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FUNDAMENTAL SOLUTION OF ONE CLASS OF DEGENERATE SECOND ORDER PARABOLIC EQUATIONS

The article deals with the fundamental solution of a linear degenerate parabolic equation with $4n$ degrees of freedom, which generalizes the Kolmogorov equation. The coefficients of the equation are continuous, bounded, and satisfy the Gelder condition with exponent $0 < \alpha \leq 1$. They used the Levi method and investigated the behavior of the volumetric potential generated by parametrix. The derivatives of the fundamental solution are established.

Key words and phrases: Fundamental solution, Kolmogorov equation, Levy method, degenerate parabolic equations, diffusion processes.

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Introduction

In this work, we build a fundamental solution of degenerate second order linear parabolic equations with three groups of parabolic degeneracy. Such equations describe diffusion systems with 4n degrees of freedom.

We use the explicit form of the fundamental solution (parametrix) of the equation with parameters and apply to it the method of Levy E.E., this work is a generalization of work [1], it has one $n--$ dimensional group of degenerate variables corresponding to the system of diffusion processes with $2n$ degrees of freedom.

This problem for equations with one group of parabolic degeneracy variables is solved in [9] with two groups of variables [1],[2],[3]. Interest in such equations is caused by their application in economic theory [8]-[9].

1 Designation and task statement. Main results.

Let's denote $n \in N$, $x \in R^{4n}$, $x = (x_1, x_2, x_3, x_4)$, $x_j \in R^n$, $j = \overline{1,4}$, $\xi \in R^{4n}$, $\xi_j \in R^n$, $j = \overline{1,4}$, $0 < \tau < t \leq T < +\infty$,$\overline{x} := (x_1, x_2)$, $\overline{\overline{x}} := (x_1 x_2 x_3)$.

$$\rho_{1k}(x_1, t; \xi_1, \tau) := 2^{-1}(x_{1k} - \xi_{1k})(t-\tau)^{-\frac{1}{2}},$$

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$$\rho_{2k}\left(\overline{x},t;\overline{\xi},\tau\right) := \sqrt{3}\left(x_{2k}-\xi_{2k}+2^{-1}(x_{1k}+\xi_{1k})(t-\tau)\right)(t-\tau)^{-\frac{3}{2}},$$

$\rho_{3k}\left(\overline{\overline{x}},t;\overline{\overline{\xi}},\tau\right) := 6\sqrt{5}(t-\tau)^{-\frac{3}{2}}(x_{3k}-\xi_{3k}+2^{-1}\left(x_{2k}+\xi_{2k}\right)(t-\tau)+\left(x_{1k}-\xi_{1k}\right)$ $(t-\tau)^2 12^{-1}$,

$\rho_{4k}\left(x,t;\xi,\tau\right) := 60\sqrt{7}(t-\tau)^{-\frac{7}{2}}(x_{4k}-\xi_{4k}+2^{-1}\left(x_{3k}+\xi_{3k}\right)(t-\tau)+$ $10^{-1}(x_{2k}-\xi_{2k})(t-\tau)^2+(x_{1k}+\xi_{1k})(t-\tau)^3 120^{-1}),$

$$\rho_k^2\left(x,t;\xi,\tau\right)=\rho_{1k}^2\left(x_1,t;\xi_1,\tau\right)+\rho_{2k}^2\left(\overline{x},t;\overline{\xi},\tau\right)+\rho_{3k}^2\left(\overline{\overline{x}},t;\overline{\overline{\xi}},\tau\right)+\rho_{4k}^2\left(x,t;\xi,\tau\right),\quad k=\overline{1,n}.$$

$r_1\left(t-\tau;x_1,\xi_1\right)=\left(x_1-\xi_1\right)(t-\tau)^{-\frac{1}{2}}, r_2\left(t-\tau;\overline{x},\xi_2\right)=\left(x_2-\xi_2+x_1\left(t-\tau\right)\right)(t-\tau)^{-\frac{3}{2}},$
$r_3\left(t-\tau;\overline{\overline{x}},\xi_3\right)=\left(x_3-\xi_3+x_2\left(t-\tau\right)+x_1(t-\tau)^2 2^{-1}\right)(t-\tau)^{-\frac{5}{2}},$

$$r_4\left(t-\tau;x,\xi_4\right)=\left(x_4-\xi_4+x_3\left(t-\tau\right)+x_2(t-\tau)^2 2^{-1}+x_1(t-\tau)^3(3!)^{-1}\right)(t-\tau)^{-\frac{7}{2}}.$$

Let

$$w^{y,\beta}\left(x,t;\xi,\tau\right)=\sum_{l,j=1}^{n}a^{l\ j}\left(y,\beta\right)\left(\rho_{1j}\left(x_1,t;\xi_1,\tau\right)\rho_{1l}\left(x_1,t;\xi_1,\tau\right)+\rho_{2j}\left(\overline{x},t;\overline{\xi},\tau\right)\times\right.$$
$$\left.\rho_{2l}\left(\overline{x},t;\overline{\xi},\tau\right)+\rho_{3j}\left(\overline{\overline{x}},t;\overline{\overline{\xi}},\tau\right)\rho_{3l}\left(\overline{\overline{x}},t;\overline{\overline{\xi}},\tau\right)+\rho_{4j}\left(x,t;\xi,\tau\right)\rho_{4l}\left(x,t;\xi,\tau\right)\right),$$

where $\left(a^{l\ j}\right)_{l,j=1}^{n}$– inverted matrix to matrix $(a_{lj})_{l,j=1}^{n}$, $a_{lj}=a_{jl}$, $y\in R^{4n}$, $\beta\in[\tau,T]$, (y,β)– fixed point [4].

Consider the equation:

$$Lu\left(x,t\right)=\sum_{k,l=1}^{n}a_{kl}\left(x,t\right)\partial^2_{x_{1k}x_{1l}}u\left(x,t\right)+\sum_{k=1}^{n}a_k\left(x,t\right)\partial_{x_{1k}}u\left(x,t\right)+a\left(x,t\right)u\left(x,t\right)+$$
$$\sum_{j=1}^{3}\sum_{\mu=1}^{n}x_{j\mu}\partial_{x_{j+1\ \mu}}u\left(x,t\right)-\partial_t u\left(x,t\right)=0, \qquad (1)$$

where $\left(a_{kl}\left(x,t\right)\right)_{k,l=1}^{n}$ a positively defined matrix in the band $\Pi_{[0,T]}=\{(x,t)\,,\ x\in R^{4n},\ t\in[0,T]\}$.

We assume

a_1) there are positive steels $v_0>0, v_1>0$, such that for any real $\xi_1\in R^n$

$$v_0|\xi_1|^2=\sum_{l,j=1}^{n}a_{lj}(x,t)\xi_{1l}\xi_{1j}\le v_1|\xi_1|^2, \qquad (2)$$

$$v_0|\xi|^2=\sum_{l,j=1}^{n}a^{lj}(x,t)\xi_{1l}\xi_{1j}\le v_1|\xi_1|^2,$$

to all $(x,t)\in\Pi_{[0,T]}$, where $a^{lj}\left(x,t\right)$– elements of the inverse matrix up to $\left(a_{lj}\left(x,t\right)\right)_{l,j=1}^{n}$

a_2) Coefficient $a_{lj}\left(x,t\right), a_l\left(x,t\right), a\left(x,t\right)$ continuous and limited in $\Pi_{[0,T]}$.

a_3) For any two points $(x,t)\in\Pi_{[0,T]}, (x_0,t_0)\in\Pi_{[0,T]}$ there is a constant $A>0,\ \ i\ \ 0<\alpha\leq 1$, what

$$|a_{lj}(x,t)-a_{lj}(x_0,t_0)|\leq A\left(|x-x_0|^{\alpha}+|\ t-t_0|^{\frac{\alpha}{2}}\right);$$

$$|a_l(x,t)-a_l(x_0,t_0)|\leq A|x-x_0|^{\alpha};$$

$$|a(x,t)-a(x_0,t_0)|\leq A|x-x_0|^{\alpha}.$$

Denote by $t>\tau$

$$G_0(x,t;\xi,\tau,\xi,\tau)=\Phi(\xi,\tau)(t-\tau)^{-8n}\exp\left\{-\mathrm{w}^{y,\beta}(x,t;\xi,\tau)\right\},$$

$$G_0(x,t;\xi,\tau,\xi,\tau)=0,\qquad t\leq\tau,$$

where

$$\Phi(x,t)=\left(\pi^{-2}180\sqrt{105}\right)^{n}/(\det(a^{lj}(t,x)))^{-2}. \tag{3}$$

The main result

Theorem 1. *If 1) $f(x,t)$ satisfies the uniform condition of Gelder on x with the index $\alpha,\ 0<\alpha\leq 1$ in $\Pi_{[0,T]}$*

2) continuous and limited in $\Pi_{[0,T]}$, then the volume potential

$$V(x,t)=\int_{\tau}^{t}d\beta\int_{R^{4n}}G_0(x,t;\xi,\beta,\xi,\beta)f(\xi,\beta)\,d\xi, \tag{4}$$

has continuous derivatives $\partial_t\ V(x,t),\ \partial_{x_{jk}}V(x,t),\ \partial^2_{x_{1k}x_{1l}}V(x,t),\ j=\overline{1,4},\ \ k=\overline{1,4}$ and correct formulas

$$\partial_t\ V(x,t)=\ f(x,t)+\int_{\tau}^{t}d\beta\int_{R^{4n}}\partial_t\ G_0(x,t;\xi,\beta,\xi,\beta)\left[f(\xi,\beta)-f(x,\beta)\right]d\xi+ \tag{5}$$
$$\int_{\tau}^{t}f(x,\beta)\,d\beta\int_{R^{4n}}\partial_t\ G_0(x,t;\xi,\beta,\xi,\beta)d\xi.$$

$$\partial_{x_{1k}}V(x,t)=\int_{\tau}^{t}d\beta\int_{R^{4n}}\partial_{x_{1k}}G_0(x,t;\xi,\beta,\xi,\beta)f(\xi,\beta)\,d\xi,\quad k=\overline{1,n}. \tag{6}$$

$$\partial_{x_{jk}}\ V(x,t)=\int_{\tau}^{t}d\beta\int_{R^{4n}}\partial_{x_{jk}}\ G_0(x,t;\xi,\beta,\xi,\beta)\left[f(\xi,\beta)-f(x,\beta)\right]d\xi \tag{7}$$
$$+\int_{\tau}^{t}f(x,\beta)\,d\beta\int_{R^{4n}}\partial_{x_{jk}}\ G_0(x,t;\xi,\beta,\xi,\beta)d\xi,$$

$$\partial^2_{x_{1k}x_{1l}}\ V(x,t)=\int_{\tau}^{t}d\beta\int_{R^{4n}}\partial^2_{x_{1k}x_{1l}}\ G_0(x,t;\xi,\beta,\xi,\beta)\left[f(\xi,\beta)-f(x,\beta)\right]d\xi \tag{8}$$
$$+\int_{\tau}^{t}f(x,\beta)\,d\beta\int_{R^{4n}}\partial^2_{x_{1k}x_{1l}}\ G_0(x,t;\xi,\beta,\xi,\beta)d\xi,$$

Theorem 2. *If the conditions are satisfied: $a_1) - a_3)$ then equation* (1) *has a fundamental solution* $\Gamma(x,t;\xi,\tau)$.

$$\Gamma(x,t;\xi,\tau) = G_0(x,t;\xi,\tau,\xi,\tau) + \int_\tau^t d\beta \int_{R^{4n}} G_0(x,t;\gamma,\beta,\gamma,\beta)\varphi\text{up}(\gamma,t;\xi,\tau)\,d\gamma, \qquad (9)$$

where $\varphi(x,t;\xi,\tau)$– *the desired function is such that* $L\Gamma(x,t;\xi,\tau) \equiv 0, \quad t < \tau.$

The volumetric potential of the function $f(x,t)$ relatively parametrix $G_0(x,t;\xi,\tau,\xi,\tau)$.
Consider the Cauchy problem

$$L_0 u(x,t) = \sum_{k,l=1}^{n} a_{kl}(y,\beta)\partial^2_{x_{1k}x_{1l}} u(x,t) + \sum_{j=1}^{3}\sum_{\mu=1}^{n} x_{j\mu}\partial_{x_{j+1\ \mu}} u(x,t) - \partial_t u(x,t) = 0,$$

$(x,t) \in \Pi_{[0,T]}$, (y,β)– fixed point, parameter $(y,\beta) \in \Pi_{[0,T]}$.

$$u(x,t)\,|_{t=\tau} = u_0(x), \qquad 0 \le \tau < t \le T < +\infty.$$

Using the Fourier transform and [5], we find the fundamental solution of the Cauchy problem for L_0 при $t < \tau$, we define it 0, we get it $G_0(x,t;\xi,\tau,y,\beta)$ and substituting $(y,\beta) = (\xi,\tau)$ we get the formula (3):
$G_0(x,t;\xi,\tau,\xi,\tau) = \Phi(\xi,\tau)(t-\tau)^{-8n}\exp\left\{-\mathrm{w}^{y,\beta}(x,t;\xi,\tau)\right\}$, at $t > \tau$.
$G_0(x,t;\xi,\tau,\xi,\tau) = 0$ at $t < \tau$
For $G_0(x,t;\xi,\tau,\xi,\tau)$ and its derivatives are correct estimates:

$$|\partial_t^m \partial_x^s G_0(x,t;\xi,\tau,\xi,\tau)| \le C_{ms}(t-\tau)^{-8n-\frac{7m}{2}-\sum_{j=1}^{4}\frac{(2j-1)|s_j|}{2}}$$

$$\sum_{j=1}^{3}(t-\tau)^{(3-j)m}|x_j|^m \exp\{-c_0\rho^2(x,t;\xi,\tau)\}, \qquad (10)$$

$$s = \sum_{j=1}^{4}|s_j|, \quad s_j = (s_{ji},\ldots,s_{jn}), \quad s_{jk} \in N \cup \{0\}, \quad m \in N \cup \{0\}, \quad k = \overline{1,n}.$$

$$|\partial_t^m \partial_r^s G_0(x,t;\xi,\tau,\xi,\tau) - \partial_t^m \partial_x^s G_0(x,t;\xi,\tau,\xi,\tau)| \le \qquad (11)$$

$$C(t-\tau)^{-8n-\frac{7m}{2}-\sum_{j=1}^{4}\frac{(2j-1)|s_j|}{2}}|\xi - \xi'|\sum_{j=1}^{4}(t-\tau)^{(4-j)m}|x_j|^m \exp\left\{-c_0\rho^2(x,t;\xi,\tau)\right\},$$

$t > \tau$. Because it exists $c_1,\ c_1 > 0$, where $r^2(t-\tau;x,\xi) = \rho^2(x,t;\xi,\tau)$, then in (10), (11) can be replaced ρ^2 on r^2.

It follows from estimates (10) that $\partial_t G_0(x,t;\xi,\beta,\xi,\beta)$, $\partial_{x_{jk}} G_0(x,t;\xi,\beta,\xi,\beta)$, $j = \overline{2,n}$, $k = \overline{1,n}$, have a non-integral feature at $t = \tau$.

Consider the boundary of the relation:
$\varphi_1(t-\tau,\gamma) = (t-\tau)^{-2}\exp\left\{-c_0\gamma^2(t-\tau)^{-2}\right\}$ and

$\varphi_2(t-\tau,\gamma) = (t-\tau)^{-p}\exp\left\{-c_0\gamma^2(t-\tau)^{-p}\right\}$, $p = 3,5,7$, $\gamma^2 > 0$ at $t \to \tau$. The limit is zero therefore at $0 < t-\tau < \frac{1}{2}$.

$$(t-\tau)^{-p}\exp\left\{-c_0\gamma^2(t-\tau)^{-p}\right\} \le (t-\tau)^{-2}\exp\left\{-c_0\gamma^2(t-\tau)^{-2}\right\}.$$

For $\frac{1}{2} \le t-\tau \le T$ any $\gamma \in R$ because $\varphi_1(t-\tau,\gamma)$ and $\varphi_2(t-\tau,\gamma)$ – continuous, limited, positive, there is a constant c_1^*, $c_1^* \ge \max\{c^*,1\}$, what

$$\varphi_1(t-\tau,\gamma) \le c_1^*\varphi_2(t-\tau,\gamma). \tag{12}$$

It follows from (12) that for the convergence of re-integrals from derivatives $\partial_t\, G_0(x,t;\xi,\beta,\xi,\beta)$, $\partial_{x_{jk}}\, G_0(x,t;\xi,\beta,\xi,\beta)$, $j=\overline{2,n}$, $k=\overline{1,n}$, suffice the conditions of Theorem 1.

In particular at $j=4$, let's take $h>0$

$$\begin{aligned}\partial_{x_{4k}}\, V_h(x,t) &:= \int_\tau^{t-h} d\beta \int_{R^{4n}} \partial_{x_{4k}}\, G_0(x,t;\xi,\beta,\xi,\beta) f(\xi,\beta)\, d\xi = \\ &= \int_\tau^{t-h} d\beta \int_{R^{4n}} \partial_{x_{4k}}\, G_0(x,t;\xi,\beta,\xi,\beta)\,[f(\xi,\beta) - f(x,\beta)]d\xi \\ &+ \int_\tau^{t-h} d\beta \int_{R^{4n}} \partial_{x_{4k}}\, G_0(x,t;\xi,\beta,\xi,\beta)d\xi\; f(x,\beta) = I_1 + I_2.\end{aligned} \tag{13}$$

Because

$$\begin{aligned}&|\partial_{x_{4k}}G_0(x,t;\xi,\tau,\xi,\tau) - \partial_{x_{1k}}G_0(x,t;\xi,\tau,y,\tau)|_{y=x}| \le \\ &C(t-\tau)^{-8n-\frac{7}{2}}|\xi - x|^{\alpha}\exp\{-c_0\rho^2(x,t;\xi,\tau)\}\end{aligned} \tag{14}$$

using the Gauss-Ostrogradsky formula [6], we have

$$\int_{R^{4n}} \partial_{x_{4k}}\, G_0(x,t;\xi,\tau,y,\tau)\,|_{y=x}d\xi = -\int_{R^{4n}} \partial_{\xi_{4k}}\, G_0(x,t;\xi,\tau,x,\tau)\,d\xi = 0. \tag{15}$$

Iз (10), (12), (14), (15) we have

$$|I_2| \le CM\int_\tau^{t-h}(t-\beta)^{-\left(1-\frac{\alpha}{2}\right)}d\beta \le \mathrm{C}\left(h^{\frac{\alpha}{2}} + (t-\tau)^{\frac{\alpha}{2}}\right). \tag{16}$$

Similarly from (10) we obtain:

$$|I_1| \le \int_\tau^{t-h} d\beta \int_{R^{4n}} (t-\beta)^{-8n-\frac{7}{2}}(t-\beta)^{-7+\frac{\alpha}{2}}\exp\left\{-c\sum_{l=1}^{n}\sum_{j=1}^{4}\gamma_{jl}^2(t-\beta)^{-(2j-1)}\right\}d\gamma.$$

We use (12) then

$$\begin{aligned}|I_1| \le C\int_\tau^{t-h} d\beta \int_{R^{4n}} (t-\tau)^{-8n+\frac{7}{2}}(t-\beta)^{-2+\frac{\alpha}{2}} \\ \exp\left\{-c_0^*\sum_{j=1}^{4}\sum_{k\ne l}^{n}\gamma_{jl}^2(t-\beta)^{-(2j-1)} - c_0^*\gamma_{4k}^2(t-\beta)^{-2}\right\}d\gamma \le C_1 h^{\frac{\alpha}{2}}, 0 < c_0^* < c_0.\end{aligned} \tag{17}$$

With (16), (17) follows existence $\lim\limits_{h\to 0}\partial_{x_{4k}}V_h(x,t)$, we show the equality of the boundary of the expression (7) at j=4, that is I.

$$I=\int_\tau^t d\beta\int_{R^{4n}}\partial_{x_{4k}}G_0(x,t;\xi,\beta;\xi,\beta)[f(\xi,\beta)-f(x,\beta)]d\xi$$

$$+\int_\tau^t(\int_{R^{4n}}\partial_{x_{4k}}G_0(x,t;\xi,\beta;\xi,\beta)d\xi)f(x,\beta)d\beta$$

Consider the difference $I-\partial_{x_{4k}}V_h(x,t)$ and appreciate:

$$|I-\partial_{x_{4k}}V_h\left(x,t\right)|\le C\int_{t-h}^t d\beta\int_{R^{4n}}(t-\beta)^{-8n+\frac{7}{2}}(t-\beta)^{-2+\frac{\alpha}{2}} \tag{18}$$
$$\exp\left\{-c_0^*\sum_{l\neq k}^n\sum_{j=1}^4\gamma_{jl}^2(t-\beta)^{-(2j-1)}-c_0^*\gamma_{4k}^2(t-\tau)^{-2}\right\}d\gamma\le Ch^{\frac{\alpha}{2}}.$$

From (18) follows formula (7) in the case $j=4$.

In the case of $j=3,2$ the proof of (7) is similar, using that

$$\partial_{x_{4k}}G_0\left(x,t;\xi,\tau;x,\tau\right)=-\partial_{\xi_{4k}}G_0\left(x,t;\xi,\tau;x,\tau\right), \tag{19}$$

$$\partial_{x_{3k}}G_0\left(x,t;\xi,\tau;x,\tau\right)=-\partial_{\xi_{3k}}G_0\left(x,t;\xi,\tau;x,\tau\right)-\left(t-\tau\right)\partial_{\xi_{4k}}G_0\left(x,t;\xi,\tau;x,\tau\right); \tag{20}$$

$$\partial_{x_{2k}}G_0\left(x,t;\xi,\tau;x,\tau\right)=-\partial_{\xi_{2k}}G_0\left(x,t;\xi,\tau;x,\tau\right)-\left(t-\tau\right)\partial_{\xi_{3k}}G_0\left(x,t;\xi,\tau;x,\tau\right)+ \tag{21}$$
$$0.9\ \partial_{\xi_{4k}}G_0\left(x,t;\xi,\tau;x,t\right);$$

In the case of $j=1$

$$|\partial_{x_{1k}}G_0\left(x,t;\xi,\tau;x,\tau\right)|\le C(t-\tau)^{-8n+\frac{1}{2}}\exp\left\{-c_0\rho^2\left(x,t;\xi,\tau\right)\right\},\ t>t,\quad k=\overline{1,n}.$$

Therefore, in the case of linear parabolic equations [7], we establish existence and continuity $\partial_{x,_{1k}}V(x,t)$,

$$\partial_{x_{1k}}V\left(x,t\right)=\int_\tau^t\int_{R^{4n}}\partial_{x_{1k}}G_0\left(x,t;\xi,\beta;\xi,\beta\right)f\left(\xi,\beta\right)d\xi d\beta.$$

To establish formula (8) we write $\partial^2_{x_{1i}x_{1j}}G_0\left(x,t;\xi,\tau;x,\tau\right)$ through the derivatives ∂_{ξ_l} $\quad l=\overline{1,4}$. In particular

$$\partial_{x_{1j}}G_0\left(x,t;\xi,\tau;x,\tau\right)=-\sum_{i=1}^n a^{(ij)}(-\rho_{1i}+2^{-1}(t-\tau)\rho_{2i}$$

$$10^{-1}(t-\tau)^2\rho_{3i}+120^{-1}(t-\tau)^3\rho_{4i})e^{-w^{(y,\beta)}(x,t,\xi,\tau)}\Phi\left(y,\beta\right)\Bigg|_{\substack{y=x\\ \beta=\tau}}.$$

$$\partial_{\xi_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) = -\sum_{i=1}^{n} a^{(ij)}(\rho_{1i} + 2^{-1}(t-\tau)\rho_{2i}$$

$$10^{-1}(t-\tau)^2\rho_{3i} + 120^{-1}(t-\tau)^3\rho_{4i})e^{-w^{(y,\beta)}(x,t,\xi,\tau)}\Phi\left(y,\beta\right)\Bigg|_{\substack{y=x\\ \beta=\tau}} .$$

Given (19) - (21), we obtain

$$\partial_{x_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) = -\partial_{\xi_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) \tag{22}$$
$$-2\sum_{i=1}^{n} a^{(ij)}(2^{-1}\left(t-\tau\right)\rho_{2i} + 120^{-1}(t-\tau)^3\rho_{4i})e^{-w^{(y,\beta)}(x,t,\xi,\tau)} =$$
$$-\partial_{\xi_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) - \left(t-\tau\right)\partial_{\xi_{2j}} G_0\left(x,t;\xi,\tau;x,\tau\right)$$
$$-2^{-1}(t-\tau)^2\partial_{\xi_{3j}} G_0\left(x,t;\xi,\tau;x,\tau\right) - 24^{-1}(t-\tau)^3\partial_{\xi_{4j}} G_0\left(x,t;\xi,\tau;x,\tau\right).$$

Given (22) we have

$$\partial^2_{x_{1l}x_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) = -\partial^2_{x_{1i}\xi_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) \tag{23}$$
$$-\left(t-\tau\right)\partial^2_{x_{1i}\xi_{2j}} G_0\left(x,t;\xi,\tau;x,\tau\right) - 2^{-1}(t-\tau)^2\partial^2_{x_{1i}\xi_{3j}} G_0\left(x,t;\xi,\tau;x,\tau\right)$$
$$-24^{-1}\left(t-\tau\right)^3\partial^2_{x_{1i}\xi_{4j}} G_0\left(x,t;\xi,\tau;x,\tau\right).$$

So we have Given (24) we have

$$\int_{R^{4n}} \left|\partial^2_{x_{1l}x_{1j}} G_0\left(x,t;\xi,\tau;\xi,\tau\right) - \partial^2_{x_{1l}x_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right)\right| d\xi \le C(t-\tau)^{-1+\frac{\alpha}{2}}. \tag{24}$$

It follows from (23) and the Gauss-Ostrogradsky theorem:

$$\int_{R^{4n}} \partial^2_{x_{1l}x_{1j}} G_0\left(x,t;\xi,\tau;x,\tau\right) d\xi = 0. \tag{25}$$

Because

$$\left|\partial^2_{x_{1i}x_{1j}} G_0\left(x,t;\xi,\tau;\xi,\tau\right)\left[f\left(\xi,\tau\right) - f\left(x,\tau\right)\right]\right| \le \tag{26}$$
$$C(t-\tau)^{-8n-1+\frac{\alpha}{2}}\exp\left\{-c_0\rho^2\left(x,t;\xi,\tau\right)\right\},$$

then from (24) - (26) follows the existence of all integrals in the form (8) at $h \to 0$.

Let us prove (5). Because

$$\partial_t\, G_0\left(x,t;\xi,\tau,\xi,\tau\right) = \sum_{j=1}^{3}\sum_{\mu=1}^{n} x_{j\mu}\partial_{x_{j+1\ \mu}} G_0\left(x,t;\xi,\tau,\xi,\tau\right)$$

$$+\sum_{k,l=1}^{n} a^{kl}\left(\xi,\tau\right)\partial^2_{x_{1k}x_{1l}} G_0\left(x,t;\xi,\tau,\xi,\tau\right),$$

then it follows existence

$$\lim_{h\to 0}\int_{\tau}^{t-h} d\beta \int_{R^{4n}} \partial_t \; G_0\left(x,t;\xi,\beta,\xi,\beta\right) \, d\beta f\left(x,\beta\right).$$

Formula (5) is set in the same way as (6) - (8). When differentiating $\int_{\tau}^{t-h} d\beta \int_{R^{4n}} \; G_0\left(x,t;\xi,\beta,\xi,\beta\right) f(\xi,\beta) d\xi$ on the upper bound we have

$$\int_{R^{4n}} G_0\left(x,t;\xi,t-h,\xi,t-h\right) f\left(\xi,t-h\right) d\xi =$$

$$\int_{R^{4n}} G_0\left(x,t;\xi,t-h,\xi,t-h\right) d\xi f(x,t-h)+$$

$$\int_{R^{4n}} G_0\left(x,t;\xi,t-h,\xi,t-h\right)\left[f\left(\xi,t-h\right)-f(x,t-h)\right] d\xi \to f\left(x,t\right),$$

evenly across t when fixed x. In particular, the first addition goes to $f(x,t)$, the second supplement does not exceed $Ch^{\frac{\alpha}{2}}$, so it goes to 0 at $h\to 0$. As a result we have $V(x,t)$ satisfies the equation

$$L_0 V\left(t,x\right) = f\left(x,t\right) + \int_{\tau}^{t} d\beta \int_{R^{4n}} L_0 G_0\left[f\left(\xi,\beta\right) - f\left(x,\beta\right)\right] d\xi$$

$$+\int_{\tau}^{t} d\beta \int_{R^{4n}} L_0 \; G_0\left(x,t;\xi,\beta,\xi,\beta\right) d\xi f\left(x,\beta\right) = f\left(x,t\right).$$

Method E.E. Levy Finding the Fundamental Solution of Equation (1)

We are looking for a fundamental solution $\Gamma\left(x,t;\xi,\tau\right)$ by the method of E.E. Levy in the form

$$\Gamma\left(x,t;\xi,\tau\right) = G_0\left(x,t;\xi,\tau,\xi,\tau\right) + \int_{\tau}^{t} d\beta \int_{R^{4n}} G_0\left(x,t;\lambda,\beta,\lambda,\beta\right)\varphi\left(\lambda,\beta;\xi,\tau\right) d\lambda = G_0 + U, \qquad (27)$$

where $\varphi\left(x,t;\xi,\tau\right)$ – the desired function that a priori when $t>\tau$ continuous and satisfies inequalities

$$|\varphi(x,t;\xi,\tau)| \leq C(t-\tau)^{-8n-1+\frac{\alpha}{2}} \exp\{-c\rho^2(x,t;\xi,\tau)\} \qquad (28)$$

$$|\Delta x_{ji}\varphi(x,t;\xi,\tau)| = |\varphi(x,t;\xi,\tau)-\varphi(x',t;\xi,\tau)| \leq C|x-x'|^{\frac{\alpha_1}{2j-1}}(t-\tau)^{-8n-1+\frac{\alpha_2}{2}} \times \max\{\{\exp -c'\rho^2(x,t;\xi,\tau)\},\{\exp c\rho^2(x',t;\xi,\tau)\}\}, \qquad (29)$$

$\alpha_1<\alpha,\ \ \alpha_2=\alpha-\alpha_1,\ x'=x+\Delta x_{ij}$. From the a priori assumptions of continuity φ and (28), (29) imply that for any continuous and bounded function $u_0\left(x\right)$, we have

$$\lim_{t\to\tau}\int_{R^{4n}} \Gamma\left(x,t;\xi,\tau\right) \; u_0\left(\xi\right) d\xi = \lim_{t\to\tau^+}\int_{R^{4n}} G_0\left(x,t;\xi,\tau,\xi,\tau\right) \; u_0\left(\xi\right) d\xi,$$

$\int_{R^{4n}} Uu_0\left(\xi\right)d\xi$ has a minor minor feature since

$$|U\left(x,t;\xi,\tau\right)|\leq C(t-\tau)^{-8n+\frac{\alpha_1}{7}}\exp\left\{-c\rho^2\left(x,t;\xi,\tau\right)\right\},$$

With proper choice $\varphi\left(x,t;\xi,\tau\right)$ function $\Gamma\left(x,t;\xi,\tau\right)$ satisfies the equation $L\Gamma\left(x,t;\xi,\tau\right)=0$. Let's calculate the estimate $LG_0\left(x,t;\xi,\tau,\xi,\tau\right)=\mathrm{K}\left(x,t;\xi,\tau\right)$,

$$|K\left(x,t;\xi,\tau\right)|=\left|\sum_{k,l=1}^{n}\left[a_{kl}\left(\xi,\tau\right)-a_{kl}\left(x,t\right)\right]\partial^2_{x_{1k}x_{1l}}G_0\left(x,t;\xi,\tau,\xi,\tau\right)-\right. \tag{30}$$

$$\left.\sum_{k=1}^{n}a_k\left(x,t\right)\partial_{x_{1k}}G_0\left(x,t;\xi,\tau,\xi,\tau\right)-a(x,t)G_0\left(x,t;\xi,\tau,\xi,\tau\right)\right|\leq$$

$$A_1(t-\tau)^{-8n-1+\frac{\alpha}{2}}\exp\left\{-c\rho^2\left(x,t;\xi,\tau\right)\right\}.$$

Let's consider $LU\left(x,t;\xi,\tau\right)$:

$$LU\left(x,t;\xi,\tau\right)=\varphi\left(x,t;\xi,\tau\right)+\int_\tau^t d\beta\int_{R^{4n}}\left(\sum_{k,l=1}^{n}a_{kl}\left(x,t\right)\partial^2_{x_{1k}x_{1l}}+\sum_{j=1}^{3}\sum_{\mu=1}^{n}x_{j\mu}\partial_{x_{j+1\mu}}-\partial_t\right)$$

$$G_0\left(x,t;\lambda,\beta,\lambda,\beta\right)\left[\varphi\left(\lambda,\beta;\xi,\tau\right)-\varphi\left(x,\beta;\xi,\tau\right)\right]d\lambda+\int_\tau^t d\beta\int_{R^{4n}}\left(\sum_{k=1}^{n}a_k\left(x,t\right)\partial_{x_{1k}}\right.$$

$$\left.+a\left(x,t\right)\right)G_0\left(x,t;\lambda,\beta,\lambda,\beta\right)\ \varphi\left(\lambda,\beta;\xi,\tau\right)d\lambda+$$

$$\int_\tau^t d\beta\int_{R^{4n}}\left[\sum_{k,l=1}^{n}a_{kl}\left(x,t\right)\partial^2_{x_{1k}x_{1l}}+\sum_{j=1}^{3}\sum_{\mu=1}^{n}x_{j\mu}\partial_{x_{j+1\ \mu}}-\partial_t\right]G_0\left(x,t;\lambda,\beta,\lambda,\beta\right)\varphi\left(x,\beta;\xi,\tau\right).$$

Given the estimate (30) for $\varphi\left(x,t;\xi,\tau\right)$

$$L_0G_0\left(x,t;\xi,\tau\right)=\left[\sum_{k,l=1}^{n}a_{kl}\left(\xi,\tau\right)\partial^2_{x_{1k}x_{1l}}+\sum_{j=1}^{3}\sum_{\mu=1}^{n}x_{j\mu}\partial_{x_{j+1\ \mu}}-\partial_t\right]G_0\left(x,t;\xi,\tau,\xi,\tau\right)=0.$$

we obtain the integral equation

$$\varphi\left(x,t;\xi,\tau\right)=-K\left(x,t;\xi,\tau\right)+\int_\tau^t d\beta\int_{R^{4n}}-K\left(x,t;\lambda,\beta\right)\varphi\left(\lambda,\beta;\xi,\tau\right)d\lambda. \tag{31}$$

We solve the integral equation (31) by the method of successive approximations

$$\varphi\left(x,t;\xi,\tau\right)=\sum_{m=1}^{\infty}K_m\left(x,t;\xi,\tau\right). \tag{32}$$

$$K_1\left(x,t;\xi,\tau\right)=K\left(x,t;\xi,\tau\right)$$

$$K_m\left(x,t;\xi,\tau\right)=\int_\tau^t d\beta\int_{R^{4n}}K\left(x,t;\lambda,\beta\right)K_{m-1}\left(\lambda,\beta;\xi,\tau\right)d\lambda.$$

Using (30), we find an estimate $K_2(x,t;\xi,\tau)$.

$$|K_2(x,t;\xi,\tau)| \le A_1^2 \int_\tau^t \frac{d\beta}{[(t-\beta)(\beta-\tau)]^{1-\frac{\alpha}{2}}} \int_{R^{4n}} (t-\beta)^{-8n} \exp\left\{-c\rho^2(x,t;\lambda,\beta)\right\} \tag{33}$$

$$(\beta-\tau)^{-8n}\exp\left\{-c\rho^2(\lambda,\beta;\xi,\tau)\right\} = A_1^2 B\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)\left(\frac{\pi}{c}\right)^{2n}(t-\tau)^{-8n-1+\alpha}$$
$$\exp\left\{-c\rho^2(x,t;\xi,\tau)\right\}.$$

$B\left(\frac{\alpha}{2},\frac{\alpha}{2}\right)-$ Euler's beta function. Estimation of approximation $K_3(x,t;\xi,\tau)$, $K_4(x,t;\xi,\tau)$ etc. we will carry out similarly, by means of mathematical induction we prove that for any m:

$$|K_m(x,t;\xi,\tau)| \le \frac{\Gamma^m\left(\frac{\alpha}{2}\right)A_1^m\left(\frac{\pi}{c}\right)^{2n(m-1)}(t-\tau)^{-8n-1+\frac{m\alpha}{2}}}{\Gamma\left(\frac{n\alpha}{2}\right)}\exp\left\{-c\rho^2(x,t;\xi,\tau)\right\}. \tag{34}$$

From estimates (30), (34) at $t-\tau \ge \varepsilon > 0$ a uniform and absolute convergence of series (32) and a correct estimate are obtained (28).

We show the estimate (29). At $(t-\tau)^{\frac{1}{2}} < \left|x_{ji}-x'_{ji}\right|^{\frac{1}{2j-1}}$ inequality (29) follows from (28). So let's consider the case $\left|x_{ji}-x'_{ji}\right|^{\frac{1}{2j-1}} \le (t-\tau)^{\frac{1}{2}}$. First let's evaluate $\Delta_{x_{ji}}K(x,t;\xi,\tau)$:

$$\left|\Delta_{x_{ji}}K(x,t;\xi,\tau)\right| \le \left|\sum_{k,l=1}^{n}\left[\Delta_{x_{ji}}a_{kl}(x,t)\partial^2_{x_{1k}x_{1l}}G_0(x,t;\xi,\tau)\right.\right.$$
$$\left.+(a_{kl}(\xi,\tau)-a_{kl}\left(x',t\right))\Delta_{x_{ji}}\partial^2_{x_{1k}x_{1l}}G_0(x,t;\xi,\tau)\right]$$
$$-\sum_{k=1}^{n}\left(\Delta_{x_{ji}}a_k(x,t)\partial_{x_{1k}}G_0(x,t;\xi,\tau)+a_k\left(x',t\right)\Delta_{x_{j_i}}\partial_{x_{1k}}G_0(x,t;\xi,\tau)\right)$$
$$\left.-\Delta_{x_{ji}}a(x,t)G_0(x,t;\xi,\tau)-a\left(x',t\right)\Delta_{x_{ji}}G_0(x,t;\xi,\tau)\right|.$$

Using the estimates (28), (30), we evaluate the terms $\Delta_{x_{ji}}\partial^2_{x_{1k}x_{1l}}G_0(x,t;\xi,\tau)$, $\Delta_{x_{ji}}\partial_{x_{1k}}G_0(x,t;\xi,\tau)$, $\Delta_{x_{ji}}G_0(x,t;\xi,\tau)$ using the mean and inequality theorem $|\Delta x_{ij}| \le (t-\tau)^{\frac{2j-1}{2}}$ i $-1<\Theta\le 1$.

$$-1+\left|\frac{x_{1i}-\xi_{1i}}{(t-\tau)^{\frac{1}{2}}}\right| \le \left|\frac{(x_{1i}-\xi_{1i})}{(t-\tau)^{\frac{1}{2}}}+\frac{\Theta\Delta x_{ij}}{(t-\tau)^{\frac{1}{2}}}\right| \le \left|\frac{x_{1i}-\xi_{1i}}{(t-\tau)^{\frac{1}{2}}}\right|+1;$$

$$-1+\left|\frac{x_{2i}-\xi_{2i}}{(t-\tau)^{\frac{3}{2}}}+\frac{x_{1i}-\xi_{1i}}{2(t-\tau)^{\frac{1}{2}}}\right| \le \left|\frac{x_{2i}-\xi_{2i}}{(t-\tau)^{\frac{3}{2}}}+\frac{x_{1i}-\xi_{1i}}{2(t-\tau)^{\frac{1}{2}}}+\frac{\Theta\Delta x_{ij}}{2(t-\tau)^{\frac{1}{2}}}\right|;$$

$$\le \left|\frac{x_{2i}-\xi_{2i}}{(t-\tau)^{\frac{3}{2}}}+\frac{x_{1i}-\xi_{1i}}{2(t-\tau)^{\frac{1}{2}}}\right|+1;$$

$$-1+\left|\frac{x_{3i}-\xi_{3i}}{(t-\tau)^{\frac{5}{2}}}+\frac{x_{2i}-\xi_{2i}}{2(t-\tau)^{\frac{3}{2}}}+\frac{x_{1i}-\xi_{1i}}{12(t-\tau)^{\frac{1}{2}}}\right| \le$$

$$\left| \frac{x_{3i} - \xi_{3i}}{(t-\tau)^{\frac{5}{2}}} + \frac{x_{2i} - \xi_{2i}}{2\,(t-\tau)^{\frac{3}{2}}} + \frac{x_{1i} - \xi_{1i}}{12\,(t-\tau)^{\frac{1}{2}}} + \frac{\Theta \Delta x_{ij}}{2(t-\tau)^{\frac{1}{2}}} \right|;$$

$$\leq \left| \frac{x_{3i} - \xi_{3i}}{(t-\tau)^{\frac{5}{2}}} + \frac{x_{2i} - \xi_{2i}}{2\,(t-\tau)^{\frac{3}{2}}} + \frac{x_{1i} - \xi_{1i}}{12\,(t-\tau)^{\frac{1}{2}}} \right| + 1;$$

$$-1 + \left| \frac{x_{4i} - \xi_{4i}}{(t-\tau)^{\frac{7}{2}}} + \frac{x_{3i} - \xi_{3i}}{2(t-\tau)^{\frac{5}{2}}} + \frac{x_{2i} - \xi_{2i}}{10\,(t-\tau)^{\frac{3}{2}}} + \frac{x_{1i} - \xi_{1i}}{120\,(t-\tau)^{\frac{1}{2}}} \right| \leq$$

$$\left| \frac{x_{4i} - \xi_{4i}}{(t-\tau)^{\frac{7}{2}}} + \frac{x_{3i} - \xi_{3i}}{2(t-\tau)^{\frac{5}{2}}} + \frac{x_{2i} - \xi_{2i}}{10\,(t-\tau)^{\frac{3}{2}}} + \frac{x_{1i} - \xi_{1i}}{120\,(t-\tau)^{\frac{1}{2}}} + \frac{\Theta \Delta x_{ij}}{2(t-\tau)^{\frac{1}{2}}} \right|;$$

$$\leq \left| \frac{x_{4i} - \xi_{4i}}{(t-\tau)^{\frac{7}{2}}} + \frac{x_{3i} - \xi_{3i}}{2(t-\tau)^{\frac{5}{2}}} + \frac{x_{2i} - \xi_{2i}}{10\,(t-\tau)^{\frac{3}{2}}} + \frac{x_{1i} - \xi_{1i}}{120\,(t-\tau)^{\frac{1}{2}}} \right| + 1;$$

since

$$\left|\Delta_{x_{ji}} K\left(x, t; \xi, \tau\right)\right| \leq \mathrm{C} |\Delta x_{ij}|^{\frac{\alpha_1}{2j-1}} (t-\tau)^{-8n-1+\frac{\alpha-\alpha_1}{2}} \exp\left\{-c\rho^2\left(x, t; \xi, \tau\right)\right\}. \tag{35}$$

Using inequalities (28), (35), we estimate $\Delta_{x_{ji}} U(x, t; \xi, \tau)$:

$$\left|\Delta_{x_{ji}} U(x, t; \xi, \tau)\right| = \left| \int_\tau^t d\beta \int_{R^{4n}} K\left(x, t; \gamma, \beta\right) \varphi\left(\gamma, \beta; \xi, \tau\right) d\gamma \right| \leq \ \mathrm{C} A_1 |\Delta x_{ij}|^{\frac{\alpha_1}{2j-1}}$$

$$\int_\tau^t \frac{d\beta}{(t-\tau)^{1-\frac{\alpha_2}{2}} (\beta-\tau)^{1-\frac{\alpha_2}{2}}} \int_{R^{4n}} (t-\tau)^{-8n} \left[\exp\left\{-c\rho^2\left(x, t; \gamma, \beta\right)\right\} + \right.$$

$$\left.\exp\left\{-c\rho^2\left(x', t; \gamma, \beta\right)\right\}\right] \exp\left\{-c\rho^2\left(\gamma, \beta; \xi, \tau\right)\right\} \ (\beta-\tau)^{-8n} d\gamma \leq$$

$$\mathrm{C}_2 |\Delta x_{ij}|^{\frac{\alpha_1}{2j-1}} (t-\tau)^{-8n-1+\frac{\alpha-\alpha_1}{2}} \exp\left\{-c\rho^2\left(x, t; \xi, \tau\right)\right\}.$$

Since estimates (28), (29) prove that the function $\Gamma\left(x, t; \xi, \tau\right)$ at $t > \tau$ s the solution of equation (1). We establish the derivatives estimates $\Gamma\left(x, t; \gamma, \beta\right)$, it is enough to estimate the derivatives $U(x, t; \xi, \tau)$. By Theorem 1, there are all derivatives that are included in equation (1) by breaking the point $t_1 = \frac{t+\tau}{2}$ integral for two integrals.

Let's consider $\partial^2_{x_{1k} x_{1l}} U(x, t; \xi, \tau)$:

$$\left|\partial^2_{x_{1k} x_{1l}} U(x, t; \xi, \tau)\right| \leq \left| \int_\tau^{t_1} d\beta \int_{R^{4n}} \partial^2_{x_{1k} x_{1l}} G_0\left(x, t; \gamma, \beta, \gamma, \beta\right) \right.$$

$$\left[\varphi\left(\gamma, \beta; \xi, \tau\right) - \varphi\left(x, \beta; \xi, \tau\right)\right] d\gamma +$$

$$+ \int_{t_1}^t d\beta \int_{R^{4n}} \partial^2_{x_{1k} x_{1l}} G_0\left(x, t; \gamma, \beta, \gamma, \beta\right) \varphi\left(\gamma, \beta; \xi, \tau\right) d\gamma +$$

$$\left. \int_\tau^{t_1} d\beta \int_{R^{4n}} \partial^2_{x_{1k} x_{1l}} G_0\left(x, t; \gamma, \beta, \gamma, \beta\right) d\gamma \varphi\left(x, \beta; \xi, \tau\right) \right| \leq$$

$$C C_2 \int_\tau^t \frac{d\beta}{(t-\beta)(\beta-\tau)^{\frac{2-\alpha}{2}}} \int_{R^{4n}} \left[\exp\left\{-c\rho^2\left(x, t; \gamma, \beta\right)\right\} \ - \exp\left\{-c\rho^2\left(\gamma, t; \xi, \tau\right)\right\}\right.$$

$$[(t-\beta)(\beta-\tau)]^{-8n}d\gamma + CC_2\int_\tau^t d\beta \int_{R^{4n}} \frac{|x-\gamma|^{\frac{\alpha_1}{7}}}{(t-\tau)^{8n+1}}\left[\exp\left\{-c\rho^2(x,t;\gamma,\beta)\right\}\right.$$

$$-\exp\left\{-c\rho^2(\gamma,t;\xi,\tau)\right\}(\beta-\tau)^{-8n-1+\frac{\alpha}{2}}d\gamma + CC'\int_{t_1}^t [(t-\beta)(\beta-\tau)]^{-1+\frac{\alpha}{7}}$$

$$\exp\left\{-c\rho^2(x,\beta;\xi,\tau)\right\}d\beta \le \tilde{C}(t-\tau)^{-8n+\frac{\alpha}{2}-1}\exp\left\{-c\rho^2(x,t;\xi,\tau)\right\}.$$

Similarly, estimates for $\partial_{x_{ij}}U(x,t;\xi,\tau)$, $\partial_t U(x,t;\xi,\tau)$ back for $\Gamma(x,t;\xi,\tau)$ on its derivatives are the correct estimate.

$$\left|\partial_t^m \partial_{x_j}^{m_j} \partial_{x_1}^{m_1}\Gamma(x,t;\xi,\tau)\right| \le C_m(t-\tau)^{-8n-\frac{(2j-1)}{2}|m_j|-\frac{7}{2}m_0\sum_{j=1}^3 (t-\tau)^{(3-j)}\left|x_j\right|^{m_0}}.$$

$$\exp\left\{-c\rho^2(x,t;\xi,\tau)\right\},\quad j=\overline{1,4},\ |m_1|=\{0,1,2\},\ 0\le m_0\le 1,\ 0\le m_j\le 1,\ j=\overline{2,4}.$$

If any $m_j \ne 0$, $j=\overline{0,4}$ then $m_k = 0$, $k \ne j$.

2 Prospects for further research

In the future, it is advisable to construct a fundamental solution of an equation with an arbitrary number of groups of variables in which the coefficients depend on all the variables by which the parabolic degeneracy, which generalizes the Kolmogorov equation.

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В статті побудовано фундаментальний розв'язок лінійного виродженого параболічного рівняння, що має 4n ступенів свободи, узагальнює рівняння Колмогорова. Коефіцієнти рівняння неперервні, обмежені і задовольняють умову Гельдера з показником $0 < \alpha \leq 1$. Використали метод Леві та дослідили поведінку об'ємного потенціалу породженого параметриксом. Встановлено оцінки похідних фундаментального розв'язку.