,q)-Lupaş Bernstein operator

m (for $p = 0.95, q = 0.85$)	$\delta_m(u)$ at $u = 0.01$	Absolute error bound $2\omega(f : \delta_m(u))$
5	0.0493	0.6726
10	0.0392	0.5392
15	0.0356	0.4851

Table 3. Error estimation table for (p,q)-Lupaş Bernstein operators

m (for $p = 0.95, q = 0.85$)	$\delta_m^*(u)$ at $u = 0.01$	Absolute error bound $2\omega^*(f : \delta_m^*(u))$
5	0.0136	0.1817
10	0.0125	0.1678
15	0.0123	0.1678

Table 4. Error estimation table for King type modification of (p,q)-Lupaş Bernstein operators

7 Conclusion

From the above Tables 1–4, one can see easily that King type modification of (p,q)-Lupaş Bernstein operators have better rate of convergence than the classical (p,q)-Lupaş Bernstein operators within the interval $\left[0, \frac{p^{m-1}}{2[m]_{p,q}}\right]$. Similarly this modified operator (3) gives less error bound than the operator (1) for fixed values of m.

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Нісар К.С., Шарма В., Хан А. *Про модифікацію типу Кінґа (p,q)-Лупашових операторів Бернштейна з покращеними оцінками* // Карпатські матем. публ. — 2023. — Т.15, №1. — C. 20–30.

Метою цієї статті є модифікація (p,q)-Лупашових операторів Бернштейна за допомогою техніки Кінґа та встановлення результатів щодо збіжності цих операторів, використовуючи модуль неперервності та клас ліпшицевих функцій. Отримано деякі апроксимаційні результати для цих нових послідовностей операторів. Показано, що швидкість збіжності модифікації типу Кінґа є кращою у порівнянні з (p,q)-Лупашовими операторами Бернштейна. Модифікація операторів типу Кінґа також забезпечує кращу оцінку похибки всередині деякого підінтервалу відрізка $[0,1]$ у порівнянні з (p,q)-Лупашовими операторами Бернштейна. В останньому розділі ми представили деякі рисунки та таблиці з метою моделювання за допомогою MATLAB (R2015a).

Ключові слова і фрази: постквантове числення, (p,q)-Лупашовий оператор Бернштейна, модуль неперервності, апроксимація типу Кінґа, оцінка похибки.