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On equitable near-proper coloring of some derived graph classes

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An equitable near-proper coloring of a graph G is a defective coloring in which the number of vertices in any two color classes differ by at most one and the bad edges obtained is minimized by restricting the number of color classes that can have adjacency among their own elements. This paper investigates the equitable near-proper coloring of some derived graph classes like Mycielski graphs, splitting graphs and shadow graphs.

Key words and phrases: equitable near-proper coloring, Mycielski graph, splitting graph, shadow graph.

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Introduction

For general concepts and notations in graphs, we refer to [6, 18]. Unless stated otherwise, we consider connected, simple, finite and undirected graphs in this paper.

The notion of *equitable coloring* of graphs was introduced in [10]. An *equitable coloring* of a graph G is a proper vertex coloring in which the cardinalities of the color classes are either equal or differ by 1. Equitable coloring of graphs has many applications real-life situations, especially in scheduling and distribution problems. Allowing conflict to a certain level gives rise to defective equitable coloring problems. An *improper coloring* or a *defective coloring* of a graph G is a vertex coloring of it, with respect to which adjacent vertices can have the same color if required and as a result of that, there may be some edges whose end vertices receive the same color. These edges are called *bad edges*. A *near-proper coloring* of G is a defective coloring in which the bad edges are minimised by implementing certain restrictions on the number of color classes that can have adjacency among their own elements. The number of bad edges resulting from a near-proper coloring of G is denoted by $b_k(G)$. Certain results on this direction can be viewed in [1–4, 9]. In light of these studies, the idea of *equitable near-proper coloring* of graphs are introduced in [7] and as follows.

An *equitable near-proper coloring* (or ENP-coloring) of a graph G is a defective coloring in which the vertex set can be partitioned into k color classes $V_1, V_2, \ldots, V_k$ with cardinalities $n_1, n_2, \ldots, n_k$ respectively, such that $|n_i - n_j| \leq 1$ for any $1 \leq i \neq j \leq k$, and the number of bad edges is minimised by restricting the number of color classes that can have adjacency among their own elements. The minimum number of bad edges resulting from an ENP-coloring of G is defined as *equitable defective number* and is denoted by $b^k_{\chi_e}(G)$.

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Motivated by the studies mentioned above, in this paper, we discuss the ENP-coloring of some derived graph classes. A derived graph is a construction of a graph from a given graph under certain conditions. This paper discusses the ENP-coloring of Mycielski graphs, splitting graphs and shadow graphs of certain graph classes. Throughout this paper, the set $c_1, c_2, \ldots, c_k$ represents the available k colors and $V_1, V_2, \ldots, V_k$ be the corresponding color classes. In an ENP-coloring, we consider the cases from $k = 2$ to $k = \chi_e(G) - 1$. In all diagrams in this paper, the bad edges are represented by dotted lines.

1 ENP-Coloring of Mycielski graphs

The *Mycielski graph* or the *Mycielskian* of a graph is defined (see [11]) as follows. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. The Mycielski graph of a graph G, denoted by $\mu(G)$ is the graph with vertex set $V(\mu(G)) = \{v_1, v_2, v_3, \ldots, v_n, u_1, u_2, u_3 \ldots, u_n, w\}$ such that

$$\begin{aligned} v_iv_j \in E(\mu(G)) &\iff v_iv_j \in E(G) \\ v_iu_j \in E(\mu(G)) &\iff v_iv_j \in E(G) \\ u_iw \in E(\mu(G)) & \end{aligned}$$

for $i = 1, 2, \ldots, n$.

The coloring of Mycielskian of some fundamental graph classes was discussed in [5,11,17]. The ENP-coloring of Mycielskian of some graph classes was discussed in [8]. The equitable defective number of complete graphs was determined in [7], as follows.

Theorem 1 ([7]). *For a complete graph K_n, the equitable defective number is given by*

$$b^k_{\chi_e}(K_n) = \frac{r\lceil \frac{n}{k} \rceil \lceil \frac{n}{k} - 1 \rceil}{2} + \frac{(k-r)\lfloor \frac{n}{k} \rfloor \lfloor \frac{n}{k} - 1 \rfloor}{2},$$

where $n \equiv r \pmod{k}$.

The following theorem discusses the ENP-coloring and the corresponding equitable defective number of Mycielski graph of complete graphs.

Theorem 2. *The equitable defective number of Mycielski graph of complete graph $\mu(K_n)$ is given by*

(i) if $k = 2$, then $b^k_{\chi_e}(\mu(K_n)) = b^k_{\chi_e}(K_n) + \frac{n(n-1)}{2}$,

(ii) if $k = 3$, then $b^k_{\chi_e}(\mu(K_n)) = \begin{cases} b^k_{\chi_e}(K_n) + \frac{n(n-2)}{3}, & \text{if } n \equiv 0 \pmod{3}, \\ b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor^2, & \text{if } n \equiv 1 \pmod{3}, \\ b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (3\lfloor \frac{n}{3} \rfloor + 2), & \text{if } n \equiv 2 \pmod{3}, \end{cases}$

(iii) if $k \geq 4$, then $b^k_{\chi_e}(\mu(K_n)) = b^k_{\chi_e}(K_n) + 2(n-k) + \lfloor \frac{n}{k} \rfloor$.

Proof. Let $\mu(K_n)$ be the Mycielskian of complete graphs with $2n+1$ vertices. Let $\{v_1, \cdots, v_n\}$ be the set of vertices of the complete graph K_n and $\{u_1, u_2, \cdots, u_n\}$ be the set of vertices corresponding to the v_i's. According to the construction of Mycielski graph of complete graphs, each vertex v_i is adjacent to all u_j, $i \neq j$, and the vertex w is adjacent to all u_i's where $1 \leq i \leq n$. We know that the equitable chromatic number of Mycielski graph of a complete graph is $n+1$.

We consider $2 \leq k \leq n$ for an ENP-coloring. The equitable defective number of complete graphs have already been investigated in [7]. Thus, we need to further count only the bad edges between v_i's and u_i's and between u_i's and w.

Case 1. When $k = 2$, we have to look in to the following subcases.

Subcase 1.1. Let n be even. Assign the two available colors c_1 and c_2 to the v_i's in a clockwise manner and the corresponding u_i's can be assigned with the same colors as received by v_i. Here, each u_i is adjacent to $\frac{n}{2} - 1$ number of same colored vertices and since there are n number of u_i's, the total number of bad edges obtained is $n[\frac{n}{2} - 1]$. And assign the vertex w with either c_1 or c_2, we obtain $\frac{n}{2}$ bad edges which are incident with w. Along with that we have the bad edges resulting from the ENP-coloring of a complete graph K_n and thus the equitable defective number is $b^k_{\chi_e}(K_n) + \frac{n(n-1)}{2}$.

Subcase 1.2. When n is odd, repeat the coloring procedure as in Subcase 1.1 for all v_i's and u_i's. It can be observed that there are $\lfloor \frac{n}{2} \rfloor$ number of c_1 colored u_i's and each of these vertices is adjacent to $\lfloor \frac{n}{2} \rfloor$ number of c_1 colored v_i's. Also, among the $\lceil \frac{n}{2} \rceil$ number of c_2 colored u_i's, $\lfloor \frac{n}{2} \rfloor$ number of u_i's are adjacent to $\lfloor \frac{n}{2} \rfloor - 1$ number of c_2 colored vertices and one u_i is adjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Assign the vertex w with color c_2 to satisfy the equitability condition, we obtain $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with w. Along with the bad edges resulting from the adjacency within v_i's, the equitable defective number is $b^k_{\chi_e}(K_n) + \frac{n(n-1)}{2}$.

Case 2. Let $k = 3$. In this case, we have to consider the following subcases.

Subcase 2.1. When $n \equiv 0 \pmod 3$, start assigning the v_i's with the three available colors c_1, c_2 and c_3 in a cyclic order. Now, assign the corresponding u_i's with the same colors as assigned to v_i and vertex w can be assigned with any of the three colors. Here, it can be observed that each u_i is adjacent to $\frac{n}{3} - 1$ number of same colored v_i's and since there are $\frac{n}{3}$ number of u_i's we obtain $\frac{n}{3}\left[\frac{n}{3} - 1\right]$ bad edges between the v_i's and u_i's. Since the vertex w is adjacent to all u_i's $(1 \leq i \leq n)$ we obtain $\frac{n}{3}$ bad edges which are incident with w. Along with the bad edges obtained from the complete graph, we get the equitable defective number as $b^k_{\chi_e}(K_n) + \frac{n(n-2)}{3}$.

Subcase 2.2. When $n \equiv 1 \pmod 3$, repeat the coloring pattern as in Subcase 2.1 for all v_i's and u_i's $(1 \leq i \leq n-1)$ and the remaining vertices v_n, u_n and w can be colored with c_1, c_2 and c_3 respectively to satisfy the equitability condition. Here, we observe that among the u_i's, color c_1 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ number of c_1 colored v_i's. Also, color c_2 is repeated $\lceil \frac{n}{3} \rceil$ times and among those vertices, $\lfloor \frac{n}{3} \rfloor$ number of c_2 colored u_i's are adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ same colored v_i's and one c_2 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ vertices. Again, color c_3 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each c_3 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ number of c_3 colored v_i's. Further, we obtain $\lfloor \frac{n}{3} \rfloor$ bad edges which are incident with w. Along with the bad edges obtained from the complete graph, the equitable defective number is $b^k_{\chi_e}(K_n) + 3\lfloor \frac{n}{3} \rfloor^2$.

Subcase 2.3. When $n \equiv 2 \pmod 3$, assign the v_i's and u_i's as in Subcase 2.1 and assign the vertex w with color c_3 to satisfy the equitability condition. Now, it can be observed that among the u_i's, the colors c_1 and c_2 are repeated $\lceil \frac{n}{3} \rceil$ times and adjacent to $\lfloor \frac{n}{3} \rfloor$ number of same colored v_i's. Furthermore, color c_3 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each c_3 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ number of c_3 colored v_i's. Since, $\lfloor \frac{n}{3} \rfloor$ number of u_i's receive color c_3, we obtain $\lfloor \frac{n}{3} \rfloor$ bad edges incident with w. Therefore, the equitable defective number is $b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (2\lceil \frac{n}{3} \rceil + \lfloor \frac{n}{3} \rfloor)$.

Case 3. When $k \geq 4$, assign the vertices as in the previous cases considering the equitability condition we obtain the equitable defective number as $b^k_{\chi_e}(K_n) + 2(n-k) + \lfloor \frac{n}{k} \rfloor$. □

Figure 1 depicts a 3-ENP-coloring of Mycielskian of complete graphs.

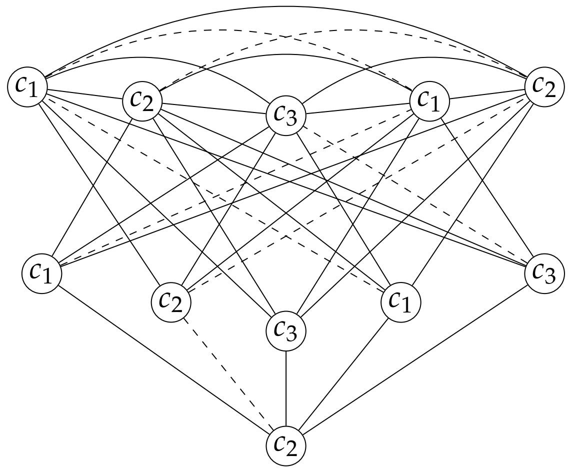


Figure 1 *Mycielskian of complete graphs with 3-ENP-coloring.*

The following theorem discusses the ENP-coloring and the corresponding equitable defective number of Mycielski graph of star graphs.

Theorem 3. *The equitable defective number of Mycielski graph of star graphs $\mu(K_{1,n})$ is given by*

$$b^k_{\chi_e}(\mu(K_{1,n})) = \begin{cases} n, & \text{if } k = 2, \\ \lfloor \frac{n}{3} \rfloor - 2, & \text{if } k \geq 3. \end{cases}$$

Proof. Let v_0 be the central vertex of the star graph $K_{1,n}$ and $\{v_1, v_2, \cdots, v_n\}$ be the set of vertices which are adjacent to v_0. Let $\{u_0, u_1, u_2, \cdots, u_n\}$ be the set of vertices corresponding to v_i's and w be the vertex which is adjacent to all u_i's, $0 \leq i \leq n$. Thus, the Mycielski graph of star graph consists of $2n + 3$ vertices. Let $\{c_1, c_2, \cdots, c_k\}$ be the set of available colors in an ENP-coloring and let $\{V_1, V_2, \cdots, V_n\}$ be the collection of corresponding color classes. Here, we consider the following cases.

Case 1. When $k = 2$, we have two available colors c_1 and c_2 and let us partition the vertex set as follows: $V_1 = \{v_0, u_0, u_1, u_2, \cdots, u_n\}$, $V_2 = \{v_1, v_2, \cdots, v_n, w\}$.

Assign the vertices in V_1 with color c_1 and the vertices in V_2 with color c_2. Since v_0 is adjacent to $u_1, u_2, \cdots, u_n$, we obtain n bad edges which are incident with v_0. All other vertices are colored properly and $||V_1| - |V_2|| = 1$. Hence, we conclude that the equitable defective number is n.

Case 2. When $k \geq 3$, let us partition the vertex set into k color classes and consider the vertex set with minimum cardinality, $\lfloor \frac{2n+3}{k} \rfloor$ vertices. Let us restrict the bad edges into this particular color class so as to minimise the number of bad edges. Place the vertex u_0 in this color class and the remaining vertices of this color class are considered to be $v_0, v_1, v_2, \cdots, v_{\lfloor \frac{2n+3}{k} \rfloor - 2}$. All the remaining vertices can be colored properly using the other two available colors (see Figure 2 for illustration). Since u_0 is adjacent to $\lfloor \frac{2n+3}{k} \rfloor - 2$ number of same colored vertices, the equitable defective number is $\lfloor \frac{2n+3}{3} \rfloor - 2$. □

Figure 2 depicts a 3-ENP-coloring of Mycielskian of stars.

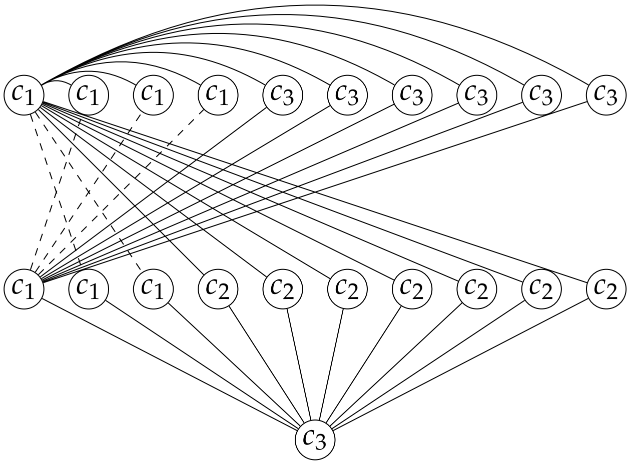


Figure 2 *Mycielskian of star graphs with* 3*-ENP-coloring.*

2 ENP-coloring of splitting graphs

A *splitting graph* denoted by $S(G)$ has been introduced in [16]. For each vertex v_i of a graph G, take a new vertex u_i and join u_i to the neighbouring vertices of v_i. The graph $S(G)$ thus obtained is called the splitting graph of G.

In the following theorem, we discuss the ENP-coloring and the corresponding equitable defective number of splitting graph of paths.

Theorem 4. *The equitable defective number of splitting graph of paths $S(P_n)$ when n is odd is* 1*.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of the path P_n and let $u_1, u_2, \ldots, u_n$ be the vertices corresponding to the vertices $v_1, v_2, \ldots, v_n$ such that each vertex v_i (where $2 \leq i \leq n-1$) is adjacent to u_{i-1} and u_{i+1}, v_1 is adjacent to u_2 and v_n is adjacent to u_{n-1}. The equitable chromatic number of splitting graph of paths $S(P_n)$ is 2 when n is even and 3 when n is odd. Thus, in an ENP-coloring of splitting graph of paths, we consider only one case $k = 2$ when n is odd. When $k = 2$, assign the path P_n with the available colors c_1 and c_2 and we can properly color the path with these two colors. Now, assign the u_i's where $1 \leq i \leq n-1$ with c_1 and c_2 such that both v_i and the corresponding vertex u_i receive the same color. Also, assign u_n with color c_2 to satisfy the equitability condition, we obtain only one $v_{n-1}u_n$ bad edge and thus the equitable defective number is 1. □

Figure 3 depicts a 2-ENP-coloring of splitting graph of paths.

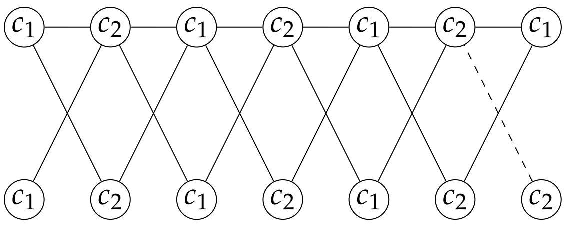


Figure 3 *Splitting graph of paths with* 2*-ENP-coloring.*

The following result discusses the ENP-coloring and hence determine the corresponding equitable defective number of splitting graph of cycles.

Theorem 5. *For a cycle C_n, $b^k_{\chi_e}(S(C_n)) = 3$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the cycle C_n and let $u_1, u_2, \ldots, u_n$ be the set of corresponding vertices. Here, each v_i (where $2 \le i \le n-1$) is adjacent to u_{i-1} and u_{i+1}, v_1 is adjacent to u_2 and u_n, v_n is adjacent to u_1 and u_n. The equitable chromatic number of splitting graph of cycles $S(C_n)$ is 2 when n is even and 3 when n is odd. Thus, in the ENP-coloring, we consider only one case $k = 2$ for n is odd. When $k = 2$, assign the cycle C_n with the available colors c_1 and c_2, we obtain one v_1v_n bad edge. Further, assign the u_i's as in Theorem 4, we obtain two more bad edges u_1v_n and $v_{n-1}u_n$. Hence, the equitable defective number is 3. □

Figure 4 illustrates a 2-ENP-coloring of the splitting graph of a cycle.

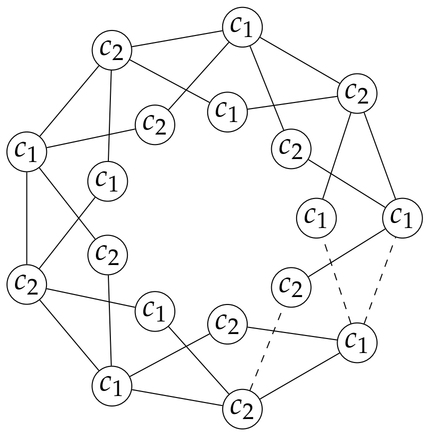


Figure 4 *Splitting graph of cycles with 2-ENP-coloring.*

The following theorem determines the equitable defective number of splitting graph of complete graphs.

Theorem 6. *The equitable defective number of splitting graph of complete graph $S(K_n)$ is given by*

(i) if $k=2$, then $b^k_{\chi_e}(S(K_n)) = \begin{cases} b^k_{\chi_e}(K_n) + \frac{n(n-2)}{2}, & \text{if } n \text{ is even,} \\ b^k_{\chi_e}(K_n) + 2\lfloor \frac{n}{2} \rfloor^2, & \text{if } n \text{ is odd,} \end{cases}$

(ii) if $k=3$, then $b^k_{\chi_e}(S(K_n)) = \begin{cases} b^k_{\chi_e}(K_n) + \frac{n(n-3)}{3}, & \text{if } n \equiv 0 \pmod 3, \\ b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (3\lfloor \frac{n}{3} \rfloor - 1), & \text{if } n \equiv 1 \pmod 3, \\ b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (3\lfloor \frac{n}{3} \rfloor + 1), & \text{if } n \equiv 2 \pmod 3, \end{cases}$

(iii) if $k \ge 4$, then $b^k_{\chi_e}(S(K_n)) = b^k_{\chi_e}(K_n) + 2(n-k)$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the complete graph and $u_1, u_2, \ldots, u_n$ be the corresponding vertices. The equitable chromatic number of splitting graph of complete graphs is n. Hence, in an ENP-coloring, we need to consider all the cases from $k = 2, 3, \ldots, n-1$ as below. The equitable defective number of complete graphs have been investigated in [7]. Thus, we consider only the bad edges between v_i's and u_i's.

Case 1. Let $k = 2$ and consider the following subcases.

Subcase 1.1. When $k = 2$ and n is even, assign the v_i's (where $1 \leq i \leq n$) with the two available colors c_1 and c_2 in a cyclic order and assign the corresponding u_i's with the same colors such that both v_i and u_i receive the same color. Here, it can be observed that each u_i is adjacent to $\frac{n}{2} - 1$ number of same colored vertices and since there are n number of u_i's, the total number of bad edges obtained is $\frac{n^2}{2} - n$. Along with that, we have the bad edges resulting from the ENP-coloring of a complete graph K_n and thus the equitable defective number is $b^k_{\chi_e}(K_n) + \frac{n(n-2)}{2}$.

Subcase 1.2. When $k = 2$ and n is odd, assign the v_i's ($1 \leq i \leq n$) and u_i's ($1 \leq i \leq n-1$) as in Subcase 1.1 and assign u_n with color c_2 to satisfy the equitability condition. We observe that there are $\lfloor \frac{n}{2} \rfloor$ number of c_1 colored u_i's and each of these vertices is adjacent to $\lfloor \frac{n}{2} \rfloor$ number of c_1 colored v_i's. Also, among the $\lceil \frac{n}{2} \rceil$ number of c_2 colored u_i's, $\lfloor \frac{n}{2} \rfloor$ number of u_i's are adjacent to $\lfloor \frac{n}{2} \rfloor - 1$ number of c_2 colored vertices and one u_i is adjacent to $\lfloor \frac{n}{2} \rfloor$ vertices. Hence, the equitable defective number is $2\lfloor \frac{n}{2} \rfloor^2$. Along with the bad edges resulting from the adjacency within v_i's, the equitable defective number is $b^k_{\chi_e}(K_n) + 2\lfloor \frac{n}{2} \rfloor^2$.

Case 2. Assume that $k = 3$. Here, we consider the following three subcases.

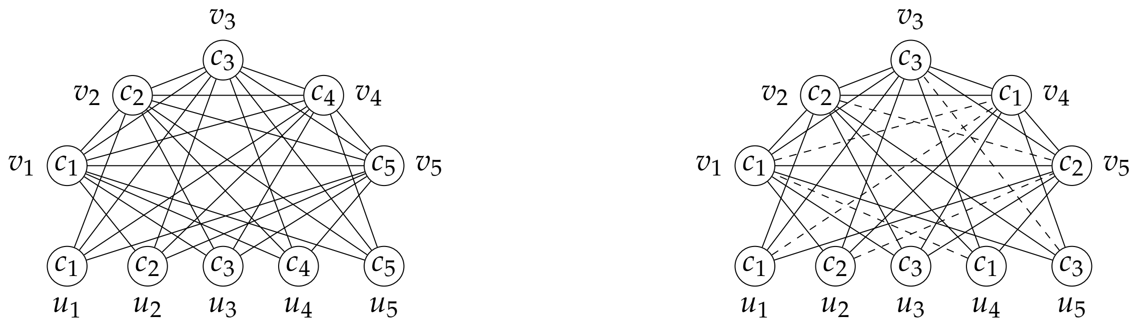
Subcase 2.1. When $k = 3$ and $n \equiv 0 \pmod 3$, assign the v_i's with the three available colors c_1, c_2 and c_3 alternatively and assign the corresponding u_i's with the same colors assigned to v_i. Here, when we consider the u_i's, each u_i repeats $\frac{n}{3}$ times and adjacent to $\frac{n}{3} - 1$ number of same colored v_i's. Along with the bad edges obtained from the complete graph, the equitable defective number is $b^k_{\chi_e}(K_n) + \frac{n(n-3)}{3}$.

Subcase 2.2. When $k = 3$ and $n \equiv 1 \pmod 3$, assign the v_i's and u_i's where $1 \leq i \leq n-1$ as in Subcase 2.1 and assign the vertices v_n and u_n with c_1 and c_2 respectively. When we consider the u_i's, it can be observed that, color c_1 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each c_1 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ number of c_1 colored v_i's. On the other hand, color c_2 is repeated $\lceil \frac{n}{3} \rceil$ times and among those vertices, $\lfloor \frac{n}{3} \rfloor$ number of c_2 colored u_i's are adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ same colored v_i's and one c_2 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ vertices. Moreover, color c_3 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each c_3 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ number of c_3 colored v_i's. Along with that we have the bad edges resulting from the ENP-coloring of complete graphs and hence the equitable defective number is $b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (3\lfloor \frac{n}{3} \rfloor - 1)$.

Subcase 2.3. When $k = 3$ and $n \equiv 2 \pmod 3$, assign the v_i's and u_i's where $1 \leq i \leq n-2$ as in the previous subcases and assign v_{n-1} and u_{n-1} with color c_1. To maintain the equitability condition, assign the vertex v_n with color c_2 and u_n with color c_3. Here, we have the following observations. Considering the colors assigned to u_i's, color c_1 is repeated $\lceil \frac{n}{3} \rceil$ times and each c_1 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ number of c_1 colored v_i's. And color c_2 is repeated $\lfloor \frac{n}{3} \rfloor$ times and each c_2 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ number of c_2 colored v_i's. Also color c_3 is repeated $\lceil \frac{n}{3} \rceil$ times and among those vertices, $\lfloor \frac{n}{3} \rfloor$ number of c_3 colored u_i's are adjacent to $\lfloor \frac{n}{3} \rfloor - 1$ number of c_3 colored v_i's and one c_3 colored u_i is adjacent to $\lfloor \frac{n}{3} \rfloor$ number of same colored vertices. Along with the bad edges resulting from the complete graph, the equitable defective number is, $b^k_{\chi_e}(K_n) + \lfloor \frac{n}{3} \rfloor (3\lfloor \frac{n}{3} \rfloor + 1)$.

Case 3. When $k \geq 4$, assign the vertices $v_1, v_2, \ldots, v_{k\lfloor \frac{n}{k} \rfloor}$ with the available k colors in a cyclic order and assign the corresponding u_i's with the same colors. Further, assign the remaining vertices with the available colors in an equitable manner and along with the bad edges obtained from the complete graph, the equitable defective number is $b^k_{\chi_e}(K_n) + 2(n-k)$. This completes the proof. $\square$

Figure 5 depicts a 5 equitable coloring and 3-ENP-coloring of splitting graph of complete graphs.



(a) *$S(K_5)$ with 5-equitable coloring* ***(b)*** *3-ENP-coloring of $S(K_5)$*

Figure 5 *Splitting graph of complete graphs with 5 equitable coloring and 3 ENP-coloring.*

Theorem 7. *For a star graph $K_{1,n}$,* $b^k_{\chi_e}(S(K_{1,n})) = \begin{cases} n-1, & \text{if } k=2, \\ \lfloor \frac{2n+2}{k} \rfloor - 2, & \text{if } k \geq 3. \end{cases}$

Proof. Splitting graph of star graph consists of $2n+2$ vertices. Let v_0 be the central vertex of the star graph $K_{1,n}$ and $v_1, v_2, \cdots, v_n$ be the set of vertices which are adjacent to v_0 and let $u_0, u_1, u_2, \cdots, u_n$ be the vertices corresponding to these vertices. Let $c_1, c_2, \cdots, c_k$ be the available colors in an ENP-coloring and let $V_1, V_2, \cdots, V_n$ be the corresponding color classes. Here, we consider the following cases.

Case 1. When $k=2$, we have the two available colors c_1 and c_2 and both color classes contain $n+1$ vertices. Assign the vertices in such a way that both v_0 and u_0 receive the same color and other vertices can be colored in an equitable manner. Here, the equitable defective number is $n-1$.

Case 2. When $k \geq 3$, assign the available colors by considering the equitability condition we obtain $\lfloor \frac{2n+2}{k} \rfloor - 2$ bad edges. $\square$

The following result describes the ENP-coloring and the corresponding equitable defective number of splitting graph of wheel graphs.

Theorem 8. *The equitable defective number of splitting graph of wheels $S(W_{1,n})$ is given by*

$$b^k_{\chi_e}(S(W_{1,n})) = \begin{cases} \frac{3n}{2}, & \text{if } k=2,\ n \text{ is even}, \\ \frac{3(n+1)}{2}, & \text{if } k=2,\ n \text{ is odd}, \\ \lfloor \frac{2n+2}{k} \rfloor - 2, & \text{if } k \geq 3. \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the rim of the wheel graph and let v_0 be the central vertex. Let $u_0, u_1, u_2, \ldots, u_n$ be the corresponding vertices of v_i's. In an ENP-coloring of splitting graph of wheels, we consider the following cases.

Case 1. When $k=2$ and n is even, assign the vertices of the rim of the wheel with the two available colors c_1 and c_2. Assign the central vertex v_0 with color c_2 which results in $\frac{n}{2}$ bad edges. Now, assign the vertices u_i's with the same colors which are assigned to v_i's, where

$1 \le i \le n$. Further, assign u_0 with color c_1 to satisfy the equitability condition. We observe that there are $\frac{n}{2}$ bad edges which are incident with u_0 and $\frac{n}{2}$ bad edges which are incident with v_0. Hence, the equitable defective number is $\frac{3n}{2}$.

Case 2. When $k = 2$ and n is odd, assign the vertices $v_1, v_2, \ldots, v_n$ as in Case 1 and assign the central vertex v_0 with color c_2, we obtain $\lceil \frac{n}{2} \rceil$ bad edges. Now, assign the u_i's with the same colors which are assigned to v_i's where $0 \le i \le n$. We observe that there are $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with v_0 and $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with u_0. Also there are two more bad edges that is $v_1 u_n$ and $u_1 v_n$. Hence, the equitable defective number in this case is $\frac{3(n+1)}{2}$.

Case 3. When $k \ge 3$, assign v_0 and u_0 with color c_1 and assign other v_i's with the remaining available colors alternatively. In an ENP-coloring, each color class should contain either $\lceil \frac{2n+2}{k} \rceil$ vertices or $\lfloor \frac{2n+2}{k} \rfloor$ vertices. Assume that the central vertex v_0 lies in the color class with minimum cardinality $\lfloor \frac{2n+2}{k} \rfloor$ and since only one v_i received color c_1, $\lfloor \frac{2n+2}{k} \rfloor - 1$ number of u_i's should receive color c_1. Since u_0 is colored with color c_1, $\lfloor \frac{2n+2}{k} \rfloor - 2$ number of u_i's other than u_0 should receive color c_1 and since v_0 is adjacent to all u_i's except u_0, we get $\lfloor \frac{2n+2}{k} \rfloor - 2$ bad edges. Thus, the equitable defective number is $\lfloor \frac{2n+2}{k} \rfloor - 2$. □

The following theorem discusses the ENP-coloring of splitting graph of helm graphs.

Theorem 9. *The equitable defective number of splitting graph of helm $S(H_{1,n})$ is given by*

$$b^k_{\chi_e}(S(H_{1,n})) = \begin{cases} \frac{3n}{2}, & \text{if } k = 2, n \text{ is even,} \\ \frac{3n+5}{2}, & \text{if } k = 2, n \text{ is odd,} \\ 3, & \text{if } k = 3, n \text{ is odd.} \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the rim of the wheel graph and let v_0 be the central vertex. Let $v'_1, v'_2, \ldots, v'_n$ be the pendent vertices such that each v_i is adjacent to v'_i for $i = 1, 2, \ldots, n$. Let $u_0, u_1, u_2, \ldots, u_n, u'_1, u'_2, \ldots, u'_n$ be the vertices corresponding to all v_i's. In an ENP-coloring, we consider the following cases.

Case 1. When $k = 2$ and n is even, assign the rim vertices $v_1, v_2, \ldots, v_n$ with colors c_1 and c_2 alternatively. Now, assign the pendent vertices $v'_1, v'_2, \ldots, v'_n$ such that if v_i is assigned with color c_1 (or c_2), then assign v'_i with color c_2 (or c_1). Also, assign the central vertex v_0 with color c_1, so that we obtain $\frac{n}{2}$ bad edges among the spokes. Further, for $i = 1, 2, \ldots, n, 1', 2', \ldots, n'$, assigning the u_i's with the same colors assigned to v_i's and assign u_0 with color c_2 to satisfy the equitability condition. Thus, we obtain $\frac{n}{2}$ bad edges which are incident with v_0 and $\frac{n}{2}$ bad edges which are incident with u_0. Hence, in this case the equitable defective number is $\frac{3n}{2}$.

Case 2. When $k = 2$ and n is odd, assign the v_i's and u_i's with the available colors as in Case 1 except for v_0 and u_0. Now, assign the central vertex v_0 with color c_2 we obtain $\lfloor \frac{n}{2} \rfloor$ bad edges among the spokes and one bad edge on the rim. Further, assign u_0 with color c_1, we obtain $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with v_0 and $\lceil \frac{n}{2} \rceil$ bad edges which are incident with u_0. Along with those, we get two more bad edges which are $u_1 v_n$ and $v_1 u_n$. Hence, the equitable defective number is $\frac{3n+5}{2}$.

Case 3. For even n, the equitable chromatic number of splitting graph of helm is 3. Therefore, in an ENP-coloring, we consider only one case, $k = 3$ and n is odd. Assign the rim vertices of the helm $v_1, v_2, \ldots, v_{n-1}$ with colors c_1 and c_2 alternatively and assign v_n with color c_3. Now, assign the vertices $v'_1, v'_2, \ldots, v'_{n-1}$ such that if v_i is assigned with color c_1 or c_2, then assign v'_i with color c_2 or c_3 and assign v'_n with color c_1. Further, assign the central vertex v_0 with color c_3 which results in only one bad edge among the spokes. Then, assign the u_i's where $i = 0, 1, 2, \ldots, n$ with the same colors assigned to v_i's, where $0 \leq i \leq n$. Also, assign the remaining u_i's where $i = 1', 2', \ldots, n'$ with the available three colors by satisfying the equitability condition, we obtain two bad edges ($v_0 u_n$ and $u_0 v_n$). Hence, the equitable defective number in this case is 3. □

3 ENP-Coloring of Shadow Graphs

The *shadow graph* of a graph G denoted by $Sh(G)$, is the graph obtained by taking two copies of G say G' and G'', and join each vertex v' in G' to the neighbours of the corresponding vertex v'' in G''.

The next result discusses the ENP-coloring of shadow graph of paths. The shadow graph of a path is obtained by taking two copies of paths and by joining each vertex of the first copy of the path to the neighbours of the corresponding vertex of the second copy of the path.

Theorem 10. *For a path P_n, $b^k_{\chi_e}(Sh(P_n)) = 2$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of the path P'_n and let $u_1, u_2, \ldots, u_n$ be the vertices of the corresponding path P''_n. According to the definition of shadow graph of paths, the neighbours of v_i should be adjacent to u_i $\forall i$. We observe that the equitable chromatic number of shadow graph of paths is 2 when n is even and 3 when n is odd. Thus, in an ENP-coloring, we consider only one case as $k = 2$ and n is odd. Here, we assign the vertices v_i's with the available colors c_1 and c_2 alternatively. Now, assign the corresponding u_i's where $1 \leq i \leq n-1$ with the same colors as assigned to v_i's. Further, assign u_n with color c_2 to satisfy the equitability condition, we obtain only two bad edges $u_{n-1}u_n$ and $v_{n-1}u_n$. Thus, the equitable defective number is 2. □

Figure 6 depicts a 2-ENP-coloring of shadow graph of paths.

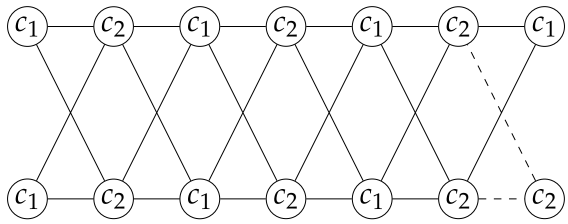


Figure 6 *Shadow graph of paths with* 2*-ENP-coloring.*

The following theorem investigates the ENP-coloring of shadow graph of cycles.

Theorem 11. *For a cycle C_n, $b^k_{\chi_e}(Sh(C_n)) = 4$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of the first copy of the cycle C'_n and let $u_1, u_2, \ldots, u_n$ be the vertices of the second copy of the cycle C''_n. Here, each v_i, $1 \leq i \leq v_{n-1}$, in C'_n is adjacent to u_{i-1} and u_{i+1} in C''_n, v_n is adjacent to u_{n-1} and u_1, v_1 is adjacent to u_n and u_1. The equitable chromatic number of shadow graph of cycles is 2 when n is even and 3 when n is odd. Thus, in an ENP-coloring we consider only one case as $k = 2$ when n is odd. Now, assign the vertices with the available colors as in Theorem 10 we obtain four bad edges such as $v_1 v_n$, $u_1 v_n$, $u_{n-1} u_n$ and $v_{n-1} u_n$. Hence, the equitable defective number is 4. □

Figure 7 depicts a 2-ENP-coloring of shadow graph of a cycle.

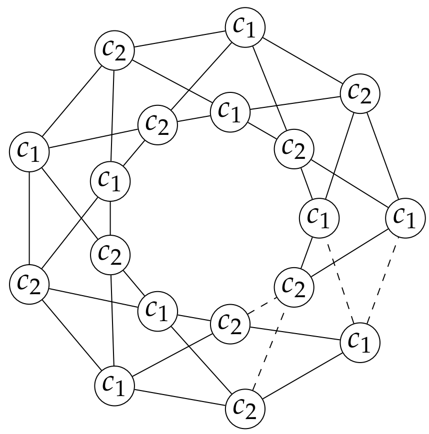


Figure 7 *Shadow graph of cycles with* 2*-ENP-coloring.*

The following theorem discusses the ENP-coloring of shadow graph of complete graphs. A shadow graph of a complete graph is obtained by taking two copies of complete graphs and by joining the vertices depending on the adjacency in the original graph. Thus, we see that the following theorem is an immediate consequence of Theorem 2.

Theorem 12. *The equitable defective number of shadow graph of complete graphs $Sh(K_n)$ is $b^k_{\chi_e}(Sh(K_n)) = b^k_{\chi_e}(K_n) + b^k_{\chi_e}(S(K_n))$.*

Proof. The proof follows from Theorem 2 and by the rule of shadow graph construction. □

Theorem 13. *For a star graph $K_{1,n}$,*

$$b^k_{\chi_e}(Sh(K_{1,n})) = \begin{cases} 2(n-1), & \text{if } k = 2, \\ 2[\lfloor \frac{2n+2}{k} \rfloor - 2], & \text{if } k \geq 3. \end{cases}$$

Proof. Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of the first copy of the shadow graph of star graph and $u_0, u_1, u_2, \ldots, u_n$ be the vertices of the second copy. Here, we consider the following cases.

Case 1. When $k = 2$, among the two color classes, each color class contains $n + 1$ vertices. Assign v_0 and u_0 with the same color and assign the remaining vertices in an equitable manner, we obtain $n - 1$ bad edges among the v_i's and u_i's and $n - 1$ bad edges between the v_i's and u_i's. Thus, the equitable defective number is $2(n - 1)$.

Case 2. When $k \geq 3$, assign the available colors by considering the equitability condition, we obtain $\lfloor \frac{2n+2}{k} \rfloor - 2$ bad edges among the u_i's and v_i's and $\lfloor \frac{2n+2}{k} \rfloor - 2$ bad edges between the u_i's and v_i's. Thus, the equitable defective number is $2[\lfloor \frac{2n+2}{k} \rfloor - 2]$. □

In view of Theorem 7, we can restate Theorem 13 as follows.

Theorem 14. *For a star graph $K_{1,n}$, $b^k_{\chi_e}(Sh(K_{1,n})) = 2b^k_{\chi_e}(S(K_{1,n}))$.*

Figure 8 depicts a 3-ENP-coloring of the shadow graphs of star graphs.

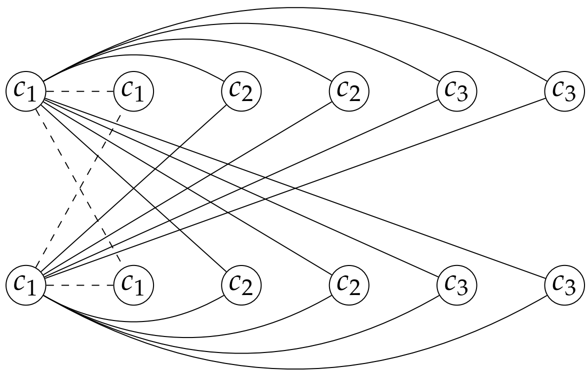


Figure 8 *Shadow graph of star graphs with 3-ENP-coloring.*

In the next theorem, we examine the ENP-coloring of shadow graph of wheel graphs.

Theorem 15. *The equitable defective number of shadow graph of wheels $Sh(W_{1,n})$ is*

$$b^k_{\chi_e}(Sh(W_{1,n})) = \begin{cases} 2n, & \text{if } k = 2,\ n \text{ is even,} \\ 2(n+1), & \text{if } k = 2,\ n \text{ is odd,} \\ 2\lfloor \frac{2n+2}{3} \rfloor - 4, & \text{if } k \geq 3. \end{cases}$$

Proof. Let $v_0, v_1, v_2, \ldots, v_n$ be the vertices of the first copy of the wheel graph and let $u_0, u_1, u_2, \ldots, u_n$ be the corresponding vertices in the second copy of wheel graph. In an ENP-coloring, we consider the following cases.

Case 1. When $k = 2$ and n is even, assign the v_i's with the available colors c_1 and c_2 alternatively and assign the central vertex v_0 with color c_2, we obtain $\frac{n}{2}$ bad edges among the spokes. Now, assign the u_i's, $1 \leq i \leq n$, with the same colors assigned to v_i's and assign u_0 with color c_1 to satisfy the equitability condition. Here, we obtain $\frac{n}{2}$ bad edges which are incident with v_0 and $\frac{n}{2}$ bad edges which are incident with u_0. Along with that we obtain $\frac{n}{2}$ bad edges among the spokes of the second copy of the wheel. Hence, the equitable defective number is $2n$.

Case 2. When $k = 2$ and n is odd, assign all the v_i's as in Case 1, we obtain $\lfloor \frac{n}{2} \rfloor$ spokes bad edges and one bad edge on the rim of the wheel. Now, assign the u_i's where $0 \leq i \leq n$ with the same colors assigned to v_i's we obtain $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with v_0 and $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with u_0. Also, there are two more bad edges $v_1 u_n$ and $u_1 v_n$. Now, the second copy of the wheel also result in $\lceil \frac{n}{2} \rceil$ bad edges and hence the equitable defective number is $2(n+1)$.

Case 3. When $k \geq 3$, assign v_0, u_0, v_n and u_n with color c_1. Assign the remaining v_i's using all available colors except c_1. Thus, we get a $v_0 v_n$ bad edge and $u_0 v_n$ bad edge. In an ENP-coloring, each color class should contain either $\lceil \frac{2n+2}{k} \rceil$ vertices or $\lfloor \frac{2n+2}{k} \rfloor$ vertices. Considering the color class containing color c_1 with minimum cardinality, c_1 has to be repeated $\lfloor \frac{2n+2}{k} \rfloor$ times. Since two v_i's receive color c_1, $\lfloor \frac{2n+2}{k} \rfloor - 2$ number of u_i's should receive color c_1. Thus, assign the u_i's, $1 \leq i \leq n-1$, using the available colors and c_1 alternatively considering the equitability condition. Since v_0 is adjacent to all u_i's where $1 \leq i \leq n$, we obtain $\lfloor \frac{2n+2}{k} \rfloor - 3$ bad edges which are incident with v_0. Since u_0 is the central vertex, $\lfloor \frac{2n+2}{k} \rfloor - 3$ bad edges are there among the spokes of the second copy of the wheel. Hence, the equitable defective number is $2\lfloor \frac{2n+2}{k} \rfloor - 4$. $\square$

In the following result, we discuss the ENP-coloring of shadow graph of helm graphs.

Theorem 16. *The equitable defective number of shadow graph of helm $Sh(H_{1,n})$ is given by*

$$b^k_{\chi_e}(Sh(H_{1,n})) = \begin{cases} 2n, & \text{if } k = 2,\ n \text{ is even}, \\ 2(n+2), & \text{if } k = 2,\ n \text{ is odd}, \\ 4, & \text{if } k = 3,\ n \text{ is odd}. \end{cases}$$

Proof. Let the labelling of vertices be given in Theorem 9 and consider the following cases.

Case 1. When $k = 2$ and n is even, assign the vertices $v_1, v_2, \ldots, v_n$ with the available colors c_1 and c_2 alternatively and assign $v'_1, v'_2, \ldots, v'_n$ in such a way that if v_i is assigned with color c_1 (or c_2) assign the corresponding u_i with color c_2 (or c_1). Now, assign v_0 with color c_1 and u_0 with color c_2 we observe that we obtain $\frac{n}{2}$ bad edges resulting from the first copy of the helm, $\frac{n}{2}$ bad edges which are incident with v_0, $\frac{n}{2}$ bad edges which are incident with u_0 and $\frac{n}{2}$ bad edges from the second copy of the helm. Hence, the equitable defective number is $2n$.

Case 2. When $k = 2$ and n is odd, assign the vertices as in Case 1 we obtain $\lceil \frac{n}{2} \rceil$ bad edges resulting from the first copy of helm, $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with v_0 and $\lfloor \frac{n}{2} \rfloor$ bad edges which are incident with u_0. Along with that we obtain two more bad edges $v_1 u_n$ and $u_1 v_n$. Also, considering the $\lceil \frac{n}{2} \rceil + 1$ bad edges from the second copy of the helm the equitable defective number is $2(n+2)$.

Case 3. When $k = 3$ and n is odd, assign $v_1, v_2, \ldots, v_{n-1}$ with the colors c_1 and c_2 alternatively and assign the corresponding u_i's with the same colors as assigned to v_i's. And assign the vertices v_0, v_n, u_0, u_n with color c_3, we obtain one bad edge among the spokes of each copy of helm. Now, all the remaining vertices can be assigned by the three available colors in an equitable manner we obtain two bad edges $v_0 u_n$ and $u_0 v_n$. Thus, the equitable defective number is 4. □

Conclusion

In this paper, we discussed the ENP-coloring of Mycielski graph of complete graphs and star graphs, splitting graph and shadow graph of paths, cycles, complete graphs, star graphs, wheel graphs and helm graphs. The equitable defective number of these graphs are also determined. This study can be extended to other derived graph classes, graph products and graph powers. Further investigation is possible for graph operations such as union, intersection, complement and join of fundamental graph classes.

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Хосе С., Надуват С. *Про рівномірне майже правильне розфарбування деяких похідних класів графів* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 529–542.

Рівномірне майже правильне розфарбування графа G — це дефектне розфарбування, в якому кількість вершин у будь-яких двох колірних класах відрізняється більш ніж на одиницю, а отримані погані ребра мінімізуються шляхом обмеження кількості класів кольорів, які можуть мати суміжність серед власних елементів. У цій статті досліджується рівномірне майже правильне розфарбування деяких похідних класів графів, таких як графи Мичельського, розщеплювальні графи та тіньові графи.

Ключові слова і фрази: рівномірне майже правильне розфарбування, граф Мичельського, розщеплюваний граф, тіньовий граф.