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Torsion locally nilpotent groups with non-Dedekind norm of Abelian non-cyclic subgroups

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The authors study relations between the properties of torsion locally nilpotent groups and their norms of Abelian non-cyclic subgroups. The impact of the norm of Abelian non-cyclic subgroups on the properties of the group under the condition of norm non-Dedekindness is investigated. It was found that for these restrictions, torsion locally nilpotent groups are finite extensions of a quasi-cyclic subgroup and the structure of such groups is completely described.

Key words and phrases: torsion locally nilpotent group, norm of a group, norm of Abelian non-cyclic subgroups, non-Dedekindness.

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Introduction

In group theory, research related to the properties of some, as a rule, characteristic subgroups and the study of the impact of these subgroups on the properties and structure of the whole group are in focus. Such subgroups fully include different norms of a group.

The first types of norms appeared long ago in the research of R. Baer [1], W. Kappe [9] and H. Wielandt [22], but the general concept of Σ-norm (norm defined by a system of subgroups Σ) was introduced by F. M. Lyman at the end of the XX century.

Following [7, 12], *the* Σ*-norm of a group* G is the intersection of the normalizers of all subgroups of a group G which belong to the system Σ of all subgroups of a group with a certain theoretical-group property. In particular, if Σ is the system of all subgroups of G, then the corresponding Σ-norm is called *the norm of a group* G and is denoted by $N(G)$ [1]. Narrowing the system Σ of subgroups, we can obtain other Σ-norms, which are generalizations of the norm $N(G)$ (see, for example, [2, 8, 12, 13, 15]).

It is clear that each Σ-norm is the characteristic subgroup of a group, contains its center and the norm $N(G)$ of a group. Moreover, in groups that have a non-empty system of subgroups Σ and coincide with their Σ-norm, all subgroups of the system Σ are normal. Such groups were actively studied in the second half of the XX century (see, in particular, [5, 11, 14]) and for many systems their properties and structure are well known.

The authors continue to study the properties of groups depending on the properties of Σ-norm – the norm N_G^A of Abelian non-cyclic subgroups of a group, started in [6, 16–20].

The norm N_G^A *of Abelian non-cyclic subgroups of* G is a Σ-norm, provided that the system Σ

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is non-empty and consists of all Abelian non-cyclic subgroups of this group. In other words, the norm of Abelian non-cyclic subgroups of a group is the intersection of the normalizers of all Abelian non-cyclic subgroups of the group, provided that the system of such subgroups is non-empty (see, for example, [16]).

The purpose of this article is to study the properties of torsion locally nilpotent groups with non-Dedekind norm of Abelian non-cyclic subgroups. Note that the locally finite p-groups (p is prime) with the non-Dedekind norm N_G^A were fully studied in [6,16,18–20]. Therefore, in this article we consider mainly torsion non-primary locally nilpotent groups with the mentioned above restriction on the norm N_G^A.

As will be shown below, such groups are finite extensions of a quasi-cyclic p-subgroup (p is prime, $p \in \pi(G)$) and the direct products of the Sylow p-subgroup G_p with non-Dedecind norm $N_{G_p}^A$ of Abelian non-cyclic subgroups and finite p'-subgroup, all Abelian subgroups of which are cyclic (Theorem 2). This result is correlates with the research of F. M. Lyman [14] about the properties of torsion locally nilpotent groups with the condition $G = N_G^A$, as well as with the results of [6,16,18–20], where primary locally finite groups with non-Dedekind norm N_G^A are characterized.

1 Preliminary results

In this paper, only groups with a non-empty system of Abelian non-cyclic subgroups with the respect to the definition of N_G^A, are studied. Most of our notation is standard and can be found in [10].

At first let us present some preliminary results regarding the properties of groups with non-Dedekind N_G^A norm of Abelian non-cyclic subgroups and properties of norm N_G^A in such groups. We get the following result by the definition of the norm of Abelian non-cyclic subgroups of a group.

Lemma 1. *Let G be a group with non-empty system of Abelian non-cyclic subgroups. Then the following propositions take place:*

1) *$N_G^A \geq N(G) \geq Z(G)$, where $N(G)$ is the norm, $Z(G)$ is the center of a group G;*
2) *if the subgroup N_G^A contains an Abelian non-cyclic subgroup, then $N_G^A = N_{N_G^A}^A$;*
3) *every Abelian non-cyclic subgroup of the group N_G^A is its normal subgroup;*
4) *if H is a subgroup in G, which contains at least one Abelian non-cyclic subgroup, then $H \bigcap N_G^A \leq N_H^A$;*
5) *if $N_G^A \leq H$ and the system of Abelian non-cyclic subgroups of the group H is non-empty, then $N_G^A \leq N_H^A$;*
6) *if $H \subseteq C_G(N_G^A)$ and the group $G_1 = H \cdot N_G^A$ contains Abelian non-cyclic subgroups, then G_1 is a Dedekind group or a non-Hamiltonian one, in which all Abelian non-cyclic subgroups are normal and $G_1 = N_{G_1}^A$ in both cases.*

The following lemma determines sufficient conditions for the Dedekindness of the norm of Abelian non-cyclic subgroups in an arbitrary group and is the direct consequence of Lemma 1.2 ([17]).

Lemma 2. *If a group G contains an Abelian non-cyclic subgroup M and*

$$M \cap N_G^A = E,$$

then the subgroup N_G^A is Dedekind.

It is clear that every Abelian non-cyclic subgroup of the norm N_G^A is its normal subgroup. However, the subgroup N_G^A may not contain Abelian non-cyclic subgroups and be a unit subgroup. The example of such a group is an infinite torsion Frobenius group, constructed by V. M. Busarkin and A. I. Starostin in [3] (Example 3).

It will be proved below that in the class of torsion locally nilpotent groups with non-Dedekind norm N_G^A the condition of existence of Abelian non-cyclic subgroups in a group is equivalent to the condition of existence of such subgroups in the norm N_G^A.

By the definition of the norm N_G^A, all Abelian non-cyclic subgroups are normal in a group G, which contains at least one Abelian non-cyclic subgroup and coincides with N_G^A. Non-Abelian groups with this property were studied in detail in [14] and were called $\overline{HA}$*-groups* (respectively, $\overline{HA}_p$*-groups*, if they are p-groups).

The structure of torsion locally nilpotent non-Hamiltonian $\overline{HA}$-groups is characterized in Propositions 1 and Proposition 2 (see [14]).

Proposition 1. *Any torsion $\overline{HA}_p$-group (p is prime) is locally finite and is a group of one of the following types:*

1) $G = (\langle a\rangle \times \langle b\rangle) \rtimes \langle c\rangle$, where $|a| = p^n$, $|b| = |c| = p$, $[a,b] = [a,c] = 1$, $[b,c] = a^{p^{n-1}}$;

2) $G = \langle a\rangle \rtimes \langle b\rangle$, where $|a| = p^n$, $|b| = p^m$, $n \geq 2, m \geq 1$, $[a,b] = a^{p^{n-1}}$;

3) $G = (\langle a\rangle \times \langle b\rangle)\langle d\rangle$, where $|a| = |d| = 9$, $|b| = 3$, $[a,b] = 1$, $[a,d] = b$, $[b,d] = d^3 = a^{-3}$;

4) $G = (H \times \langle b\rangle) \rtimes \langle c\rangle$, where $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2$, $|b| = |c| = 2$, $[h_1,h_2] = h_1^2$, $[H,\langle b\rangle] = [H,\langle c\rangle] = E$, $[b,c] = h_1^2$;

5) $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle$, where $|a| = |b| = |c| = 4$, $c^2 = a^2b^2$, $[c,b] = c^2$, $[c,a] = a^2$;

6) $G = (\langle a\rangle \times \langle b\rangle)\langle c\rangle\langle d\rangle$, where $|a| = |b| = |c| = |d| = 4$, $c^2 = d^2 = a^2b^2$, $[a,c] = [d,c] = a^2$, $[b,d] = b^2$, $[c,b] = [d,a] = c^2$;

7) $G = H \times \langle c\rangle$, where $H = \langle h_1,h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2$, $[h_1,h_2] = h_1^2$, $|c| = 2^n$, $n \geq 2$;

8) $G = H \times Q$, where H and Q are quaternion groups of order 8;

9) $G = (H \times \langle b\rangle)\langle c\rangle$, where $H = \langle h_1,h_2\rangle$, $|h_1| = |h_2| = |b| = |c| = 4$, $[h_1,h_2] = h_1^2 = h_2^2$, $[H,\langle c\rangle] = E$, $c^2 = b^2h_1^2$, $[b,c] = b^2$;

10) $G = (\langle h_2\rangle \times \langle c\rangle)\langle h_1\rangle$, where $|h_1| = |h_2| = 4$, $[h_1,h_2] = h_1^2 = h_2^2$, $|c| = 2^n > 2$, $[c,h_1] = c^{2^{n-1}}$;

11) $G = (H \times \langle a\rangle)\langle b\rangle$, where $H = \langle h_1,h_2\rangle$, $|h_1| = |h_2| = 4$, $|a| = 2$, $|b| = 8$, $[a,b] = [h_1,h_2] = h_1^2 = h_2^2$, $b^2 = h_1$, $[h_2,b] = a$;

12) $G = \langle a\rangle\langle b\rangle$, where $|a| = 2^n, n > 2$, $|b| = 8$, $b^4 = a^{2^{n-1}}$, $b^{-1}ab = a^{-1}$;

13) *$G = A\langle b\rangle$, A is a quasi-cyclic 2-group, $|b| = 4$, $b^2 \in A$, $b^{-1}ab = a^{-1}$ for any element $a \in A$;*

14) *$G = A\langle b\rangle$, A is a quasi-cyclic 2-group, $|b| = 8$, $b^4 \in A$, $b^{-1}ab = a^{-1}$ for any element $a \in A$;*

15) *$G = A \times H$, A is a quasi-cyclic 2-group, $H = \langle h_1, h_2\rangle$, $|h_1| = |h_2| = 4$, $h_1^2 = h_2^2$, $[h_1, h_2] = h_1^2$;*

16) *$G = (A \times \langle b\rangle) \leftthreetimes \langle c\rangle$, A is a quasi-cyclic p-group, $|b| = |c| = p$, $[A, \langle c\rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = p$.*

Proposition 2. *Torsion locally nilpotent non-Hamiltonian group G is $\overline{HA}$-group if and only if*

$$G = G_p \times B,$$

where G_p is the Sylow p-subgroup of a group G, which is a non-Hamiltonian $\overline{HA}_p$-group, and B is a finite Dedekind group, which all Abelian subgroups are cyclic.

Thus, the structure of torsion locally nilpotent groups with the condition $G = N_G^A$ is actually known. Therefore, a natural question arises regarding the characterization of the properties of such groups provided that N_G^A is their proper non-Dedekind subgroup. As mentioned above, the corresponding studies were carried out by the authors in the class of locally finite p-groups (see, for example, [16,19,20]) provided that the N_G^A of such groups is a non-Dedekind group.

Structural characterization of infinite locally finite p-groups (p is prime) with the given restriction on the norm of Abelian non-cyclic subgroups, is in Proposition 3.

Proposition 3. *Any infinite locally finite p-group G has non-Dedekind norm N_G^A if and only if it is a group of one of the following types:*

1) *G is an infinite locally finite $\overline{HA}_p$-group of one of the types 13) – 16) of Proposition 1;*

2) *$G = (A \times \langle b\rangle) \leftthreetimes \langle c\rangle \leftthreetimes \langle d\rangle$, where A is a quasi-cyclic 2-group, $|b| = |c| = |d| = 2$, $[A, \langle c\rangle] = E$, $[b, c] = [b, d] = [c, d] = a_1 \in A$, $|a_1| = 2$, $d^{-1}ad = a^{-1}$ for any element $a \in A$; $N_G^A = (\langle a_2\rangle \times \langle b\rangle) \leftthreetimes \langle c\rangle$, $a_2 \in A$, $|a_2| = 4$;*

3) *$G = (A\langle y\rangle)Q$, where A is a quasi-cyclic 2-group, $[A, Q] = E$, $Q = \langle q_1, q_2\rangle$, $|q_1| = 4$, $q_1^2 = q_2^2 = [q_1, q_2]$, $|y| = 4$, $y^2 = a_1 \in A$, $y^{-1}ay = a^{-1}$ for any element $a \in A$, $[\langle y\rangle, Q] \subseteq \langle a_1\rangle \times \langle q_1^2\rangle$; $N_G^A = \langle a_2\rangle \times Q$, $a_2 \in A$, $|a_2| = 4$.*

Therefore, further attention will be focused on the study of non-primary torsion locally nilpotent groups with non-Dedekind norm N_G^A.

2 Properties of torsion locally nilpotent groups with non-Dedekind norm of Abelian non-cyclic subgroups

In this section, the properties of torsion locally nilpotent groups with non-Dedekind norm N_G^A of Abelian non-cyclic subgroups will be considered. Let us prove that in such groups the existence of Abelian non-cyclic subgroups directly depends on the existence of Abelian non-cyclic subgroups in the norm N_G^A. Further the following lemma will be used actively.

Lemma 3 ([14, Lemma 1.2]). *Locally finite p-group G does not contain elementary Abelian subgroups of order p^2 if and only if it is a group of one of the following types:*

1) G is a locally cyclic p-group;

2) G is a quaternion 2-group (finite or infinite).

Theorem 1. *Torsion locally nilpotent group G with the non-Dedekind norm N_G^A of Abelian non-cyclic subgroups contains Abelian non-cyclic subgroups if and only if the N_G^A contains such subgroups.*

Proof. The sufficiency of the condition of the theorem is evident, so we will prove only their necessities.

Let a group G contain Abelian non-cyclic subgroups and have the non-Dedekind norm N_G^A. Suppose that, contrary to the assumption of the theorem, the system of Abelian non-cyclic subgroups of the norm N_G^A is empty. Since G is locally nilpotent, it is the direct product of its Sylow p-subgroups by Proposition 1.4 [4]. Its norm N_G^A of Abelian non-cyclic subgroups is also locally nilpotent and is a non-Dedekind subgroup by the condition. By the assumption of Theorem 1, Lemma 3, and Propositions 1, 2 the norm N_G^A is the direct product of a finite quaternion 2-group Q of order greater than 8 and a cyclic Sylow $2'$-subgroup $(N_G^A)_{2'} = \langle h \rangle$:

$$N_G^A = Q \times \langle h \rangle,$$

where $Q = \langle a \rangle \langle b \rangle, |a| = 2^n, n \geq 3, |b| = 4, a^{2^{n-1}} = b^2, b^{-1}ab = a^{-1}, |h| = m, (m, 2) = 1.$

Let us assume that G contains Abelian non-cyclic $2'$-subgroups and let us denote any of such a subgroup by M. Then

$$[\langle q \rangle, M] \subseteq M \cap Q = E$$

for any element $q \in Q$. But in this case $\langle q \rangle \times M$ is Abelian non-cyclic and N_G^A-admissible subgroup. So

$$\langle q \rangle = (\langle q \rangle \times M) \cap Q \lhd Q,$$

that is impossible. Thus, G does not contain Abelian non-cyclic, so infinite Abelian, $2'$-subgroups.

It is known (see, for example, [10, p. 499]), that any infinite locally finite group contains infinite Abelian subgroup. By this and proved above the Sylow $2'$-subgroup $G_{2'}$ of a group G is finite. Since any Sylow p-subgroup ($p \neq 2$) of a group G contains unique subgroup of prime order, it is cyclic by Lemma 3. So Sylow $2'$-subgroup $G_{2'}$, which is the direct product its Sylow p-subgroups ($p \neq 2$), is finite cyclic and coinsides with Sylow $2'$-subgroup of the norm N_G^A,

$$G_{2'} = (N_G^A)_{2'} = \langle h \rangle.$$

Let us consider Sylow 2-subgroup G_2 of a group G. It is evident that $Q \lhd G$ and $a_1 = a^{2^{n-1}} \in Q$ is central involution of a group G. Suppose that G_2 contains an involution $x \notin Q$. Then the subgroup $\langle x, a_1 \rangle = \langle x \rangle \times \langle a_1 \rangle$ is Abelian non-cyclic and

$$[\langle x \rangle, Q] \subseteq \langle x, a_1 \rangle \cap Q = \langle a_1 \rangle.$$

Let $\langle y \rangle$ be not invariant subgroup of order 4 from the subgroup Q. If $[y, x] = 1$, then the subgroup $\langle x \rangle \times \langle y \rangle$ is N_G^A-admissible and

$$[\langle y \rangle, Q] \subseteq (\langle x \rangle \times \langle y \rangle) \cap Q = \langle y \rangle.$$

Then $\langle y\rangle \lhd Q$, which contradicts the choice of the subgroup $\langle y\rangle$. Thus, $[y,x]\neq 1$ and $[y,x]=a_1=y^2$. Then $x^{-1}bx=b^{-1}$ and $|xb|=2$. So

$$x^{-1}abx=(ab)^{-1}=b^{-1}a^{-1}.$$

On the other hand,

$$x^{-1}abx=(x^{-1}ax)(x^{-1}bx)=x^{-1}axb^{-1}.$$

Thus, $x^{-1}axb^{-1}=b^{-1}a^{-1}$, so $x^{-1}ax=b^{-1}a^{-1}b=a$ and $[x,a]=1$. Since the subgroup $\langle a_1,xb\rangle=\langle a_1\rangle\times\langle xb\rangle$ is Abelian non-cyclic, it is Q-admissible. But,

$$[a,xb]=[a,b]=\langle a^2\rangle\not\subseteq\langle a_1,xb\rangle.$$

We get a contradiction. Thus, G_2 contains unique involution $a_1\in Q$ and is a quaternion 2-group (finite or infinite) by Lemma 3.

If G_2 is a finite quaternion 2-group, then it (and the whole group G) does not contain any Abelian non-cyclic subgroups, which contradicts the condition. So this case is impossible. Thus, G_2 is an infinite quaternion 2-group. Then it is a $\overline{HA}_2$-group by Proposition 1 and $G=N_G^A$ by locally nilpotency. But, the norm N_G^A also contains Abelian non-cyclic subgroups, which contradicts the assumption. So the assumption is wrong and the theorem is proved. ☐

Taking into account Theorem 1 and the definition of the norm N_G^A of Abelian non-cyclic subgroups of a group, the study of the properties of torsion locally nilpotent groups with non-Dedekind norm N_G^A will be considered provided that the N_G^A contains at least one Abelian non-cyclic subgroup, so it is a non-Hamiltonian $\overline{HA}$-group and satisfies the condition of Proposition 2.

Let us give some statements that characterize the properties of infinite non-primary locally nilpotent groups with non-Dedekind norm N_G^A.

Lemma 4. *If the norm N_G^A of Abelian non-cyclic subgroups of a torsion non-primary locally nilpotent group G is non-Dedekind, then G contains Abelian non-cyclic p-subgroups of unique prime p. Its Sylow q-subgroups (where q is an odd prime not equal to p, $q\in\pi(G)$) are cyclic, Sylow 2-subgroup ($p\neq 2$) is either cyclic or a finite quaternion 2-group.*

Proof. By the condition of lemma and Theorem 1 the norm N_G^A is a non-Hamiltonian locally nilpotent $\overline{HA}$-group. By Proposition 2

$$N_G^A=(N_G^A)_p\times B,$$

where $(N_G^A)_p$ is Sylow p-subgroup of the norm N_G^A, which is a non-Hamiltonian $\overline{HA}_p$-group of one of the types of Proposition 1, B is a finite Dedekind group, which all Abelian subgroups are cyclic and $(|B|,p)=1$.

Let us denote Sylow p- and p'-subgroups of a group G by G_p and $G_{p'}$. Then

$$G=G_p\times G_{p'},$$

and $(N_G^A)_p\subseteq G_p$ by the locally nilpotency of a group. So Sylow p-subgroup G_p of G is a group with Abelian non-cyclic subgroups. Let us prove that all Abelian subgroups of the group $G_{p'}$

are cyclic. Let $A \leq G_{p'}$ be an Abelian non-cyclic subgroup. Since the subgroup $(N_G^A)_p$ is characteristic and the subgroup A is N_G^A-admissible,

$$[\langle x \rangle, A] \subseteq (N_G^A)_p \cap A = E$$

for any element $x \in (N_G^A)_p$. Taking into account that $\langle x, A \rangle = \langle x \rangle \times A$ is Abelian non-cyclic and N_G^A-admissible, we get

$$(\langle x \rangle \times A) \cap (N_G^A)_p = \langle x \rangle \lhd (N_G^A)_p.$$

But in this case $(N_G^A)_p$ is Dedekind, which contradicts the condition. Thus, all Abelian p'-subgroups of a group G are cyclic.

Following M. Kargapolov, F. Hall and C. R. Kulatilaka (see, for example, [10, p. 499]), any infinite locally finite group contains an infinite Abelian subgroup. Therefore, taking into account the previous considerations, we conclude that $G_{p'}$ is finite, and all its Abelian subgroups are cyclic. So, by Lemma 3 all Sylow q-subgroups of a group G ($q \in \pi(G)$, $q \neq p$) are cyclic for odd prime, but Sylow 2-subgroup ($p \neq 2$) is either cyclic or a finite quaternion 2-group. Lemma is proved. ☐

Corollary 1. *If the norm of Abelian non-cyclic subgroups of a torsion non-primary locally nilpotent group G is non-Dedekind and $2 \notin \pi(G)$, then G contains non-cyclic Sylow p-subgroup for unique prime $p \in \pi(G)$.*

Lemma 5. *Any infinite torsion locally nilpotent group G with non-Dedekind norm N_G^A satisfies the minimality condition and is a Chernikov group.*

Proof. Let a group G and its norm N_G^A of Abelian non-cyclic subgroups satisfy the lemma condition. Then N_G^A is a non-Hamiltonian locally nilpotent $\overline{HA}$-group. By the description of such groups (Propositions 1, 2) the norm N_G^A is either finite or a finite extension of a quasi-cyclic p-subgroup for some prime $p \in \pi(G)$.

Suppose that G does not satisfy the minimality condition for Abelian subgroups. Then it contains an Abelian subgroup M, which is direct product of infinitely many subgroups of prime order.

Let

$$M_1 = N_G^A \cap M.$$

By the structure of the norm N_G^A, we get $|M_1| < \infty$. Thus, we can find an infinite Abelian subgroup M_2 in M, which $N_G^A \cap M_2 = E$. But in this case by Lemma 2 the norm N_G^A is Dedekind, contrary to the condition. Thus, G is a group with the minimality condition for Abelian subgroups.

Since G is a locally finite group, it is satisfies the minimality condition for all subgroups and is a finite extension of the direct product of finitely many of quasi-cyclic subgroups (that is, a Chernikov group) following V. Shunkov [21]. Lemma is proved. ☐

Lemma 6. *Infinite torsion locally nilpotent group G with the non-Dedekind norm N_G^A and the non-Hamiltonian Sylow p-subgroup $(N_G^A)_p$ is a finite extension of quasi-cyclic p-subgroup.*

Proof. Let a group G satisfy the lemma condition. Then it is a Chernikov group and is a finite extension of a divisible Abelian subgroup P by Lemma 5. Since any Sylow q-subgroup $(q \neq p)$ of a group G is either cyclic or a quaternion 2-subgroup by Lemma 4, P is the direct product of finitely many of quasi-cyclic p-subgroups.

Let $P \supseteq (A_1 \times A_2)$, where A_1 and A_2 are quasi-cyclic p-subgroups. Since

$$N_G^A \lhd G_1 = (A_1 \times A_2) \cdot N_G^A$$

and G_1/N_G^A is a divisible Abelian subgroup, the center of the group G_1 coincides a divisible Abelian subgroup A, $|A \cap N_G^A| < \infty$ and

$$G_1 = A \cdot N_G^A$$

by [4, Theorem 1.16]. Thus, G_1 is a locally nilpotent group, which is an extension of the norm N_G^A by a central subgroup. By Lemma 1 G_1 is a $\overline{HA}$-group, so $P = A$ is a quasi-cyclic p-subgroup, which is maximal divisible subgroup of a group G by the discription of such groups (Propositions 1, 2). Lemma is proved. $\square$

Corollary 2. *Any torsion locally nilpotent group G with infinite non-Dedekind norm N_G^A is a finite extension of this norm.*

The following theorem gives the complete description of the structure of torsion locally nilpotent groups with the non-Dedekind norm of Abelian non-cyclic subgroups and actually finishes the study of such groups.

Theorem 2. *Any torsion locally nilpotent group G has non-Dedekind norm N_G^A of Abelian non-cyclic subgroups if and only if*

$$G = G_p \times G_{p'},$$

where

- *G_p is Sylow p-subgroup of a group G with non-Dedekind norm $N_{G_p}^A$ of Abelian non-cyclic subgroups $(p \in \pi(G))$;*
- *$G_{p'}$ is a finite p'-subgroup, which all Abelian subgroup are cyclic, and $G_{p'} = \langle y \rangle \times H$, where $\langle y \rangle$ is a cyclic subgroup of odd coprime with p (in particular, unit) order, H is a finite cyclic (in particular, unit subgroup) or a quaternion 2-group $(p \neq 2)$.*

In this case

$$N_G^A = N_{G_p}^A \times \langle y \rangle \times H_1,$$

where

- *$H_1 = H$, if H is a cyclic 2-group or the quaternion group of order 8;*
- *$H_1 = \langle h_1^{2^{n-2}} \rangle, n > 2$, if $H = \langle h_1, h_2 \rangle$ is a quaternion 2-group of order greater then 8, $h_1^{2^{n-1}} = h_2^2$, $|h_2| = 4$, $h_2^{-1} h_1 h_2 = h_1^{-1}$.*

Proof. The sufficiency of the condition of the first statement of the theorem is evident. Let us prove their necessity. Let the norm N_G^A of Abelian non-cyclic subgroups be non-Dedekind. By Proposition 2

$$N_G^A = (N_G^A)_p \times B,$$

where $(N_G^A)_p$ is a Sylow p-subgroup of the group N_G^A, which is a non-Hamiltonian $\overline{HA}_p$-group, B is a finite Dedekind group, which all Abelian subgroups are cyclic and $(|B|, p) = 1$.

Since G is a torsion locally nilpotent group, it is the direct product of Sylow subgroups G_p and $G_{p'}$:

$$G = G_p \times G_{p'}.$$

By Lemma 4 the subgroup $G_{p'}$ is finite and does not contain non-cyclic Abelian subgroups. By locally nilpotency of a group and Lemma 3 the subgroup $G_{p'}$ is either cyclic or the direct product of a cyclic group of odd (in particular, unit) order and a finite cyclic or quaternion 2-group.

Let us prove that the Sylow p-subgroup $(N_G^A)_p$ of the norm N_G^A coincides with the norm $N_{G_p}^A$ of Abelian non-cyclic subgroups of the group G_p,

$$(N_G^A)_p = N_{G_p}^A.$$

By Lemma 1 $(N_G^A)_p \subseteq N_{G_p}^A$, so, the norm $N_{G_p}^A$ of the group G_p is non-Dedekind. Taking into account that every Abelian non-cyclic subgroup of a group G can be presented as $A \times \langle h \rangle$, where A is an Abelian non-cyclic p-subgroup, $(|h|, p) = 1$, and $N_{G_p}^A$ normalizers all such subgroups, we get

$$(N_G^A)_p = N_{G_p}^A.$$

Thus, the necessity of the condition of the first statement of the theorem is proved.

Let us prove that the norm N_G^A of a torsion locally nilpotent group G is of the type mentioned in the theorem. By the proved above Sylow p-subgroup $(N_G^A)_p$ of the norm N_G^A coincides with the norm $N_{G_p}^A$ of Abelian non-cyclic subgroups of the subgroup G_p, $(N_G^A)_p = N_{G_p}^A$.

Let us consider the Sylow p'-subgroup $G_{p'}$. If $2 \notin \pi(G_{p'})$, then $G_{p'}$ is cyclic and

$$G_{p'} \subseteq Z(G) \subseteq N_G^A.$$

Let $2 \in \pi(G_{p'})$. If Sylow 2-subgroup G_2 of a group G is cyclic or the quaternion group of order 8, then $G_2 \subseteq N_G^A$. Let G_2 be a generalized quaternion group of order greater then 8:

$$G_2 = \langle h_1, h_2 \rangle,$$

where $|h_1| = 2^n > 4, h_1^{2^{n-1}} = h_2^2, h_2^{-1}h_1h_2 = h_1^{-1}$.

Let us prove that in this case the subgroup $H_1 = \left\langle h_1^{2^{n-2}} \right\rangle$ normalizers every Abelian non-cyclic subgroup of a group G. If A is an Abelian non-cyclic $2'$-subgroup of a group G, then the subgroup $A_1 = \left\langle h_1^\alpha h_2^\beta \right\rangle \times A$ (where α, β are integer, $h_1^\alpha h_2^\beta \neq 1$) is N_G^A-admissible. So the subgroup $\left\langle h_1^\alpha h_2^\beta \right\rangle$ is also N_G^A-admissible. Since

$$[h_1^\alpha h_2^\beta, h_1^{2^{n-2}}] = h_2^{2\beta} \in \left\langle h_1^\alpha h_2^\beta \right\rangle,$$

$H_1 \subset N_G(A_1)$ and $H_1 \subset N_G^A$.

On the other hand, if $(\alpha, 2) = 1$, then no other non-central subgroup of the group H normalizers the subgroup $\langle h_1^{\alpha} h_2 \rangle$ and $H_1 = H \cap N_G^A$. Finally we get

$$N_G^A = N_{G_p}^A \times \langle y \rangle \times H_1.$$

The theorem is proved. □

Note that the result of Theorem 2 is also true for torsion nilpotent groups with non-Dedekind norm N_G^A. Thus, any torsion locally nilpotent (or nilpotent) group with the non-Dedekind norm N_G^A is the direct product of the Sylow p-subgroup with the non-Dedekind norm $N_{G_p}^A$ and a finite p'-subgroup of the group $G_{p'}$, which all Sylow q-subgroups are cyclic (q is odd prime not equal to p), and the Sylow 2-subgroup is either cyclic or a quaternion 2-group. The structure of such groups is actually determined by the structure of the Sylow p-subgroup G_p and is described by Proposition 3 for infinite case.

Let us consider some statements that follow from Theorem 2 and characterize the structure of infinite torsion locally nilpotent groups with non-Dedekind norm N_G^A.

Corollary 3. *Any infinite torsion locally nilpotent group G with non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and $2 \notin \pi(G)$ is a group of the type*

$$G = (A \times \langle b \rangle) \rightthreetimes \langle c \rangle \times \langle y \rangle,$$

where A is a quasi-cyclic p-group, $p \neq 2, |b| = |c| = p$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = p$, $|y| = m, (m, 2p) = 1$. Moreover $N_G^A = G$ and all Abelian non-cyclic subgroups of a group G are normal in it.

Proof. By the condition and Theorem 2, G is the direct product of infinity Sylow p-subgroup G_p with the non-Dedekind norm $N_{G_p}^A$ and finite p'-subgroup $G_{p'}$, all Abelian subgroups of which are cyclic,

$$G = G_p \times G_{p'}.$$

Since $2 \notin \pi(G)$, Sylow q-subgroups (where $q \in \pi(G)$, $q \neq p$) of a group G are cyclic, therefore, the Sylow p'-subgroup $G_{p'}$ (which is the direct product of Sylow q-subgroups) is cyclic, $G_{p'} = \langle y \rangle$. By the last condition and Proposition 3 the p-subgroup G_p is an infinite $\overline{HA}_p$-group of the type 16) of Proposition 1 for $p \neq 2$.

Therefore,

$$G = (A \times \langle b \rangle) \rightthreetimes \langle c \rangle \times \langle y \rangle,$$

where A is a quasi-cyclic p-group, $p \neq 2, |b| = |c| = p$, $[A, \langle c \rangle] = E$, $[b, c] = a_1 \in A$, $|a_1| = p$, $|y| = m, (m, 2p) = 1$. It is clear that this group is a $\overline{HA}$-group and therefore $N_G^A = G$. □

Corollary 4. *Any infinite torsion locally nilpotent group G has non-Dedekind norm N_G^A of Abelian non-cyclic subgroups and an Abelian non-cyclic 2-subgroup if and only if*

$$G = G_2 \times \langle y \rangle,$$

where $|y| = m, (m, 2) = 1$ and G_2 is a group of one of the following types:

1) *G_2 is an infinite $\overline{HA_2}$-group of one of the types 13) – 16) of Proposition 1; $N_G^A = G$;*
2) *G_2 is a 2-group of one of the types 2) – 3) of Proposition 3; $N_G^A = N_{G_2}^A \times \langle y \rangle$.*

Proof. The sufficiency of the condition of this corollary is obvious, so let us prove only the necessity.

Let G be an infinite torsion locally nilpotent group with an Abelian non-cyclic 2-subgroup and the non-Dedekind norm of Abelian non-cyclic subgroups. By Theorem 2 $G = G_p \times G_{p'}$, where G_p is Sylow p-subgroup of G with non-Dedekind norm $N_{G_p}^A$ of Abelian non-cyclic subgroups and $G_{p'}$ is a finite p'-subgroup, all Abelian subgroups of which are cyclic.

Since G is infinite and all Abelian non-cyclic subgroups are in G_p, $p = 2$ and G_2 is an infinite 2-group with non-Dedekind norm $N_{G_2}^A$ by the condition. Then by Theorem 2 the Sylow $2'$-subgroup $G_{2'}$ is cyclic. To complete the proof it remains to apply Proposition 3, which characterizes the structure of the Sylow 2-subgroup G_2. □

Corollary 5. *If an infinite torsion locally nilpotent group G does not contain any quaternion 2-subgroups of order greater then 8 and has infinite non-Dedekind norm N_G^A of Abelian non-cyclic subgroups, then*

$$G = N_G^A$$

and all Abelian non-cyclic subgroups of a group G are normal in it.

Proof. Let G satisfy the condition of the lemma. Then by Theorem 2 G is the direct product of infinite Sylow p-subgroup G_p with the non-Dedekind norm $N_{G_p}^A$ and finite p'-subgroup $G_{p'}$, all Abelian subgroups of which are cyclic.

Since G does not contain quaternion 2-subgroups of order greater than 8, $G_{p'} = \langle y \rangle \times H$, where $\langle y \rangle$ is a cyclic subgroup of odd coprime with p (in particular, unit) order and H is the finite cyclic or the quaternion 2-group (if $p \neq 2$) of order 8 (in particular, H is the unit subgroup).

On the other hand, the subgroup G_p satisfies the conditions of Proposition 3, so it is a group of one of the types 1) − 3). However, groups of the types 2) and 3), as well as the group of the type 1), which is $\overline{HA}_2$-group of the type 13) of Proposition 1, contain a quaternion group of order greater than 8. Therefore, G_p is an infinite $\overline{HA}_p$-group of one of the types 14) − 16) of Proposition 1. Taking into account the structure of the Sylow p'-subgroup, Proposition 2 and Theorem 2, we conclude that $G = N_G^A$, that is, all Abelian non-cyclic subgroups are normal in G. □

Corollary 6. *If Sylow 2-subgroup of infinite torsion locally nilpotent group G is not a finite quaternion 2-group of order greater then 8 and*

$$1 < [G : N_G^A] < \infty,$$

then the norm N_G^A is Dedekind.

Proof. Assume that N_G^A is non-Dedekind. By Theorem 2 we have

$$G = G_p \times G_{p'},$$

where G_p is the infinite Sylow p-subgroup of G with the non-Dedekind norm $N_{G_p}^A$, $G_{p'}$ is a finite p'-subgroup, all Abelian subgroups of which are cyclic. In this case, if $p \neq 2$, then the Sylow 2-subgroup is cyclic or the finite quaternion 2-group.

Since $1 < [G : N_G^A] < \infty$, the subgroup N_G^A is infinite and $N_G^A \neq G$. If under these condition a group G does not contain quaternion 2-subgroups of order greater than 8, then $N_G^A = G$ by Corollary 5, which is impossible. Thus, G contains quaternion subgroups of order greater than 8.

Taking into account that the Sylow 2-subgroup G_2 of G is not a finite quaternion 2-group of order greater than 8, we conclude that it is infinite. So, in this case $p = 2$ and G_2 is an infinite locally finite 2-group with non-Dedekind norm $N_{G_2}^A$ of Abelian non-cyclic subgroups. By Proposition 3 either $N_{G_2}^A$ is a group of the type 13) of Proposition 1 and $G = N_G^A$, or $|N_{G_2}^A| < \infty$ (for groups of the types 2) or 3) of Proposition 3) and $|N_G^A| < \infty$. In both cases we have contradiction. So, the norm N_G^A is Dedekind. □

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Авторами вивчаються взаємозв'язки між властивостями періодичних локально нільпотентних груп та їх норм абелевих нециклічних підгруп. Досліджено вплив норми абелевих нециклічних підгруп на властивості групи за умови недедекіндовості цієї норми. Встановлено, що за вказаних обмежень періодичні локально нільпотентні групи є скінченнимим розширеннями квазіциклічної підгрупи й повністю описано будову таких груп.

Ключові слова і фрази: періодична локально нільпотентна група, норма групи, норма абелевих нециклічних підгруп, недедекіндовість.