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On a Family of Infinite Series with Reciprocal Catalan Numbers

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Abstract: We study a certain family of infinite series with reciprocal Catalan numbers. We first evaluate two special candidates of the family in closed form, where we also present some Catalan–Fibonacci relations. Then, we focus on the general properties of the family and prove explicit formulas, including two types of integral representations.

Keywords: Catalan numbers; infinite series; Fibonacci numbers; Lucas numbers; Stirling numbers; Mellin transform

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1. Introduction and Motivation

The famous Catalan numbers $C_n, n \geq 0$ are defined by $C_n = \frac{1}{n+1}\binom{2n}{n}$. They can be also expressed by the recursion

$$C_n = \frac{2(2n-1)}{n+1} C_{n-1}, \quad C_0 = 1.$$

The generating function for C_n is

$$\sum_{n=0}^{\infty} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

Catalan numbers have a long history and play an extraordinary role in combinatorics. Excellent sources on these numbers are the books by Koshy [1], Roman [2] and Stanley [3]. Some examples of recent work involving Catalan numbers and their generalizations include [4–10].

Catalan numbers form a special class of the so-called special numbers and polynomials. Other classes of these objects with comparable importance are Bernoulli numbers (polynomials), Euler numbers (polynomials), Fibonacci numbers (polynomials), etc. These objects play an important role in combinatorics, number theory and mathematical physics. The main approach in the study of these numbers is via their generating functions, which have been studied continuously. These generating functions have attracted considerable attention from many mathematicians, statisticians, physicists and engineers [11–20].

This paper was inspired by a recent paper by Amdeberhan et al. [21], who studied the function

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{C_n}, \qquad z \in [0;4),$$

which generates the reciprocals of Catalan numbers. They prove by several methods that

$$f(z)=\frac{2(z+8)}{(4-z)^2}+\frac{24\sqrt{z}\arcsin(\sqrt{z}/2)}{(4-z)^{5/2}},\qquad z\in[0;4),\tag{1}$$

This expression also appears in [22], and an equivalent form is given in [23]. The hypergeometric expression for $f(z)$ is

$$f(z)={}_2F_1\left(1,2;\frac{1}{2};\frac{z}{4}\right),$$

where

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

$(a)_n=a(a+1)\cdots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)}$, $(a)_0=1$, is the Pochhammer symbol, and $\Gamma(z)$ is the gamma function defined by $\Gamma(z)=\int\limits_0^{+\infty}t^{z-1}e^{-t}dt$ with $\Re(z)>0$.

Substituting $z=2$ and $z=3$, respectively, into (1) yields the evaluations

$$\sum_{n=0}^{\infty}\frac{2^n}{C_n}=5+\frac{3\pi}{2}\quad\text{and}\quad\sum_{n=0}^{\infty}\frac{3^n}{C_n}=22+8\sqrt{3}\pi,$$

which were stated in 2014 by Beckwith and Harbor as Problem 11765 in the American Mathematical Monthly [24] and solved by Abel [25].

From [26], we have the identities

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{C_n}=\frac{22}{5}+\frac{6(5+9\sqrt{5})\pi}{125}\omega,\qquad\sum_{n=0}^{\infty}\frac{L_{2n}}{C_n}=\frac{62}{5}+\frac{6(15+19\sqrt{5})\pi}{125}\omega,$$

where F_n and L_n are the famous Fibonacci and Lucas numbers, respectively, $\alpha=\frac{1+\sqrt{5}}{2}$ is the golden ratio and $\omega=\sqrt{\sqrt{5}\alpha}=\sqrt{2+\alpha}$. These numbers are defined for $n\geq 0$ by the recursions $F_{n+2}=F_{n+1}+F_n$ and $L_{n+2}=L_{n+1}+L_n$ with initial conditions $F_0=0, F_1=1$, $L_0=2$ and $L_1=1$, respectively. The Binet formulas are given by

$$F_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},\qquad L_n=\alpha^n+\beta^n,$$

where $\beta=-\frac{1}{\alpha}=\frac{1-\sqrt{5}}{2}$. For negative subscripts we have

$$F_{-n}=(-1)^{n+1}F_n\quad\text{and}\quad L_{-n}=(-1)^nL_n.$$

See the book by Koshy [27] for more details.

Our purpose in this paper is to study, for each integer $m\geq 0$, the following family of series:

$$g_m(z)=\sum_{n=0}^{\infty}\frac{2^{2n}n^m}{2n+1}\frac{z^n}{C_n},\quad 0\leq z<1.$$

We begin by evaluating the functions $g_0(z)$ and $g_1(z)$ explicitly for some values of z, including Fibonacci and Lucas numbers. Then, focusing on $g_m(z)$, we prove some explicit expressions for $g_m(z)$, including two integral representations.

2. The Functions $g_0(z)$ and $g_1(z)$

Sprugnoli [28] has derived some generating functions for series involving reciprocals of central binomial coefficients. His approach is built on ordinary differential equations but leaves some gaps in the derivations. For instance, he does not state the domains of the presented functions. One of Sprugnoli's results [28], Theorem 2.4, is the following identity:

$$\sum_{n=0}^{\infty} \frac{2^{2n} z^{n+1}}{(2n+1)\binom{2n}{n}} = \sqrt{\frac{z}{1-z}} \arctan\left(\sqrt{\frac{z}{1-z}}\right). \tag{2}$$

Since $\arctan\left(\sqrt{\frac{z}{1-z}}\right) = \arcsin\left(\sqrt{z}\right)$ for $0 \le z < 1$, the above identity could be stated equivalently using the arcsine function as in (1). In this paper, however, we have decided to work with the notation used by Sprugnoli. Our first goal is to give a rigorous proof of (2).

Theorem 1. *For all* $0 \le z < 1$ *we have the identity*

$$\sum_{n=0}^{\infty} \frac{2^{2n} z^{n+1}}{(2n+1)\binom{2n}{n}} = \sqrt{\frac{z}{1-z}} \arctan\left(\sqrt{\frac{z}{1-z}}\right). \tag{3}$$

Proof. For each integer $n \ge 0$, consider the integral $\int_{-1}^{1} (1-x^2)^n dx$. Then, we can evaluate the integral in two ways. First, we have

$$\int_{-1}^{1} (1-x^2)^n dx = \frac{2^{2n+1}}{(2n+1)\binom{2n}{n}}. \tag{4}$$

The result is known. It can be proved easily using integration by parts. It is, however, a special case of the more interesting fact [1] (p. 52) that

$$\int_{a}^{b} (x-a)^n (b-x)^n dx = 2 \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{3 \cdot 5 \cdot 7 \cdots (2n+1)} \cdot \left(\frac{b-a}{2}\right)^{2n+1}, \qquad n \ge 1.$$

By the binomial theorem, the integral can be also evaluated as

$$\int_{-1}^{1} (1-x^2)^n dx = \int_{-1}^{1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^{2k} dx = 2 \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{2k+1}.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{2^{2n} z^{n}}{(2n+1)\binom{2n}{n}} &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} (-1)^k \frac{z^n}{2k+1} \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \sum_{n=k}^{\infty} \binom{n}{k} z^n \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2k+1} \sum_{n=0}^{\infty} \binom{n+k}{k} z^n \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2k+1} \sum_{n=0}^{\infty} \binom{n+k}{n} z^n \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k z^k}{2k+1} \frac{1}{(1-z)^{k+1}}.
\end{aligned}$$

This shows that

$$\sum_{n=0}^{\infty} \frac{2^{2n} z^{n+1}}{(2n+1)\binom{2n}{n}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{z}{1-z}\right)^{n+1}.$$

Combining this with the fact that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

is the Taylor series of the arctangent for $-1 < x < 1$, and letting $x = \sqrt{\frac{z}{1-z}}$, we find that identity (2) holds for $0 \le z \le \frac{1}{2}$. However, since the series has a radius of convergence 1, by analytic continuation, the identity in question holds for every $z \in [0;1)$. $\square$

Differentiating both sides of (3) with respect to z gives

$$g_0(z) = \frac{1}{2(1-z)} + \frac{1}{2(1-z)^2} \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{\sqrt{\frac{z}{1-z}}}, \qquad z \in [0;1).$$

Differentiating once more and multiplying by z gives

$$g_1(z) = \frac{2z+1}{4(1-z)^2} + \frac{4z-1}{4(1-z)^3} \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{\sqrt{\frac{z}{1-z}}}, \qquad z \in [0;1). \tag{5}$$

Moreover, we see that

$$g_0(z) = \frac{1}{(1-z)^2} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2n+1} \left(\frac{z}{1-z}\right)^n,$$

and

$$g_1(z) = \frac{1}{(1-z)^3} \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2n+1} (2z+n) \left(\frac{z}{1-z}\right)^n.$$

The trigonometric versions of $g_0(z)$ and $g_1(z)$ are also useful; namely,

$$g_{t,0}(z) = \sum_{n=0}^{\infty} \frac{2^{2n} \sin^{2n} z}{(2n+1)C_n} = \frac{1}{2}\left(\frac{1}{\cos^2 z} + \frac{z}{\cos^3 z \sin z}\right) = \frac{1}{2\cos^2 z}\left(1 + \frac{2z}{\sin 2z}\right)$$

and

$$g_{t,1}(z) = \sum_{n=0}^{\infty} \frac{2^{2n} n \sin^{2n} z}{(2n+1)C_n} = \frac{1}{4\cos^4 z}\left(2 - \cos 2z + (1 - 2\cos 2z)\frac{2z}{\sin 2z}\right), \tag{6}$$

both valid for $0 \le z < \frac{\pi}{2}$.

At $z = \frac{\pi}{3}$, $z = \frac{\pi}{4}$ and $z = \frac{\pi}{6}$, function $g_{t,0}(z)$, respectively, gives

$$\sum_{n=0}^{\infty} \frac{3^n}{(2n+1)C_n} = 2 + \frac{8\pi\sqrt{3}}{9}, \qquad \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)C_n} = 1 + \frac{\pi}{2},$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)C_n} = \frac{2}{3} + \frac{4\pi\sqrt{3}}{27}.$$

We also find from (5) or (6) and the principal branch for the square-root function

$$\sum_{n=1}^{\infty} \frac{n}{(2n+1)C_n} = \frac{2}{3}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n n}{(2n+1)C_n} = \frac{2}{25}\left(1 - \frac{16}{\sqrt{5}} \ln \alpha\right),$$

$$\sum_{n=1}^{\infty} \frac{2^n n}{(2n+1)C_n} = 2 + \frac{\pi}{2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n n 2^n}{(2n+1)C_n} = \frac{\sqrt{3}}{9} \ln(2 - \sqrt{3}), \tag{7}$$

$$\sum_{n=1}^{\infty}\frac{3^n n}{(2n+1)C_n}=10+\frac{32\pi}{3\sqrt{3}},\qquad \sum_{n=1}^{\infty}\frac{(-1)^n n3^n}{(2n+1)C_n}=-\frac{2}{49}\left(1-\frac{16}{\sqrt{21}}\ln\left(\frac{5-\sqrt{21}}{2}\right)\right).$$

To offer some evaluations of $g_0(z)$ involving Fibonacci and Lucas numbers, we need the following lemma.

Lemma 1 ([29] Lemma 1, see also [30] p. 271, identities (20)–(22))**.** *We have*

$$\sin\left(\frac{\pi}{10}\right)=-\frac{\beta}{2},\quad \sin\left(\frac{3\pi}{10}\right)=\frac{\alpha}{2},\quad \cos\left(\frac{\pi}{10}\right)=\frac{\sqrt{\alpha\sqrt{5}}}{2},\quad \cos\left(\frac{3\pi}{10}\right)=\frac{1}{2}\sqrt{\frac{5}{\alpha\sqrt{5}}}.$$

Theorem 2. *With $\omega=\sqrt{\sqrt{5}\alpha}=\sqrt{2+\alpha}$, we have for any integer s*

$$5\sum_{n=0}^{\infty}\frac{F_{2n+s}}{(2n+1)C_n}=2L_{s+1}+\frac{4\sqrt{5}\pi\,\omega}{25}\big((2+2\alpha)F_s+(4-\alpha)F_{s-1}\big),$$

$$\sum_{n=0}^{\infty}\frac{L_{2n+s}}{(2n+1)C_n}=2F_{s+1}+\frac{4\sqrt{5}\pi\,\omega}{25}(2F_s+\alpha F_{s-1}).$$

Proof. Set $z=\frac{3\pi}{10}$ in $g_{t,0}(z)$ and multiply through by α^s, where s is an arbitrary integer. Using Lemma 1 yields

$$\sum_{n=0}^{\infty}\frac{\alpha^{2n+s}}{(2n+1)C_n}=\frac{2}{\sqrt{5}}\alpha^{s+1}+\frac{12\pi}{\sqrt{125\sqrt{5}}}\alpha^s\sqrt{\alpha}.$$

In a similar manner, set $z=\frac{\pi}{10}$ in $g_{t,0}(z)$ and multiply through by β^s to obtain

$$\sum_{n=0}^{\infty}\frac{\beta^{2n+s}}{(2n+1)C_n}=-\frac{2}{\sqrt{5}}\beta^{s+1}+\frac{4\pi}{\sqrt{125\sqrt{5}}}\frac{\beta^s}{\sqrt{\alpha}}.$$

The difference and sum of the above identities result in the identities

$$\sum_{n=0}^{\infty}\frac{F_{2n+s}}{(2n+1)C_n}=\frac{2}{5}L_{s+1}+\frac{4\pi}{25\sqrt{\alpha\sqrt{5}}}(\alpha L_{s+2}+L_{s-1}),$$

$$\sum_{n=0}^{\infty}\frac{L_{2n+s}}{(2n+1)C_n}=2F_{s+1}+\frac{4\pi}{\sqrt{125\alpha\sqrt{5}}}\big(\alpha(F_{s+1}+L_s)+F_s+L_{s+1}\big).$$

The stated identities follow upon simplifications. □

As examples, we have with $\omega=\sqrt{\sqrt{5}\alpha}$:

$$\sum_{n=1}^{\infty}\frac{F_{2n}}{(2n+1)C_n}=\frac{2}{5}+\frac{4(7\alpha-6)\pi\omega}{125},\qquad \sum_{n=1}^{\infty}\frac{L_{2n}}{(2n+1)C_n}=\frac{4(\alpha+2)\pi\omega}{25},$$

$$\sum_{n=1}^{\infty}\frac{F_{2n+1}}{(2n+1)C_n}=\frac{1}{5}+\frac{8(3\alpha+1)\pi\omega}{125},\qquad \sum_{n=1}^{\infty}\frac{L_{2n+1}}{(2n+1)C_n}=1+\frac{8\sqrt{5}\pi\omega}{25},$$

$$\sum_{n=1}^{\infty}\frac{F_{2n-2}}{(2n+1)C_n}=\frac{3}{5}+\frac{8(4\alpha-7)\pi\omega}{125},\qquad \sum_{n=1}^{\infty}\frac{L_{2n-2}}{(2n+1)C_n}=-1+\frac{8(3-\alpha)\pi\omega}{25}.$$

Using the same idea for $g_{t,1}(z)$, we can prove the next theorem, whose proof we therefore omit.

Theorem 3. *With $\omega = \sqrt{\sqrt{5}\alpha} = \sqrt{2+\alpha}$, we have for any integer s*

$$\sum_{n=1}^{\infty} \frac{nF_{2n+s}}{(2n+1)C_n} = 2F_{s+1} + \frac{8}{5}F_s + \frac{8\sqrt{5}\pi\,\omega}{125}\left(\sqrt{5}\alpha F_{s+1} + 2F_s\right)$$

and

$$\sum_{n=1}^{\infty} \frac{nL_{2n+s}}{(2n+1)C_n} = 2L_{s+1} + \frac{8}{5}L_s + \frac{8\sqrt{5}\pi\,\omega}{125}\left((6+\alpha)F_{s+1} + 2\alpha^2 F_s\right).$$

Another interesting example for an evaluation of $g_1(z)$ with Fibonacci (Lucas) entries is the following result.

Theorem 4. *Let r be an even integer and s any integer. Then,*

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{2n} nF_{rn+s}}{(2n+1)L_r^n C_n} &= \left(F_{3r+s} + \frac{L_r}{2}F_{2r+s}\right)\frac{L_r}{2} + (L_r L_{2r+s} - 4L_{3r+s})\frac{L_r^2}{4\sqrt{5}}\arctan(\beta^r) \\ &\quad + \left(4L_{3r+s} - L_r L_{2r+s} + (4F_{3r+s} - L_r F_{2r+s})\sqrt{5}\right)\frac{\pi L_r^2}{16\sqrt{5}}\end{aligned}$$

and

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2^{2n} nL_{rn+s}}{(2n+1)L_r^n C_n} &= \left(L_{3r+s} + \frac{L_r}{2}L_{2r+s}\right)\frac{L_r}{2} + (L_r F_{2r+s} - 4F_{3r+s})\frac{L_r^2\sqrt{5}}{4}\arctan(\beta^r) \\ &\quad + \left(4L_{3r+s} - L_r L_{2r+s} + (4F_{3r+s} - L_r F_{2r+s})\sqrt{5}\right)\frac{\pi L_r^2}{16}.\end{aligned}$$

Proof. First note that $1 - \alpha^r/L_r = \beta^r/L_r$ and that if r is an even integer, then

$$\sqrt{\frac{\alpha^r/L_r}{1-\alpha^r/L_r}} = \sqrt{\frac{\alpha^r}{L_r - \alpha^r}} = \alpha^r.$$

Let s be an arbitrary integer. Consider $\alpha^s g_1(\alpha^r/L_r) \mp \beta^s g_1(\beta^r/L_r)$. We have

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{2^{2n} n(\alpha^{rn+s} \mp \beta^{rn+s})}{L_r^n(2n+1)C_n} \\ &\quad = (\alpha^{3r+s} \mp \beta^{3r+s})\frac{L_r}{2} + (\alpha^{2r+s} \mp \beta^{2r+s})\frac{L_r^2}{4} + \left(\frac{4\alpha^{2r+s}}{\beta^r} - \frac{\alpha^{r+s}}{\beta^r}L_r\right)\frac{L_r^2}{4}\arctan(\alpha^r) \\ &\quad \mp \left(\frac{4\beta^{2r+s}}{\alpha^r} - \frac{\beta^{r+s}}{\alpha^r}L_r\right)\frac{L_r^2}{4}\arctan(\beta^r),\end{aligned}$$

from which the stated identities in the theorem follow using the Binet formulas and the fact that $\arctan\alpha^r = \frac{\pi}{2} - \arctan\beta^r$ for any even integer r. □

As particular instances of Theorem 4, we have, for an even integer r,

$$\sum_{n=1}^{\infty} \frac{2^{2n+1} nF_{(n-2)r}}{(2n+1)L_r^n C_n} = F_{2r} + (L_r + 2F_r\sqrt{5})\frac{\pi L_r^2}{4\sqrt{5}} - \frac{L_r^3}{\sqrt{5}}\arctan(\beta^r),$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} nL_{(n-2)r}}{(2n+1)L_r^n C_n} = L_r^2 + (L_r + 2F_r\sqrt{5})\frac{\pi L_r^2}{8} - F_r L_r^2\sqrt{5}\arctan(\beta^r), \quad (8)$$

$$\sum_{n=1}^{\infty} \frac{2^{2n} n F_{(n-3)r}}{(2n+1)L_r^n C_n} = -\frac{1}{4}L_r^2 F_r + \left(8 - L_r^2 + F_{2r}\sqrt{5}\right)\frac{\pi L_r^2 \sqrt{5}}{80} + \left(L_r^2 - 8\right)\frac{L_r^2\sqrt{5}\arctan(\beta^r)}{20},$$

and

$$\sum_{n=1}^{\infty} \frac{2^{2n} n L_{(n-3)r}}{(2n+1)L_r^n C_n} = \left(2 + \frac{1}{2}L_r^2\right)\frac{L_r}{2} + \left(8 - (-1)^r L_r^2 + F_{2r}\sqrt{5}\right)\frac{\pi L_r^2}{16} - \frac{1}{4}L_r^3 F_r \sqrt{5}\arctan(\beta^r). \tag{9}$$

Note that both (8) and (9) give (7) when $r = 0$.

3. Integral Expressions for $g_0(z)$ and $g_1(z)$

Integral expressions for the functions $g_0(z)$ and $g_1(z)$ are derived easily using the integral identity (4).

Theorem 5. *We have*

$$g_0(z) = \frac{1}{2}\int_{-1}^{1} \frac{1}{(1 - z(1-x^2))^2}dx,$$

$$g_1(z) = z\int_{-1}^{1} \frac{1-x^2}{(1 - z(1-x^2))^3}dx.$$

Proof. From the geometric series and the above lemma, we deduce that for all $0 \le z < 1$,

$$\sum_{n=0}^{\infty} \frac{2^{2n} z^{n+1}}{(2n+1)\binom{2n}{n}} = \frac{1}{2}\int_{-1}^{1} \frac{z}{1 - z(1-x^2)}dx.$$

Differentiating produces the first equation. To obtain the second equation, we perform the operation $z(d/dz)g_0(z)$, and the proof is completed. □

It is interesting to compare the integral expressions for $g_0(z)$ and $g_1(z)$ with that for $f(z)$. This expression is not stated explicitly in [21] but can be derived as follows:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{z^n}{C_n} &= \sum_{n=1}^{\infty} \frac{(n+1)}{\binom{2n}{n}} z^n \\
&= \sum_{n=1}^{\infty} \frac{n(n+1)\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} z^n \\
&= \sum_{n=1}^{\infty} n(n+1)B(n, n+1)z^n \\
&= \sum_{n=1}^{\infty} n(n+1)\int_0^1 x^{n-1}(1-x)^n dx\, z^n \\
&= 2z\int_0^1 \frac{1-x}{(1 - zx(1-x))^3}dx,
\end{aligned}$$

where $B(a,b)$ is the beta function $B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

This proves that

$$f(z) = 1 + 2z\int_0^1 \frac{1-x}{\left(1 - zx(1-x)\right)^3}dx.$$

4. Some General Properties of $g_m(z)$

In this section, we present some general properties of $g_m(z)$, which is defined by

$$g_m(z) = \sum_{n=0}^{\infty} \frac{2^{2n} n^m}{2n+1} \frac{z^n}{C_n}, \quad 0 \le z < 1,$$

with $m \ge 0$ being an integer. We have the following result:

Theorem 6. *For each $m \ge 0$ and all $0 \le z < 1$, $g_m(z)$ possesses the representation*

$$g_m(z) = \frac{P_{1,m}(z)}{(1-z)^{m+1}} + \frac{P_{2,m}(z)}{(1-z)^{m+2}} \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{\sqrt{\frac{z}{1-z}}}, \tag{10}$$

where $P_{1,m}(z)$ and $P_{2,m}(z)$ are polynomials in z of degree m with rational coefficients. Moreover, for $m \ge 1$, the polynomials $P_{1,m}(z)$ and $P_{2,m}(z)$ can be expressed recursively according to

$$P_{1,m}(z) = z(1-z)\frac{d}{dz} P_{1,m-1}(z) + mzP_{1,m-1}(z) + \frac{1}{2} P_{2,m-1}(z), \tag{11}$$

$$P_{2,m}(z) = z(1-z)\frac{d}{dz} P_{2,m-1}(z) + (m+1)zP_{2,m-1}(z) - \frac{1}{2} P_{2,m-1}(z), \tag{12}$$

with $P_{1,0}(z) = P_{2,0}(z) = \frac{1}{2}$.

Proof. The proof of the representation (10) is easy using induction on m taking into account $g_{m+1}(z) = z(d/dz)\, g_m(z) = (zd/dz)^m g_0(z)$ and the identity

$$\frac{d}{dz} \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{\sqrt{\frac{z}{1-z}}} = \frac{1}{2z} - \frac{1}{2z(1-z)} \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{\sqrt{\frac{z}{1-z}}}.$$

The recursive expressions for $P_{1,m}(z)$ and $P_{2,m}(z)$ follow from the proof as a by-product. □

The first few polynomials have the following explicit forms:

$$P_{1,1}(z) = \frac{1}{4}(2z+1), \qquad P_{2,1}(z) = \frac{1}{4}(4z-1),$$

$$P_{1,2}(z) = \frac{1}{8}(4z^2+12z-1), \quad P_{2,2}(z) = \frac{1}{8}(16z^2-2z+1).$$

We mention that the coupled recursions (11) and (12) can be solved explicitly, but the closed forms seem not to shed enough light on their general structure. Nevertheless, we can prove the following expressions:

Proposition 1. *For each m,*

$$P_{1,m}(z) = \frac{1}{2} m! z^m + \sum_{j=0}^{m-1} \binom{m}{j} j! z^j \left(z(1-z) \frac{d}{dz} P_{1,m-(j+1)}(z) + \frac{1}{2} P_{2,m-(j+1)}(z) \right),$$

$$P_{2,m}(z) = \frac{1}{2} \prod_{j=1}^{m} \left((j+1)z - \frac{1}{2} \right) + z(1-z) \sum_{j=1}^{m} \frac{d}{dz} P_{2,m-j}(z) \prod_{k=2}^{j} \left((m+3-k)z - \frac{1}{2} \right),$$

where the empty product is one and the empty sum is zero.

Proof. We can use induction on m to prove both formulas. For $m = 0$, the statements are true. The inductive step for $P_{1,m}(z)$ is

$$\begin{aligned}
&P_{1,m+1}(z) = (m+1)zP_{1,m}(z) + z(1-z)\frac{d}{dz}P_{1,m}(z) + \frac{1}{2}P_{2,m}(z) \\
&= \frac{1}{2}(m+1)!z^{m+1} + \sum_{j=0}^{m-1}\binom{m}{j}j!(m+1)z^{j+1}\Big(z(1-z)\frac{d}{dz}P_{1,m-(j+1)}(z) + \frac{1}{2}P_{2,m-(j+1)}(z)\Big) \\
&\quad + z(1-z)\frac{d}{dz}P_{1,m}(z) + \frac{1}{2}P_{2,m}(z) \\
&= \frac{1}{2}(m+1)!z^{m+1} + \sum_{j=-1}^{m-1}\binom{m+1}{j+1}(j+1)!z^{j+1}\Big(z(1-z)\frac{d}{dz}P_{1,m-(j+1)}(z) + \frac{1}{2}P_{2,m-(j+1)}(z)\Big) \\
&= \frac{1}{2}(m+1)!z^{m+1} + \sum_{j=0}^{m}\binom{m+1}{j}j!z^{j}\Big(z(1-z)\frac{d}{dz}P_{1,m-j}(z) + \frac{1}{2}P_{2,m-j}(z)\Big).
\end{aligned}$$

Similarly, the inductive proof for $P_{2,m}(z)$ is accomplished according to

$$\begin{aligned}
P_{2,m+1}(z) &= \Big((m+2)z - \frac{1}{2}\Big)P_{2,m}(z) + z(1-z)\frac{d}{dz}P_{2,m}(z) \\
&= \Big((m+2)z - \frac{1}{2}\Big)\Bigg(\frac{1}{2}\prod_{j=1}^{m}\Big((j+1)z - \frac{1}{2}\Big) \\
&\quad + z(1-z)\sum_{j=1}^{m}\frac{d}{dz}P_{2,m-j}(z)\prod_{k=2}^{j}\Big((m+3-k)z - \frac{1}{2}\Big)\Bigg) + z(1-z)\frac{d}{dz}P_{2,m}(z) \\
&= \frac{1}{2}\prod_{j=1}^{m+1}\Big((j+1)z - \frac{1}{2}\Big) + z(1-z)\Bigg(\frac{d}{dz}P_{2,m}(z) \\
&\quad + \sum_{j=1}^{m}\frac{d}{dz}P_{2,m-j}(z)\Big((m+2)z - \frac{1}{2}\Big)\prod_{k=2}^{j}\Big((m+3-k)z - \frac{1}{2}\Big)\Bigg) \\
&= \frac{1}{2}\prod_{j=1}^{m+1}\Big((j+1)z - \frac{1}{2}\Big) + z(1-z)\sum_{j=0}^{m}\frac{d}{dz}P_{2,m-j}(z)\prod_{k=1}^{j}\Big((m+3-k)z - \frac{1}{2}\Big) \\
&= \frac{1}{2}\prod_{j=1}^{m+1}\Big((j+1)z - \frac{1}{2}\Big) + z(1-z)\sum_{j=1}^{m+1}\frac{d}{dz}P_{2,m-(j-1)}(z)\prod_{k=1}^{j-1}\Big((m+3-k)z - \frac{1}{2}\Big) \\
&= \frac{1}{2}\prod_{j=1}^{m+1}\Big((j+1)z - \frac{1}{2}\Big) + z(1-z)\sum_{j=1}^{m+1}\frac{d}{dz}P_{2,m+1-j}(z)\prod_{k=2}^{j}\Big((m+4-k)z - \frac{1}{2}\Big).
\end{aligned}$$

□

Applying Theorem 6 in the case $m = 2$ yields

$$\sum_{n=1}^{\infty}\frac{2^{2n}n^2}{2n+1}\frac{z^n}{C_n} = \frac{4z^2+12z-1}{8(1-z)^3} + \frac{16z^2-2z+1}{8(1-z)^4}\frac{\arctan\Big(\sqrt{\frac{z}{1-z}}\Big)}{\sqrt{\frac{z}{1-z}}},$$

from which we obtain

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n+1)C_n} = \frac{2}{3} + \frac{8\sqrt{3}\pi}{81},$$

$$\sum_{n=1}^{\infty}\frac{2^n n^2}{(2n+1)C_n} = 6 + 2\pi, \qquad \sum_{n=1}^{\infty}\frac{3^n n^2}{(2n+1)C_n} = 82 + \frac{272\sqrt{3}\pi}{9}.$$

Corollary 1. *For each $m \geq 0$, the sum $\sum_{n=0}^{\infty} \frac{2^n n^m}{(2n+1)C_n}$ can be expressed in the form $a + b\pi$, with a and b rational. The sums $\sum_{n=0}^{\infty} \frac{n^m}{(2n+1)C_n}$ and $\sum_{n=0}^{\infty} \frac{3^n n^m}{(2n+1)C_n}$ allow the same representation but with b being irrational.*

Theorem 7. *For each $m \geq 0$ and all $|z| < 1$, $g_m(z)$ possesses the integral representation*

$$g_m(z) = \int_{-1}^{1} \frac{Q_m(z;x)}{(1 - z(1-x^2))^{m+2}} dx,$$

where $Q_m(z;x)$ is a polynomial in z of degree m given by

$$\begin{aligned} Q_m(z;x) &= \frac{1}{2}(m+1)!(az)^m + (1-az)z\frac{d}{dz}Q_{m-1}(z;x) \\ &\quad + (1-az)z\sum_{j=1}^{m-1}\binom{m+1}{j} j!(az)^j \frac{d}{dz}Q_{m-(j+1)}(z;x), \end{aligned} \tag{13}$$

with $Q_0(z;x) = \frac{1}{2}$, and where we have set $a = 1 - x^2$.

Proof. We prove the claim by induction on m. Since

$$g_0(z) = \frac{1}{2}\int_{-1}^{1} \frac{1}{(1 - z(1-x^2))^2} dx,$$

the statement is true for $m = 0$. Now, assuming it is true for a fixed $m > 0$, we can proceed with

$$g_{m+1}(z) = z\frac{d}{dz}g_m(z) = z\int_{-1}^{1} \frac{\frac{d}{dz}Q_m(z;x)(1-az) + (m+2)aQ_m(z;x)}{\left(1 - z(1-x^2)\right)^{m+3}} dx.$$

This gives the recursion

$$Q_m(z;x) = (m+1)azQ_{m-1}(z;x) + z(1-az)\frac{d}{dz}Q_{m-1}(z;x), \qquad m \geq 1. \tag{14}$$

This recursion can be solved by standard methods to give (13). Alternatively, one can prove (13) directly by induction on m using (14). □

5. Another Integral Expression for $g_m(z)$ Using Mellin Transform

Lemma 2. *For integers $m, n \geq 0$, we have*

$$\sum_{j=0}^{m} (-1)^j S(m+1, m+1-j)(n+m-j)! = n^m n!,$$

where $S(n,k)$ are the Stirling numbers of the second kind, defined by $S(0,0) = 1$, $S(n,n) = S(n,1) = 1$ $(n \geq 1)$ and

$$S(n,k) = \frac{1}{k!}\sum_{s=0}^{k}(-1)^s\binom{k}{s}(k-s)^n.$$

Proof. Consider the known representation

$$x^m = \sum_{j=0}^{m}\binom{x}{j}S(m,j)j!.$$

Let $x = -n$. Using $\binom{s}{j} = (-1)^j \binom{-s+j-1}{j}$ we have

$$\begin{aligned}(-1)^{m+1}n^{m+1} &= \sum_{j=0}^{m+1} \binom{-n}{j} S(m+1,j)j! \\ &= \sum_{j=0}^{m+1} (-1)^j \binom{n+j-1}{j} S(m+1,j)j! \\ &= \sum_{j=0}^{m+1} (-1)^j \frac{(n+j-1)!}{(n-1)!} S(m+1,j).\end{aligned}$$

Thus, by reindexing the summation

$$\begin{aligned}(-1)^{m+1}n^{m+1}(n-1)! &= \sum_{j=0}^{m+1} (-1)^{m-j+1} S(m+1,m+1-j)(n+m-j)! \\ &= \sum_{j=0}^{m} (-1)^{m-j+1} S(m+1,m+1-j)(n+m-j)!\end{aligned}$$

as $S(n,0) = 0, n \geq 1$. □

Theorem 8. *The function $g_m(z)$ possesses the integral representation*

$$g_m(z) = \frac{1}{\sqrt{z}} \sum_{j=0}^{m} (-1)^j S(m+1,m+1-j) \int_0^\infty x^{(m-j)/2} K_{j+1-m}(2\sqrt{x}) \sinh(2\sqrt{xz}) dx,$$

where $K_v(x)$ is the modified Bessel function of the second kind, which can be defined by

$$K_v(x) = \int_0^\infty \cosh(vt) e^{-x\cosh t} dt \quad (x > 0).$$

Proof. The proof is based on ideas developed in [21]. Recall that the Mellin transform of a real-valued function $f(x)$ on $(0,\infty)$ is defined by the integral [31]

$$M[f(x)](s) = \int_0^\infty x^{s-1} f(x) dx.$$

The gamma function $\Gamma(n)$ can be interpreted as $M[e^{-x}](n)$ and thus

$$M[xe^{-x}](n+1) = (n+1)!.$$

Since

$$g_m(z) = \sum_{n=0}^{\infty} \frac{(4z)^n}{(2n+1)!} n^m (n+1)! n!,$$

we want to find a function $f_m(x)$ such that

$$n^m n!(n+1)! = M[f_m(x)] M[xe^{-x}](n+1).$$

By Lemma 2, it follows that such a function is

$$f_m(x) = \sum_{j=0}^{m} (-1)^j S(m+1,m+1-j) x^{m-j} e^{-x}.$$

Now, we are going to apply the Mellin convolution theorem:

$$M[f_1(x)] M[f_2(x)](s) = M[F(x)](s)$$

with
$$F(x) = \int_0^\infty f_1(x_1) f_2\Big(\frac{x}{x_1}\Big) \frac{dx_1}{x_1}.$$

In our case, $F(x)$ equals
$$\begin{aligned} F(x) &= \int_0^\infty \Big(\sum_{j=0}^{m} (-1)^j S(m+1, m+1-j) x_1^{m-j} e^{-x_1} \Big) \frac{x}{x_1} e^{-x/x_1} \frac{dx_1}{x_1} \\ &= \sum_{j=0}^{m} (-1)^j S(m+1, m+1-j) \int_0^\infty x \frac{e^{-(x_1+x/x_1)}}{x_1^{j+2-m}} dx_1 \\ &= 2 \sum_{j=0}^{m} (-1)^j S(m+1, m+1-j) x^{(m+1-j)/2} K_{j+1-m}(2\sqrt{x}), \end{aligned}$$

where the following representation for the modified Bessel function of the second kind [21] was used:
$$K_v(z) = \frac{1}{2}\Big(\frac{z}{2}\Big)^v \int_0^\infty \frac{e^{-(t+z^2/4t)}}{t^{v+1}} dt.$$

Finally, we calculate
$$\begin{aligned} g_m(z) &= \sum_{n=0}^{\infty} \frac{(4z)^n}{(2n+1)!} n^m (n+1)! n! \\ &= \sum_{n=0}^{\infty} \frac{(4z)^n}{(2n+1)!} M[F(x)](n+1) \\ &= \int_0^\infty F(x) \sum_{n=0}^{\infty} \frac{(4xz)^n}{(2n+1)!} dx \\ &= \int_0^\infty F(x) \frac{\sinh(2\sqrt{xz})}{2\sqrt{xz}} dx. \end{aligned}$$

□

Two special cases of the representation are
$$g_0(z) = \frac{1}{\sqrt{z}} \int_0^\infty \sinh(2\sqrt{xz}) K_1(2\sqrt{x}) dx$$
and
$$g_1(z) = \frac{1}{\sqrt{z}} \int_0^\infty \sinh(2\sqrt{xz}) \Big(\sqrt{x} K_0(2\sqrt{x}) - K_1(2\sqrt{x}) \Big) dx.$$

6. Concluding Comments

In this paper, we have studied an interesting family of infinite series involving Catalan numbers. In particular, we have evaluated these series for special arguments and provided characterizations. Before closing, we want to state two different approaches that were communicated to us by one of the referees. First, if we set $r(z) = z \arctan z$ and $k(z) = \sqrt{\frac{z}{1-z}}$, then $h(z) = r(z) \circ k(z)$, where $h(z)$ equals identity (3). This shows that the functions $g_m(z)$ can also be studied by the Faà di Bruno formula. Furthermore, as $g_{m+1}(z) = z\frac{d}{dz} g_m(z) = (z\frac{d}{dz})^m g_0(z)$, it is possible to study $g_m(z)$ by expanding the factors $(z\frac{d}{dz})^m$ according to
$$\Big(z\frac{d}{dz}\Big)^n = \sum_k S(n,k) z^k \Big(\frac{d}{dz}\Big)^k,$$
where $S(n,k)$ are the Stirling numbers of the second kind.

Finally, we remark that Sprugnoli's identity (2), which is the starting point of our exploration, can be integrated resulting in the identity

$$\sum_{n=0}^{\infty} \frac{2^{2n}}{(2n+1)(n+1)(n+2)} \frac{z^n}{C_n} = \frac{1}{2z} + \frac{1}{2z^2} \arctan^2\left(\sqrt{\frac{z}{1-z}}\right) - \frac{\arctan\left(\sqrt{\frac{z}{1-z}}\right)}{z\sqrt{\frac{z}{1-z}}},$$

and containing the evaluation

$$\sum_{n=0}^{\infty} \frac{2^n}{(2n+1)(n+1)(n+2)C_n} = \frac{\pi^2}{8} - \frac{\pi}{2} + 1$$

as a special instance (at $z = \frac{1}{2}$).

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