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POSITIVE DEFINITE BRANCHED CONTINUED FRACTIONS OF SPECIAL FORM

Research of the class of branched continued fractions of special form, whose denominators do not equal to zero, is proposed and the connection of such fraction with a certain quadratic form is established. It furnishes new opportunities for the investigation of convergence of branching continued fractions of special form.

Key words and phrases: positive definite branched continued fraction of special form, quadratic form.

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INTRODUCTION

The convergence problem of the continued fractions and their generalizations — branched continued fractions (BCF) — is that on the basis of information on the coefficients of fraction to conclude about of its convergence or divergence. Using the methods of majorants, fundamental inequalities, theorems about compact family of holomorphic functions, the convergence of some numerical and functional BCF of special form are investigated in [1, 2, 3, 5, 6, 7]. Taking into account the formulas for the numerators and denominators of approximants as determinants, the properties of positive definite BCF are defined and considered in the monograph [4, pp. 130–137]. The criteria of positive definite of the BCF established here are sufficient as opposed to one-dimensional case, where the analogous conditions are also necessary. As a result, for bounded and real multidimensional J-fractions the properties are studied and the criteria of convergence are established [4, pp. 141–146].

In this paper we have defined class of BCF whose denominators do not equal to zero — positive definite BCF of special form. Representing denominators as determinants, the connection between the about mentioned fraction and the certain quadratic form is established. Moreover, for BCF of special form the sufficient and necessary conditions of the positive definiteness are established.

1 DEFINITION OF A POSITIVE DEFINITE BCF OF SPECIAL FORM

We consider BCF of the form

$$\Phi_0+\cfrac{1}{b_{01}+z_{01}-\Phi_1+\displaystyle\mathop{\mathrm{D}}_{s=2}^{\infty}\frac{-a_{0s}^2}{b_{0s}+z_{0s}-\Phi_s}},\quad \Phi_p=\cfrac{1}{b_{1p}+z_{1p}+\displaystyle\mathop{\mathrm{D}}_{r=2}^{\infty}\frac{-a_{rp}^2}{b_{rp}+z_{rp}}},\quad p\geq 0, \qquad (1)$$

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where a_{rs}, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r+s \geq 2$, b_{rs}, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, are complex numbers, z_{rs}, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, are complex variables. Let $\mathbf{z} = (z_{10}, z_{01}, z_{20}, z_{11}, z_{02}, \ldots)$ be an infinite-dimensional vector and n be an arbitrary natural number. By curtailing the nth approximant

$$f_n(\mathbf{z}) = \Phi_0^n + \cfrac{1}{b_{01} + z_{01} - \Phi_1^{n-1} + \displaystyle\mathop{\mathrm{D}}_{s=2}^{n} \frac{-a_{0s}^2}{b_{0s} + z_{0s} - \Phi_s^{n-s}}}, \tag{2}$$

where

$$\Phi_p^{n-p} = \cfrac{1}{b_{1p} + z_{1p} + \displaystyle\mathop{\mathrm{D}}_{r=2}^{n-p} \frac{-a_{rp}^2}{b_{rp} + z_{rp}}}, \quad 0 \leq p \leq n-1,$$

of BCF (1) top-down without any shortening in the intermediate operations (see [4, pp. 15–27]), we obtain its representation as a ratio

$$f_n(\mathbf{z}) = \frac{A_n(\mathbf{z})}{B_n(\mathbf{z})}, \tag{3}$$

where $A_n(\mathbf{z})$, $B_n(\mathbf{z})$ are polynomials of variables z_{rs}, $r \geq 0$, $s \geq 0$, $1 \leq r+s \leq n$, and constant numbers a_{rs}, $r \geq 0$, $s \geq 0$, $r \neq 1$, $2 \leq r+s \leq n$, b_{rs}, $r \geq 0$, $s \geq 0$, $1 \leq r+s \leq n$.

The numerator of ratio (3) $A_n(\mathbf{z})$ is called nth numerator and denominator $B_n(\mathbf{z})$ — nth denominator of the approximant (2).

Obviously that for any $n \geq 1$ each positive integer $j \leq n(n+3)/2$ can be uniquely written as

$$j = 1 + 2 + \cdots + (r-1) + r + s, \tag{4}$$

where $1 \leq r \leq n$, $0 \leq s \leq r$. We consider the symmetric matrix

$$C_{n(n+3)/2} = \left\| \begin{array}{cccc} c_{11} & c_{12} & \cdots & c_{1,n(n+3)/2} \\ c_{21} & c_{22} & \cdots & c_{2,n(n+3)/2} \\ \multicolumn{4}{c}{\dotfill} \\ c_{n(n+3)/2,1} & c_{n(n+3)/2,2} & \cdots & c_{n(n+3)/2,n(n+3)/2} \end{array} \right\|, \quad n \geq 1, \tag{5}$$

whose elements are related to the components of BCF (1) as follows: $c_{jj} = b_{r-s,s} + z_{r-s,s}$; $c_{j,j+r+1} = c_{j+r+1,j} = -1$, $c_{j,j+r+2} = c_{j+r+2,j} = -a_{0,r+1}$, if $s = r$, i.e., $j = r(r+3)/2$; $c_{j,j+r+1} = c_{j+r+1,j} = -a_{r-s+1,s}$, if $0 \leq s < r$; $c_{ij} = 0$ otherwise; where $1 \leq i, j \leq n(n+3)/2$, $n \geq 1$, r and s are determined from the decomposition number j as (4).

By arguments similar to the proof of the lemma 4.1 [4, pp. 130–132], we can show that following lemma holds.

Lemma 1. *The denominators of the BCF* (1) *are given by the formulas* $B_n(\mathbf{z}) = \det C_{n(n+3)/2}$, $n \geq 1$, *where* $C_{n(n+3)/2}$, $n \geq 1$, *are matrices as* (5).

Let n be an arbitrary natural number,

$$X_n = (x_{10}, x_{01}, \ldots, x_{n0}, x_{n-1,1}, \ldots, x_{0n}) \in \mathbb{C}^{n(n+3)/2}.$$

We consider the system of homogeneous linear equations $C_{n(n+3)/2}X_n = 0$, namely,

$$\begin{cases} (b_{10}+z_{10})x_{10} - a_{20}x_{20} = 0, \\ (b_{01}+z_{01})x_{01} - x_{11} - a_{02}x_{02} = 0, \\ -a_{20}x_{10} + (b_{20}+z_{20})x_{20} - a_{30}x_{30} = 0, \\ -x_{01} + (b_{11}+z_{11})x_{11} - a_{21}x_{21} = 0, \\ -a_{02}x_{01} + (b_{02}+z_{02})x_{02} - x_{12} - a_{03}x_{03} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -a_{n0}x_{n-1,0} + (b_{n0}+z_{n0})x_{n0} = 0, \\ -a_{n-1,1}x_{n-2,1} + (b_{n-1,1}+z_{n-1,1})x_{n-1,1} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -a_{0n}x_{0,n-1} + (b_{0n}+z_{0n})x_{0n} = 0. \end{cases} \tag{6}$$

Let us multiply the equations (6) by $\bar{x}_{10}, \bar{x}_{01}, \dots, \bar{x}_{0n}$, respectively, and add the resulting equations. This gives

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} (b_{rs}+z_{rs})|x_{rs}|^2 - \sum_{\substack{r,s=0\\ r+s\geq 2, r\neq 1}}^{n} a_{rs}(x_{rs}\bar{x}_{r-1+\delta_{r0},s-\delta_{r0}} + x_{r-1+\delta_{r0},s-\delta_{r0}}\bar{x}_{rs}) - \sum_{s=1}^{n}(x_{1s}\bar{x}_{0s} + x_{0s}\bar{x}_{1s}) = 0, \tag{7}$$

where δ_{pq} is the Kronecker symbol. We put $\beta_{rs} = \operatorname{Im} b_{rs}$, $y_{rs} = \operatorname{Im} z_{rs}$, $r \geq 0,\ s \geq 0,\ r+s \geq 1$, $\alpha_{rs} = \operatorname{Im} a_{rs}$, $r \geq 0,\ s \geq 0,\ r \neq 1,\ r+s \geq 2$, and suppose that

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} (\beta_{rs}+y_{rs})|x_{rs}|^2 - \sum_{\substack{r,s=0\\ r+s\geq 2, r\neq 1}}^{n} \alpha_{rs}(x_{rs}\bar{x}_{r-1+\delta_{r0},s-\delta_{r0}} + x_{r-1+\delta_{r0},s-\delta_{r0}}\bar{x}_{rs}) > 0 \tag{8}$$

for

$$y_{rs} > 0, \quad r \geq 0,\ s \geq 0,\ 1 \leq r+s \leq n, \quad \sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} |x_{rs}|^2 > 0. \tag{9}$$

Lemma 2. *For an arbitrary natural number n by conditions (9) the inequality (8) is equivalent to non-negative definite of the real quadratic form*

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} \beta_{rs}\xi_{rs}^2 - 2\sum_{\substack{r,s=0\\ r+s\geq 2\\ r\neq 1}}^{n} \alpha_{rs}\xi_{rs}\xi_{r-1+\delta_{r0},s-\delta_{r0}} \geq 0, \tag{10}$$

where ξ_{rs}, $r \geq 0, s \geq 0, 1 \leq r+s \leq n$, are arbitrary real numbers.

Proof. Let n be an arbitrary natural number and let the inequality (8) holds for arbitrary complex numbers x_{rs}, $r \geq 0, s \geq 0, 1 \leq r+s \leq n$, such that the conditions (9) holds. In particular, the inequality (8) holds iff $x_{rs} = \xi_{rs}$, $r \geq 0, s \geq 0, 1 \leq r+s \leq n$. In the inequality (8) we replace the x_{rs} by the real numbers ξ_{rs} ($r \geq 0, s \geq 0, 1 \leq r+s \leq n$) and pass to limit in the both parts of this inequality as $y_{rs} \to 0$ ($r \geq 0, s \geq 0, 1 \leq r+s \leq n$). Then we obtain (10).

Let for an arbitrary natural number n the inequality (10) holds and let $x_{rs} = u_{rs} + iv_{rs}$, $r \geq 0$, $s \geq 0, 1 \leq r+s \leq n$. We then write the left-hand member of (8) in the form

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} \beta_{rs}u_{rs}^2 - 2\sum_{\substack{r,s=0\\ r+s\geq 2\\ r\neq 1}}^{n} \alpha_{rs}u_{rs}u_{r-1+\delta_{r0},s-\delta_{r0}} + \sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} \beta_{rs}v_{rs}^2 - 2\sum_{\substack{r,s=0\\ r+s\geq 2\\ r\neq 1}}^{n} \alpha_{rs}v_{rs}v_{r-1+\delta_{r0},s-\delta_{r0}} + \sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} y_{rs}|x_{rs}|^2,$$

from which (8) follows by conditions (9). □

We now make the following definition.

Definition. *The BCF* (1) *is said to be positive definite if the quadratic form* (10) *is non-negative definite for arbitrary natural number* n *and for all real values of* ξ_{rs}, $r \geq 0$, $s \geq 0$, $r+s \geq 1$.

Theorem 1. *If the BCF* (1) *is positive definite, then its denominators* $B_n(\mathbf{z})$, $n \geq 1$, *do not equal to zero for* $\operatorname{Im} z_{rs} > 0$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$.

Proof. For each natural n the system of homogeneous linear equations (6) has the trivial solution (all variables equal to zero) iff $B_n(\mathbf{z}) \neq 0$. Since (7) is corollary of the system of equations (6), obviously the system of equations has only a trivial solution, if the conditions of theorem (7) holds iff $x_{rs} = 0$, $r \geq 0$, $s \geq 0$, $1 \leq r+s \leq n$. Indeed, if (10) holds, then (8) holds via lemma 2, and thus (7) holds iff

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} |x_{rs}|^2 = 0$$

for each natural n. □

We shall now prove the following theorem, which furnishes a parametric representation for the coefficients of a positive definite BCF of special form.

Theorem 2. *The BCF* (1) *is positive definite iff both the following conditions are satisfied.*

A) The imaginary parts of the numbers b_{rs}, $r \geq 0$, $s \geq 0$, $r+s \geq 1$ *are all non-negative*

$$\beta_{rs} = \operatorname{Im} b_{rs} \geq 0, \quad r \geq 0,\ s \geq 0,\ r+s \geq 1. \tag{11}$$

B) There exist numbers g_{rs}, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, *such that*

$$0 \leq g_{rs} \leq 1, \quad r \geq 0,\ s \geq 0,\ r+s \geq 1, \tag{12}$$

and

$$\alpha_{rs}^2 = \beta_{rs}\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r \geq 0,\ s \geq 0,\ r \neq 1,\ r+s \geq 2, \tag{13}$$

where $\alpha_{rs} = \operatorname{Im} a_{rs}$, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r+s \geq 2$, δ_{pq} *is the Kronecker symbol.*

Proof. Let n be an arbitrary natural number. Let arbitrary p and q be given, such that $p \geq 0$, $q \geq 0$, $1 \leq p+g \leq n$; put in (10) $\xi_{pq} \neq 0$ and $\xi_{rs} = 0$ otherwise. Then the inequality (10) we write in the form $\beta_{pq}\xi_{pq}^2 \geq 0$. It follows that the conditions (11) are necessary. Let q be an arbitrary number, $q \geq 0$, $\xi_{rq} \neq 0$, $r \geq 1$, and all other cases $\xi_{rs} = 0$. Then according to theorem 16.2 [8, pp. 67–68] for $s = q$ the conditions (12) and (13) are necessary, i.e., there exist the numbers g_{rq}, $r \geq 1$, such that $0 \leq g_{rq} \leq 1$, $r \geq 1$, and $\alpha_{rq}^2 = \beta_{rq}\beta_{r-1,q}(1-g_{r-1,q})g_{rq}$, $r \geq 2$. If $\xi_{0s} \neq 0$, $s \geq 1$, and all other cases ξ_{rs} equal to 0, then according to theorem 16.2 [8, pp. 67–68] for $r = 0$ the conditions (12) and (13) are also necessary.

Let the conditions (11)–(13) holds. Then

$$\sum_{\substack{r,s=0\\ r+s\geq 1}}^{n} \beta_{rs}\xi_{rs}^2 - 2\sum_{\substack{r,s=0\\ r+s\geq 2\\ r\neq 1}}^{n} \alpha_{rs}\xi_{rs}\xi_{r+\delta_{r0}-1,s-\delta_{r0}} = \sum_{s=0}^{n-1} \beta_{1s}g_{1s}\xi_{1s}^2 + \sum_{\substack{r+s=n\\ r,s\geq 0}} \beta_{rs}(1-g_{rs})\xi_{rs}^2$$

$$+ \beta_{01}g_{01}\xi_{01}^2 + \sum_{\substack{r,s=0\\ r+s\geq 2\\ r\neq 1}}^{n} \left[\sqrt{\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})}\,\xi_{r+\delta_{r0}-1,s-\delta_{r0}} \pm \sqrt{\beta_{rs}g_{rs}}\,\xi_{rs}\right]^2,$$

where "$+$" is taken, if $\alpha_{rs} \leq 0$, and "$-$" is taken, if $\alpha_{rs} > 0$, from which (10) follows. □

By arguments similar to the proof of the theorem 4.6 [4, pp. 135–137], we can show that following theorem holds.

Theorem 3. *If for natural n the quadratic form* (10) *is non-negative definite, then the quadratic form*

$$\sum_{\substack{r,s=0\\r+s\geq 1}}^{n} \beta_{rs}\xi_{rs}^2 - 2\sum_{\substack{r,s=0\\r+s\geq 2\\r\neq 1}}^{n} \alpha'_{rs}\xi_{rs}\xi_{r+\delta_{r0}-1,s-\delta_{r0}}$$

is also non-negative definite for $|\alpha'_{rs}| \leq |\alpha_{rs}|$, $r \geq 0$, $s \geq 0$, $r \neq 1$, $2 \leq r+s \leq n$.

Corollary. *In theorem 2 we may replace the conditions* (13) *by the following ones*

$$\alpha_{rs}^2 \leq \beta_{rs}\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r\geq 0,\ s\geq 0,\ r\neq 1,\ r+s\geq 2, \tag{14}$$

where $0 \leq g_{rs} \leq 1$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$.

Since $|a_{rs}^2| - \mathrm{Re}(a_{rs}^2) = 2\alpha_{rs}^2$ for each rs, $r \geq 0,\ s \geq 0,\ r \neq 1,\ r+s \geq 2$, then the conditions (14) we may write in the form

$$|a_{rs}^2| - \mathrm{Re}(a_{rs}^2) \leq 2\beta_{rs}\beta_{r+\delta_{r0}-1,s-\delta_{r0}}(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs},\ \ r\geq 0,\ s\geq 0,\ r\neq 1,\ r+s\geq 2, \tag{15}$$

where $0 \leq g_{rs} \leq 1$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$.

2 THE EXAMPLES OF A POSITIVE DEFINITE BCF OF SPECIAL FORM

We consider fraction

$$\Phi_0 + \cfrac{1}{1+\Phi_1+\displaystyle\mathop{\mathrm{D}}_{s=2}^{\infty}\frac{a_{0s}}{1+\Phi_s}}, \quad \Phi_p = \cfrac{1}{1+\displaystyle\mathop{\mathrm{D}}_{r=2}^{\infty}\frac{a_{rp}}{1}}, \quad p \geq 0,$$

where a_{rs}, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r+s \geq 2$, are complex constants. By an equivalent transformation we reduce its to the form

$$i\Psi_0 + \cfrac{i}{i-\Psi_1+\displaystyle\mathop{\mathrm{D}}_{s=2}^{\infty}\frac{-c_{0s}^2}{i-\Psi_s}}, \quad \Psi_p = \cfrac{1}{i+\displaystyle\mathop{\mathrm{D}}_{r=2}^{\infty}\frac{-c_{rp}^2}{i}}, \quad p \geq 0, \tag{16}$$

where $c_{rs}^2 = a_{rs}$, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r+s \geq 2$. Than, taking into account that all $\beta_{rs} = 1$, the conditions (15) for BCF (16) we write in the form

$$|c_{rs}^2| - \mathrm{Re}(c_{rs}^2) \leq 2(1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r\geq 0,\ s\geq 0,\ r\neq 1,\ r+s\geq 2, \tag{17}$$

where $0 \leq g_{rs} \leq 1$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$. If we put $g_{rs} = 1/2$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$, this reduces to the parabola regions

$$|c_{rs}^2| - \mathrm{Re}(c_{rs}^2) \leq 1/2, \quad r\geq 0,\ s\geq 0,\ r\neq 1,\ r+s\geq 2.$$

If the c_{rs}, $r \geq 0$, $s \geq 0$, $r \neq 1$, $r+s \geq 2$, are pure imaginary, then (17) reduces to

$$|c_{rs}^2| \leq (1-g_{r+\delta_{r0}-1,s-\delta_{r0}})g_{rs}, \quad r\geq 0,\ s\geq 0,\ r\neq 1,\ r+s\geq 2,$$

where $0 \leq g_{rs} \leq 1$, $r \geq 0$, $s \geq 0$, $r+s \geq 1$.

CONCLUSION

An established connection between the positive definite BCF of special form and the certain quadratic form furnishes us new opportunities of approach to the convergence problem of the BCF of special form.

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Дмитришин Р.І. *Додатно визначені гіллясті ланцюгові дроби спеціального вигляду* // Карпатські математичні публікації. — 2013. — Т.5, №2. — С. 225–230.

Запропоновано дослідження класу гіллястих ланцюгових дробів спеціального вигляду, знаменники яких відмінні від нуля. Встановлено зв'язок такого дробу з певною квадратичною формою, що дає нові можливості для дослідження збіжності гіллястих ланцюгових дробів спеціального вигляду.

Ключові слова і фрази: додатно визначений гіллястий ланцюговий дріб спеціального вигляду, квадратична форма.

Дмитришин Р.И. *Положительно определенные ветвящиеся цепные дроби специального вида* // Карпатские математические публикации. — 2013. — Т.5, №2. — С. 225–230.

Предложены исследования класса ветвящихся цепных дробей специального вида, знаменатели которых отличны от нуля. Установлена связь такой дроби с определенной квадратичной формой, что дает новые возможности для исследования сходимости ветвящихся цепных дробей специального вида.

Ключевые слова и фразы: положительно определенная ветвящаяся цепная дробь специального вида, квадратичная форма.