Discrete Mathematics Letters
www.dmlett.com

Discrete Math. Lett. **8** (2022) 30–37
DOI: 10.47443/dml.2021.0080

Research Article

Binomial tribonacci sums

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(Received: 20 August 2021. Received in revised form: 28 September 2021. Accepted: 6 October 2021. Published online: 13 October 2021.)

Abstract

We derive expressions for several binomials sums involving a generalized tribonacci sequence. We also study double binomial sums involving this sequence. Several explicit examples involving tribonacci and tribonacci–Lucas numbers are stated to highlight the results.

Keywords: generalized tribonacci sequence; tribonacci number; tribonacci–Lucas number; binomial transform.

2020 Mathematics Subject Classification: 11B37, 11B39.

1. Introduction

There is a dearth of tribonacci summation identities including binomial coefficients. Our goal in this paper is to derive several new binomial tribonacci sums such as

$$\sum_{k=0}^{n}\binom{n}{k}G_{4k+s}=2^nG_{3n+s},\qquad \sum_{k=1}^{n}\binom{n}{k}\frac{G_{4k+s}}{k}=\sum_{m=1}^{n}\frac{2^mG_{3m+s}-G_s}{m},$$

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}G_{2k+s}=2^{n-1}\big(G_{4n+s}+(-1)^nG_{s-2n}\big),\qquad \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{2k}\frac{4^k}{n+k}G_{4n+2k+s}=\frac{G_{8n+s}+G_s}{2n},$$

and double binomial tribonacci summation identities such as

$$\sum_{k=0}^{n}\sum_{p=0}^{k}(-1)^{k+p}\binom{n}{k}\binom{k}{p}G_{5k+p+s}=3^nG_{3n+s},\qquad \sum_{k=0}^{n}\sum_{p=0}^{k}\binom{n}{k}\binom{k}{p}\frac{G_{k+5p+s}}{3^k}=\left(\frac{7}{3}\right)^nG_{3n+s}.$$

In the above identities, n denotes a non-negative integer, s and p are arbitrary integers and G_n is a generalized tribonacci number.

The generalized tribonacci sequence $G_n=G_n(c_0,c_1,c_2)$, $n\geq 0$, is defined recursively by

$$G_n=G_{n-1}+G_{n-2}+G_{n-3},\qquad n\geq 3,$$

with initial values $G_0=c_0$, $G_1=c_1$, $G_2=c_2$ not all being zero. Extension of the definition of G_n to negative subscripts is provided by writing the recurrence relation as

$$G_{-n}=G_{-(n-3)}-G_{-(n-2)}-G_{-(n-1)},$$

so that G_n is defined for all integers n.

The most prominent representatives of G_n and widely studied in the literature are $G_n(0,1,1)=T_n$ the sequence of tribonacci numbers and $G_n(3,1,3)=K_n$ the sequence of tribonacci-Lucas numbers (sequences A000073 and A001644 in [19], respectively).

The first few tribonacci numbers and tribonacci-Lucas numbers with positive and negative subscripts are given in Table 1.

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n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705
T_{-n}	0	0	1	−1	0	2	−3	1	4	−8	5	7	−20	18	9
K_n	3	1	3	7	11	21	39	71	131	241	443	815	1499	2757	5071
K_{-n}	3	−1	−1	5	−5	−1	11	−15	3	23	−41	21	43	−105	83

Table 1: Tribonacci and tribonacci–Lucas numbers.

Properties of (generalized) tribonacci sequences were investigated in the recent articles [1–4, 7, 8, 10, 12–18, 20, 21], among others. For instance, Janjić [16] found the remarkable combinatorial identity

$$T_n = 1 + \sum_{k=1}^{n-1} \sum_{i=0}^{k} \sum_{j=i}^{n-k} \binom{k}{i} \binom{j-1}{i-1} \binom{j}{n-k-2j}.$$

A generalized tribonacci number $G_n(c_0, c_1, c_2)$ is given by the Binet formula

$$G_n(c_0, c_1, c_2) = A\alpha^n + B\beta^n + C\gamma^n, \tag{1}$$

where α, β and γ are the distinct roots of the equation $x^3 - x^2 - x - 1 = 0$. The coefficients A, B and C depend on the initial values and are determined by the system

$$\begin{cases} A + B + C = c_0, \\ A\alpha + B\beta + C\gamma = c_1, \\ A\alpha^2 + B\beta^2 + C\gamma^2 = c_2. \end{cases}$$

The Binet formulas for T_n and K_n are

$$T_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)}$$

and

$$K_n = \alpha^n + \beta^n + \gamma^n\,,$$

where

$$\alpha = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3}, \qquad \beta = \frac{1 + \omega\sqrt[3]{19+3\sqrt{33}} + \omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19+3\sqrt{33}} + \omega\sqrt[3]{19-3\sqrt{33}}}{3}\,,$$

and $\omega = \frac{-1+i\sqrt{3}}{2}$ is a primitive cube root of unity.

Tribonacci and tribonacci-Lucas numbers with negative indices can be accessed directly, using the following result.

Lemma 1.1. *For integer n,*

$$T_{-n} = T_{n-1}^2 - T_{n-2}T_n, \tag{2}$$

$$K_{-n} = \frac{K_n^2 - K_{2n}}{2}. \tag{3}$$

For a proof of (2), see, for example, [8, Theorem 2.2]. The proof of (3) one can find in [6, Formula (9)].

In this article, we study binomial and double binomial sums with terms being a generalized tribonacci sequence. We derive closed forms for several such sums. We also prove a general binomial identity characterizing G_{an+b} for $a \geq 1$ and b an arbitrary integer.

2. Some auxiliary results

In this section we present some results that we will use in the sequel.

Lemma 2.1. *Let $\phi \in \{\alpha, \beta, \gamma\}$. Then, for all $n \geq 0$, we have*

$$\phi^{n+1} = \phi^2 T_n + \phi(T_{n-1} + T_{n-2}) + T_{n-1}. \tag{4}$$

For a proof of (4), see [7, Formula (6)].

Lemma 2.2. *We have*

$$(\alpha-1)^3 = 2\alpha^{-2}, \tag{5}$$
$$(\alpha+1)^3 = 2\alpha^4, \tag{6}$$
$$(\alpha^2+1)^3 = 4\alpha^5, \tag{7}$$
$$(\alpha^3-1)^3 = 2\alpha^7, \tag{8}$$
$$\alpha^4+1 = 2\alpha^3, \tag{9}$$

with identical relations for β and γ.

Proof. Since

$$1+\alpha+\alpha^2 = \alpha^3\,, \tag{10}$$

we have

$$\frac{\alpha^2+1}{\alpha^2-1} = \alpha \tag{11}$$

and

$$\frac{\alpha+1}{\alpha-1} = \alpha^2. \tag{12}$$

Addition of (11) and (12) gives

$$(\alpha+1)^2(\alpha-1) = 2\alpha^2, \tag{13}$$

while their subtraction produces

$$(\alpha-1)^2(\alpha+1) = 2. \tag{14}$$

Eliminating $\alpha+1$ between (13) and (14) gives identity (5), while the elimination of $\alpha-1$ yields (6).

Cubing identity $\alpha^2+1 = \frac{2\alpha}{\alpha-1}$ and making use of (5) gives (7). Subtracting (10) from $\alpha+\alpha^2+\alpha^3 = \alpha^4$ produces identity (8). Identity (9) follows from $\alpha^4+1 = \alpha^4+\alpha^3+\alpha^2+\alpha = (\alpha^2+1)(\alpha+1)$ with the help of (6) and (7). □

Lemma 2.3. *Let a, b, c and d be rational numbers and λ an irrational number. Then*

$$a+\lambda b = c+\lambda d \quad \Longleftrightarrow \quad a=c, \quad b=d\,.$$

3. Identities from the binomial theorem and binomial transform

The next lemma will be the key ingredient to derive many results in this paper. For a proof and some applications to Horadam numbers, see [11].

Lemma 3.1. *Let n and j be integers with $0 \le j \le n$. Then, for each $x, y \in \mathbb{C}$, we have*

$$\sum_{k=j}^{n} (\pm 1)^{k-j} \binom{k}{j}\binom{n}{k} y^k x^{n-k} = \binom{n}{j} y^j (x \pm y)^{n-j}.$$

We also mention the standard fact about sequences and their binomial transforms [5]: Let $(a_n)_{n\ge 0}$ be a sequence of numbers and $(b_n)_{n\ge 0}$ be its binomial transform. Then we have the following relations:

$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k \quad \Longleftrightarrow \quad a_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} b_k. \tag{15}$$

Furthermore, if $a_0 = 0$ (so that $b_0 = 0$ too) the binomial pair exhibits the following properties:

$$\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k} = \sum_{m=1}^{n} \frac{b_m}{m} \tag{16}$$

and

$$\sum_{k=1}^{n} \binom{n}{k} \frac{a_k}{k+1} = \frac{1}{n+1} \sum_{m=1}^{n} b_m. \tag{17}$$

Theorem 3.1. *Let j and s be integers such that s is arbitrary and $j \ge 0$. Then*

$$\sum_{k=j}^{n} \binom{k}{j}\binom{n}{k} G_{4k+s} = \binom{n}{j} 2^{n-j} G_{3n+j+s}. \tag{18}$$

Proof. Use identity (9) in Lemma 3.1 with $x = 1$ and $y = \alpha^4$, taking note of Lemma 2.3. □

Corollary 3.1. *For n a non-negative integer and s any integer,*

$$\sum_{k=0}^{n} \binom{n}{k} G_{4k+s} = 2^n G_{3n+s}, \tag{19}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-2)^k G_{3k+s} = (-1)^n G_{4n+s}, \tag{20}$$

$$\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k} = \sum_{m=1}^{n} \frac{2^m G_{3m+s} - G_s}{m} \tag{21}$$

and

$$\sum_{k=1}^{n} \binom{n}{k} \frac{G_{4k+s}}{k+1} = \frac{1}{n+1} \left(\sum_{m=1}^{n} 2^m G_{3m+s} - nG_s \right). \tag{22}$$

Proof. To obtain (19) set $j = 0$ in (18). Identities (20), (21) and (22) follow form (15), (16) and (17), respectively. □

From (19) and (20) we immediately obtain the following binomial tribonacci and tribonacci–Lucas relations.

Corollary 3.2. *For $n \geq 0$,*

$$\sum_{k=0}^{n} \binom{n}{k} T_{4k} = 2^n T_{3n}, \qquad \sum_{k=0}^{n} \binom{n}{k} K_{4k} = 2^n K_{3n},$$

$$\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n+1} = 2^n, \qquad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n+1} = 2^n,$$

$$\sum_{k=0}^{n} \binom{n}{k} T_{4k-3n} = 0, \qquad \sum_{k=0}^{n} \binom{n}{k} K_{4k-3n} = 3 \cdot 2^n.$$

Theorem 3.2. *For non-negative integer n, any integer s, we have*

$$\sum_{k=0}^{3n} \delta^k \binom{3n}{k} G_{pk+s} = \delta^n 2^{qn} G_{rn+s},$$

where the values of δ, p, q and r as given in each column in Table 2.

δ	-1	1	1	-1
p	1	1	2	3
q	1	1	2	1
r	-2	4	5	7

Table 2: Values of δ, p, q and r from Theorem 3.2.

Proof. Each of the identities (5)–(8) can be written as $(\alpha^p + \delta)^3 = 2^q \alpha^r$, where the values of δ, p, q and r in each case are as given in each column in Table 2. The identity of the theorem then follows from the binomial theorem and Lemma 2.3. □

Lemma 3.2. *For non-negative integer n and real or complex z,*

$$2 \sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} z^{2k} = (1+z)^{3n} + (1-z)^{3n},$$

$$2 \sum_{k=1}^{\lceil 3n/2 \rceil} \binom{3n}{2k-1} z^{2k-1} = (1+z)^{3n} - (1-z)^{3n}.$$

Theorem 3.3. *For non-negative integer n and any integer s,*

$$\sum_{k=0}^{\lfloor 3n/2 \rfloor} \binom{3n}{2k} G_{2k+s} = 2^{n-1} \big(G_{4n+s} + (-1)^n G_{s-2n} \big),$$

$$\sum_{k=1}^{\lceil 3n/2 \rceil} \binom{3n}{2k-1} G_{2k+s-1} = 2^{n-1} \big(G_{4n+s} - (-1)^n G_{s-2n} \big).$$

Proof. Set $z=\alpha$ in Lemma 3.2, make use of identities (5) and (6), noting Lemma 2.3 with $\lambda=\alpha$. □

Setting $s=0$ in Theorem 3.3, we immediately obtain the following.

Corollary 3.3. *For non-negative integer* n*,*

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}G_{2k}=2^{n-1}\big(G_{4n}+(-1)^nG_{-2n}\big),$$

$$\sum_{k=1}^{\lceil 3n/2\rceil}\binom{3n}{2k-1}G_{2k-1}=2^{n-1}\big(G_{4n}-(-1)^nG_{-2n}\big).$$

As special cases of formulas above we have:

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}T_{2k}=2^{n-1}\big(T_{4n}+(-1)^n(T_{2n-1}^2-T_{2n-2}T_{2n})\big),$$

$$\sum_{k=1}^{\lceil 3n/2\rceil}\binom{3n}{2k-1}T_{2k-1}=2^{n-1}\big(T_{4n}-(-1)^n(T_{2n-1}^2-T_{2n-2}T_{2n})\big)$$

and

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}K_{2k}=2^{n-2}\big(2K_{4n}+(-1)^n(K_{2n}^2-K_{4n})\big),$$

$$\sum_{k=1}^{\lfloor 3n/2\rfloor}\binom{3n}{2k-1}K_{2k-1}=2^{n-2}\big(2K_{4n}-(-1)^n(K_{2n}^2-K_{4n})\big).$$

Theorem 3.4. *For non-negative integer* n *and any integer* s*,*

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}G_{4k+s}=2^{2n-1}\big(G_{5n+s}+(-1)^nG_{2n+s}\big),$$

$$\sum_{k=1}^{\lceil 3n/2\rceil}\binom{3n}{2k-1}G_{4k+s-2}=2^{2n-1}\big(G_{5n+s}-(-1)^nG_{2n+s}\big).$$

Proof. Combining (5) with (6) yields

$$(\alpha^2-1)^3=4\alpha^2. \tag{23}$$

Now set $z=\alpha^2$ in Lemma 3.2 and make use of identities (7) and (23), noting Lemma 2.3 with $\lambda=\alpha$. □

Theorem 3.5. *For non-negative integer* n *and any integer* s*,*

$$\sum_{k=0}^{\lfloor 3n/2\rfloor}\binom{3n}{2k}G_{8k+s}=2^{3n-1}\big(G_{9n+s}+(-2)^nG_{7n+s}\big),$$

$$\sum_{k=1}^{\lceil 3n/2\rceil}\binom{3n}{2k-1}G_{8k+s-4}=2^{3n-1}\big(G_{9n+s}-(-2)^nG_{7n+s}\big).$$

Proof. Combining (7) and (23) we have

$$(\alpha^4-1)^3=16\alpha^7. \tag{24}$$

Set $z=\alpha^4$ in Lemma 3.2 and make use of identities (9) and (24), noting Lemma 2.3 with $\lambda=\alpha$. □

4. Identities from the Waring formulas

Our next result provides two combinatorial identities for generalized tribonacci numbers involving binomial coefficients.

Lemma 4.1. *The following identities hold for $n \geq 0$ and real or complex x and y:*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} (xy)^k (x+y)^{n-2k} = \frac{x^{n+1} - y^{n+1}}{x-y} \tag{25}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} \frac{n}{n-k} (xy)^k (x+y)^{n-2k} = x^n + y^n. \tag{26}$$

Formulas (25) and (26) are well-known in combinatorics and called Waring (sometimes Girard-Waring) formulas. The proof of these formulas can be found, for example, in [9].

Theorem 4.1. *Let n be a non-negative integer and s any integer. Then*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (G_{3n-2k+s+4} - G_{3n-2k+s}) = \frac{G_{4n+s+4} - G_s}{2^n}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{G_{3n-2k+s}}{n-k} = \frac{G_{4n+s} + G_s}{2^n n}.$$

Proof. Set $(x, y) = (1, \alpha^4)$ in (25) and (26), respectively, Lemma 4.1 and use identity (8) and Lemma 2.3. $\square$

Corollary 4.1. *For $n \geq 0$,*

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (G_{n-2k+4} - G_{n-2k}) = \frac{G_{2n+4} - G_{-2n}}{2^n},$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{G_{n-2k}}{n-k} = \frac{G_{2n} + G_{-2n}}{n 2^n}.$$

In particular,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (T_{n-2k+4} - T_{n-2k}) = \frac{T_{2n+4} - T_{2n-1}^2 + T_{2n-2} T_{2n}}{2^n},$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} (K_{n-2k+4} - K_{n-2k}) = \frac{2K_{2n+4} - K_{2n}^2 + K_{4n}}{2^{n+1}}$$

and

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{T_{n-2k}}{n-k} = \frac{T_{2n-1}^2 + T_{2n}(1 - T_{2n-2})}{n 2^n},$$

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \left(-\frac{1}{4}\right)^k \binom{n-k}{k} \frac{K_{n-2k}}{n-k} = \frac{K_{2n}^2 + 2K_{2n} - K_{4n}}{n 2^{n+1}}.$$

5. Double binomial tribonacci sums

Theorem 5.1. *Let n, j and s be integers with s arbitrary and $j \geq 0$. Then,*

$$\sum_{k=j}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{k}{j} \binom{n}{k} \binom{k}{p} G_{5k+p+s} = 3^{n-j} \binom{n}{j} \sum_{p=0}^{j} (-1)^{j-p} \binom{j}{p} G_{3n+2j+p+s}. \tag{27}$$

Proof. The identity can be derived from Lemma 3.1 using $3\phi^3 = \phi^6 - \phi^5 + 1$. $\square$

Corollary 5.1. *Let n and s be integers. Then,*

$$\sum_{k=0}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{n}{k} \binom{k}{p} G_{5k+p+s} = 3^n G_{3n+s},$$

$$\sum_{k=1}^{n} \sum_{p=0}^{k} (-1)^{k-p} \binom{n}{k} \binom{k}{p} k G_{5k+p+s} = n 3^{n-1} (G_{3n+s} + G_{3n+s+1}).$$

Proof. Set $j = 0$ and $j = 1$ in (27), respectively. □

Theorem 5.2. *Let j and s be integers with s arbitrary and $j \geq 0$. Then*

$$\sum_{k=j}^{n}\sum_{p=0}^{k}\binom{k}{j}\binom{n}{k}\binom{k}{p}\frac{G_{k+4p+s}}{2^k} = 2^{2n-j}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}G_{3n-2j+4m+s}, \tag{28}$$

$$\sum_{k=j}^{n}\sum_{p=0}^{k}\binom{k}{j}\binom{n}{k}\binom{k}{p}\frac{G_{k+5p+s}}{3^k} = \frac{7^{n-j}}{3^n}\binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}G_{3n-2j+5m+s}. \tag{29}$$

Proof. Use Lemma 3.1 in conjunction with $4\phi^3 = \phi^5 + \phi + 2$ and $7\phi^3 = \phi^6 + \phi + 3$, respectively. □

Corollary 5.2. *Let n and s be integers. Then,*

$$\sum_{k=0}^{n}\sum_{p=0}^{k}\binom{n}{k}\binom{k}{p}\frac{G_{k+4p+s}}{2^k} = 2^n G_{3n+s},$$

$$\sum_{k=1}^{n}\sum_{p=0}^{k}\binom{n}{k}\binom{k}{p}\frac{k\,G_{k+4p+s}}{2^k} = 2^{n-2}n(G_{3n+s+2} + G_{3n+s-2})$$

and

$$\sum_{k=0}^{n}\sum_{p=0}^{k}\binom{n}{k}\binom{k}{p}\frac{G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n G_{3n+s},$$

$$7\sum_{k=1}^{n}\sum_{p=0}^{k}\binom{n}{k}\binom{k}{p}\frac{k\,G_{k+5p+s}}{3^k} = \left(\frac{7}{3}\right)^n n(G_{3n+s+3} + G_{3n+s-2}).$$

Proof. Set $j = 0$ and $j = 1$ in (28) and (29), respectively. □

6. A general binomial sum identity

Theorem 6.1. *Let j, s and v be integers with $j, v \geq 0$, $v \neq 0$, $v \neq 1$. Then,*

$$\binom{n}{j}\sum_{m=0}^{j}\sum_{q=0}^{j-m}(-1)^{j+m+q}\binom{j}{j-m}\binom{j-m}{q}\left(\frac{T_v}{T_{v-1}}\right)^m G_{vn-j(v-1)+q+s}$$
$$= \frac{T_{v-2}^n}{T_{v-1}^j}\sum_{k=j}^{n}\sum_{p=0}^{k}\sum_{w=0}^{k-p}(-1)^{k+w+p}\binom{k}{j}\binom{n}{k}\binom{k}{k-p}\binom{k-p}{w}\left(\frac{T_{v-1}}{T_{v-2}}\right)^k\left(\frac{T_v}{T_{v-1}}\right)^p G_{k+w+s}.$$

Proof. For $v \geq 1$ and $\phi = \alpha$ write (4) in the form

$$\alpha^v = \alpha\big(T_v + T_{v-1}(\alpha - 1)\big) + T_{v-2}.$$

Now, identify $x = \alpha\big(T_v + T_{v-1}(\alpha-1)\big)$ and $a = T_{v-2}$ and use Lemma 3.1 and the binomial theorem to get

$$\sum_{k=j}^{n}\binom{k}{j}\binom{n}{k}(-1)^k T_{v-2}^{n-k}\sum_{p=0}^{k}\binom{k}{p}T_v^p T_{v-1}^{k-p}\sum_{w=0}^{k-p}\binom{k-p}{w}(-1)^{w+p}\alpha^{k+w}$$
$$= \binom{n}{j}\sum_{m=0}^{j}\binom{j}{m}T_v^m T_{v-1}^{j-m}\sum_{q=0}^{j-m}\binom{j-m}{q}(-1)^{j-(m+q)}\alpha^{vn-j(v-1)+q}.$$

Multiply both sides by α^s and combine the similar results for β and γ according to the Binet formula (1). □

Corollary 6.1. *We have*

$$\sum_{k=0}^{n}\sum_{p=0}^{k}\sum_{w=0}^{k-p}(-1)^{k+w+p}\binom{n}{k}\binom{k}{p}\binom{k-p}{w}\left(\frac{T_{v-1}}{T_{v-2}}\right)^k\left(\frac{T_v}{T_{v-1}}\right)^p G_{k+w+s} = \frac{G_{vn+s}}{T_{v-2}^n}.$$

For $v = 1$ the left-hand side collapses and we end with G_{n+s} on both sides of the equality sign. The special values for $v = 2$ and $v = 3$ are given by

$$\sum_{p=0}^{n}\sum_{w=0}^{n-p}(-1)^{w+p}\binom{n}{p}\binom{n-p}{w}G_{n+w+s} = (-1)^n G_{2n+s}$$

and

$$\sum_{k=0}^{n}\sum_{p=0}^{k}\sum_{w=0}^{k-p}(-1)^{w+p+k}\binom{n}{k}\binom{k}{k-p}\binom{k-p}{w}2^p G_{k+w+s} = G_{3n+s}.$$

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