ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2016, **8** (2), 195–210
doi:10.15330/cmp.8.2.195-210

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2016, Т.8, №2, С.195–210

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COUPLED FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS ALONG WITH CLR PROPERTY IN MENGER METRIC SPACES

Coupled fixed point problems have attracted much attention in recent times. The aim of this paper is to extend the notions of E.A. property, CLR property and $JCLR$ property for coupled mappings in Menger metric space and use this notions to establish common coupled fixed point results for four self mappings. Our work generalizes the recent results of Jian-Zhong Xiao [24] et al (2011). The main result is supported by a suitable example.

Key words and phrases: Menger metric space, t-norm of H-type, weak compatibility coupled common fixed point, CLR property, E.A. property, $JCLR$ property.

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1 INTRODUCTION

The concept of a probabilistic metric space was introduced and studied by Menger [3, 19]. Since then, many authors have studied the fixed point property for mappings defined on probabilistic metric spaces (see [2,4,7,12,24]). Jachymski [15] has proved some fixed point theorems for probabilistic nonlinear contractions with a gauge function φ and discussed the relations between several assumptions concerning φ.

Bhaskar and Lakshmikantham [24] introduced the notion of coupled fixed points and proved some coupled fixed point results in partially ordered metric spaces. The work [23] was illustrated by proving the existence and uniqueness of the solution for a periodic boundary value problem. These results were further extended and generalized by Lakshmikantham and Ćirić [8] to coupled coincidence and coupled common fixed point results for nonlinear contractions in partially ordered metric spaces.

Sedghi and al [5, 9–11] proved some coupled fixed point theorems under contractive conditions in fuzzy metric spaces. The results proved by Fang [1] for compatible and weakly compatible mappings under φ-contractive conditions in Menger spaces that provide a tool to Hu [6] for proving fixed points results for coupled mappings and these results are the genuine generalization of the result of [10].

Aamri and Moutawakil [22] introduced the concept of E.A. property in a metric space. Sintunavarat and Kuman [14] introduced a new concept of CLR property. Very recently, Chauhan et.al [13] introduced the notion of $JCLR$ property. The importance of CLR property ensures that one does not require the closeness of range subspaces.

In this paper, we give the concept of E.A. property, CLR property and $JCLR$ property for coupled mappings and prove a result which provides a generalization of the result of Zhong Xiao [24].

УДК 515.124
2010 *Mathematics Subject Classification:* 47H10, 54H25.

2 PRELIMINARIES

We now state some basic concepts and results which will be used. In the standard notation, we suppose that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^+$ be the set of positive integers.

A function $F : \overline{\mathbb{R}} \to [0, 1]$ is called a distribution function if it is non decreasing and left continuous with $F(-\infty) = F(+\infty) = 1$. The class of all distribution functions is denoted by D_∞.

Suppose that $D = \{F \in D_\infty : \inf F\, D_\infty^+ (t) = 0, \sup F(t) = 1\}$, $D_\infty^+ = \{F \in D_\infty : F(0) = 0\}$ and $D^+ = D \cap D_\infty^+$ (see [10, 17]).

A special element of D^+ is the Heaviside function H defined by

$$H(t) = \begin{cases} 1, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Definition 1 ([16, 17]). *A function $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied for any $a, b, c, d \in [0, 1]$:*

(Δ-1) $\Delta(a, 1) = a$;

(Δ-2) $\Delta(a, b) = \Delta(b, a)$;

(Δ-3) $\Delta(a, b) \geq \Delta(c, d)$, *for* $a \geq c, b \geq d$;

(Δ-4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Two examples of t-norm are $\Delta_M(a, b) = \min\{a, b\}$ and $\Delta_P(a, b) = ab$. It is evident that, as regards the point wise ordering, $\Delta \leq \Delta_M$ for each t-norm Δ.

Definition 2 ([16–18]). *A triplet (X, F, Δ) is called a generalized Menger probabilistic metric space if X is a non-empty set, Δ is t-norm and F is a mapping from $X \times X$ into D_∞^+ satisfying the following condition ($F(x, y)$ for $x, y \in X$ is denoted by $F_{x,y}$):*

(MS-1) $F_{x,y}(t) = H(t)$ *for all* $t \in \mathbb{R}$ *if and only if* $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ *for all* $x, y \in X$ *and* $t \in \mathbb{R}$;

(MS-3) $F_{x,y}(t + s) \geq T(F_{x,z}(t), F_{z,y}(s))$ *for all* $x, y, z \in X$ *and* $t, s \in \mathbb{R}^+$.

A Menger probabilistic metric space (for short, a Menger PM-space) is a generalized Menger space with $F(X \times X) \in D^+$.

Schweizer et al [1, 19] point out that if the t-norm T of a Menger PM-space satisfies the condition $\sup\limits_{0<a<1} \Delta(a, a) = 1$, then (X, F, Δ) is a first countable Hausdorff topological space in the (ε, λ) topology τ, i.e., the family of sets

$$\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in [0, 1], \ (x \in X)\}$$

is the base of neighborhoods of point x for τ, where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

By virtue of this topology τ a sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent to x (we write $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$) if $\lim\limits_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a Cauchy sequences in (X, F, Δ) if for any given $\varepsilon > 0$ and $\lambda \in [0, 1]$, there exists $N = N(\varepsilon, \lambda) \in \mathbb{Z}^+$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$; (X, F, Δ) is said to be complete if each Cauchy sequence in X is convergent to some point in X.

In the sequel, we will always assume that (X, F, Δ) is a Menger space with the (ε, λ) topology.

Lemma 1. *Let (X, d) be a usual metric space. Define a mapping $F : X \times X \to D^+$ by*

$$F_{x,y}(t) = H(t - d(x,y)) \quad \textit{for } x, y \in X \textit{ and } t > 0.$$

Then (X, F, Δ_m) is a Menger PM-space. It is called the induced Menger PM space by (X, d) and it is complete if (X, d) is complete.

An arbitrary t-norm can be extended (by (Δ-3)) in a unique way to an n-ary operation. For $(a_1, a_2, \ldots, a_n) \in [0,1]^n, n \in \mathbb{Z}^+$, the value $\Delta^n(a_1, a_2, \ldots, a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1, a_2, \ldots, a_n) = \Delta\left(\Delta^{n-1}(a_1, a_2, \ldots, a_{n-1}), a_n\right)$.

For each $a \in [0,1]$, the sequence $\{\Delta^n(a)\}_{n=1}^{\infty}$ is defined by $\Delta^1(a) = a$ and $\Delta^n(a) = \Delta\left(\Delta^{n-1}(a), a\right)$.

Definition 3. *A t-norm Δ is said to be of H-type if the sequence of functions $\{\Delta^n(a)\}_{n=1}^{\infty}$ is equicontinuous at $a = 1$.*

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{0<a<1} \Delta(a, a) = 1$.

Lemma 2. *Let (X, F, Δ) be a Menger PM-space. For each $\lambda \in (0,1]$, define a function $d_\lambda : X \times X \to \mathbb{R}^+$ by*

$$d_\lambda(x,y) = \inf\left\{t > 0 : F_{x,y}(t) > 1 - \lambda\right\}. \tag{1}$$

Then the following statements hold:

(1) $d_\lambda(x,y) < t$ if and only if $F_{x,y}(t) > 1 - \lambda$;

(2) $d_\lambda(x,y) = d_\lambda(y,x)$ for all $x, y \in X$ and $\lambda \in (0,1]$;

(3) $d_\lambda(x,y) = 0$ for all $\lambda \in (0,1]$ if and only if $x = y$.

Lemma 3. *Let (X, F, Δ) be a Menger PM-space and let $\{d_\lambda\}_{\lambda \in (0,1]}$ be a family of pseudo-metrics on X defined by (1).*

If Δ is a t-norm of H-type, then for each $\lambda \in (0,1]$ there exists $\mu \in (0, \lambda]$ such that for each $m \in \mathbb{Z}^+$,

$$d_\lambda(x_0, x_m) \leq \sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}) \quad \textit{for all } x_0, x_1, \ldots, x_m \in X.$$

Lemma 4. *Suppose that $F \in D^+$. For each $n \in \mathbb{Z}^+$, let $F_n : \mathbb{R} \to [0,1]$ be nondecreasing, and $g_n : (0, +\infty) \to (0, +\infty)$ satisfies $\lim_{n \to +\infty} g_n(t) = 0$ for any $t > 0$. If*

$$F_n(g_n(t)) \geq F(t) \quad \textit{for all} \quad t > 0,$$

then $\lim_{n \to +\infty} F_n(t) = 1$ for any $t > 0$.

Definition 4 ([20]). *An element $x \in X$ is called a common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if*

$$x = f(x,x) = g(x).$$

Definition 5 ([21]). *An element $(x,y) \in X \times X$ is called:*
(i) a coupled fixed point of the mapping $f : X \times X \to X$ if

$$f(x,y) = x, \;\; f(y,x) = y;$$

(ii) a coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$f(x,y) = g(x), \;\; f(y,x) = g(y);$$

(iii) a common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if

$$x = f(x,y) = g(x), \;\; y = f(y,x) = g(y).$$

In [22], Abbas et al introduced the concept of weakly compatible mappings. Here we give a similar concept in Menger metric spaces as follows.

Definition 6. *Let (X, F, Δ) be a Menger metric space and let $f : X \times X \to X$ and $g : X \to X$ be two mappings. f and g are said to be weakly compatible (or w-compatible) if they commute at their coupled coincidence points, i.e.; if (x,y) is a coupled coincidence point of f and g, then*

$$g(f(x,y)) = f(g(x), g(y)), \qquad g(f(y,x)) = f(g(y), g(x)).$$

Definition 7 ([23]). *Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings. Then, the pairs (B,S) and (A,T) are said to have common coupled coincidence point if there exist a, b in X such that*

$$B(a,b) = S(a) = T(a) = A(a,b) \text{ and } B(b,a) = S(b) = T(b) = A(b,a).$$

3 MAIN RESULTS

Now, we introduce the following concepts.

Definition 8. *Let (X, F, Δ) be a Menger metric space and let mappings $A : X \times X \to X$ and $S : X \to X$ are said to satisfy the E.A. property if there exist sequences $\{x_n\}, \{y_n\} \in X$ such that*

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = x \text{ and } \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = y$$

for some $x, y \in X$.

Definition 9. *Let (X, F, Δ) be a Menger metric space and let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings.*

Then the pairs (B,T) and (A,S) are said to satisfy the common $E.A.$ property if there exist sequences $\{x_n\}, \{y_n\}, \{x'_n\}, \{y'_n\} \in X$ such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} B\left(x'_n, y'_n\right) = \lim_{n\to\infty} T\left(x'_n\right) = x,$$

$$\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} B\left(y'_n, x'_n\right) = \lim_{n\to\infty} T\left(y'_n\right) = y$$

for some $x, y \in X$.

Definition 10. *Let (X, F, Δ) be a Menger metric space. The mappings $A : X \times X \to X$ and $S : X \to X$ are said to satisfy the CLR_S property if there exist sequences $\{x_n\}, \{y_n\} \in X$ such that*

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = Sx \text{ and } \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = Sy$$

for some $x, y \in X$.

Definition 11. *Let (X, F, Δ) be a Menger metric space and let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$, $S : X \to X$ be four mappings.*

Then the pairs (B, T) and (A, S) are said to satisfy the common CLR_{ST} property if there exist sequences $\{x_n\}, \{y_n\}, \{x_n'\}, \{y_n'\} \in X$ such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} B\left(x_n', y_n'\right) = \lim_{n\to\infty} T\left(x_n'\right) = x,$$

$$\lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} B\left(y_n', x_n'\right) = \lim_{n\to\infty} T\left(y_n'\right) = y,$$

where $x, y \in S(X) \cap T(X)$.

Jian-Zhong Xiao [24] proved the following result.

Theorem 1. *Let (X, F, Δ) be a complete Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $T : X \to X$ be two mappings such that*

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Tx,Tu}(t),\ F_{Ty,Tv}(t)\right)\right]^{1/2}$$

for all $x, y, u, v \in X$ and $t > 0$, where $A(X \times X) \subseteq T(X)$, T is continuous and commutative with A. Then there exists a unique $u \in X$ such that $u = Tu = A(u, u)$.

We now give our main result which provides a generalization of Theorem 1.

Theorem 2. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Su}(t),\ F_{Sy,Sv}(t)\right)\right]^{1/2}; \tag{2}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{3}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$, and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (2) we have

$$\begin{aligned} F_{A(x_n,y_n),A(p,q)}(t) &\geq F_{A(x_n,y_n),A(p,q)}(\varphi^{n_0}(t)) \\ &\geq \left[\Delta\left(F_{S(x_n),S(p)}\left(\varphi^{n_0-1}(t)\right), F_{S(y_n),S(q)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),S(p)}\left(\varphi^{n_0-1}(t)\right) F_{S(y_n),S(q)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2}. \end{aligned} \tag{4}$$

Letting $n \to \infty$ in (4), we have $F_{S(p),A(p,q)}(t) = 1$, that is, $A(p,q) = S(p) = x$. Similarly, $S(q) = A(q,p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x,y) = S(x)$ and $A(y,x) = S(y)$. Hence A and S have a coupled coincidence point.

Step 2. To show that $S(x) = y, S(y) = x$.

In fact, from (2) we have

$$\begin{aligned} F_{S(x_n),S(y)}(\varphi(t)) = F_{A(x_n,y_n),A(y,x)}(\varphi(t)) &\geq \left[\Delta\left(F_{S(x_n),S(y)}(t), F_{S(y_n),S(x)}(t)\right)\right]^{1/2} \\ &\geq \left[F_{S(x_n),S(y)}(t) F_{S(y_n),S(x)}(t)\right]^{1/2}. \end{aligned} \tag{5}$$

Similarly, we have

$$F_{S(x),S(y_n)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t) F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{6}$$

Suppose that $Q_n(t) = F_{S(x_n),S(y)}(t) F_{S(y_n),S(x)}(t)$. By (5) and (6), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$ and hence,

$$Q_n(\varphi^n(t)) \geq Q_{n-1}\left(\varphi^{n-1}(t)\right) \geq \cdots \geq Q_0(t). \tag{7}$$

Furthermore, from (5)–(7) it follows that

$$F_{S(x_n),S(y)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \text{ and } F_{S(x),S(y_n)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}. \tag{8}$$

It is evident that $[Q_0(t)]^{1/2} \in D^+$. Since $\lim_{n\to\infty} \varphi^n(t) = 0$, from (8) and Lemma 4 we have

$$\lim_{n\to\infty} S(x_n) = S(y) \text{ and } \lim_{n\to\infty} S(y_n) = S(x).$$

This shows that $S(x) = y$ and $S(y) = x$. Hence, $A(x,y) = y$ and $A(y,x) = x$.

Step 3. Next we shall show that $x = y$.

By (2) we have

$$F_{x,y}(\varphi(t)) = F_{A(y,x),A(x,y)}(\varphi(t)) \geq \left[\Delta\left(F_{S(y),S(x)}(t), F_{S(x),S(y)}(t)\right)\right]^{1/2} \geq F_{x,y}(t). \tag{9}$$

From (9) we have $F_{x,y}(\varphi^n(t)) \geq F_{x,y}(t)$. Using Lemma 4, we have $F_{x,y}(t) = 1$, i.e., $x = y$. The uniqueness of x follows from (2). So, the proof of Theorem 2 is finished. $\square$

Theorem 3. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[F_{Sx,Su}(t)\ F_{Sy,Sv}(t)\right]^{1/2}; \tag{10}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{11}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

From (10) and $\varphi(t) < t$, we obtain

$$\begin{aligned} F_{S(x_n),A(p,q)}(t) &\geq F_{S(x_n),A(p,q)}(\varphi(t)) = F_{A(x_n,y_n),A(p,q)}(\varphi(t)) \\ &\geq \left[F_{S(x_n),S(p)}(t)\, F_{S(y_n),S(q)}(t)\right]^{1/2}. \end{aligned} \tag{12}$$

Letting $n \to \infty$ in (12), we have $\lim_{n\to\infty} S(x_n) = A(p, q)$. Hence, $S(p) = A(p, q) = x$. Similarly, we can show that $S(q) = A(q, p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x, y) = S(x)$, $A(y, x) = S(y)$.

Step 2. To show that $S(x) = y, S(y) = x$.

In fact, from (10) we have

$$F_{S(x_n),S(y)}(\varphi(t)) = F_{A(x_n,y_n),A(y,x)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{13}$$

Similarly, we have

$$F_{S(x),S(y_n)}(\varphi(t)) \geq \left[F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)\right]^{1/2}. \tag{14}$$

Suppose that $Q_n(t) = F_{S(x_n),S(y)}(t)\, F_{S(y_n),S(x)}(t)$. By (13) and (14), we have

$$\begin{aligned} &Q_n(\varphi^n(t)) \geq Q_{n-1}\left(\varphi^{n-1}(t)\right) \geq \cdots \geq Q_0(t); \\ &F_{S(x_n),S(y)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \quad \text{and} \quad F_{S(x),S(y_n)}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}. \end{aligned}$$

Since $[Q_0(t)]^{1/2} \in D^+$ and $\lim_{n\to\infty} \varphi^n(t) = 0$, by Lemma 4 we conclude that

$$\lim_{n\to\infty} S(x_n) = S(y) \quad \text{and} \quad \lim_{n\to\infty} S(y_n) = S(x).$$

This shows that $S(x) = y$ and $S(y) = x$. Hence, $A(x, y) = y$ and $A(y, x) = x$.

Step 3. Finally, we prove that $x = y$.

By (10) we have

$$F_{x,y}(\varphi(t)) = F_{A(y,x),A(x,y)}(\varphi(t)) \geq \left[F_{S(y),S(x)}(t)\, F_{S(x),S(y)}(t)\right]^{1/2} \geq F_{x,y}(t). \tag{15}$$

From (15), we have $F_{x,y}(\varphi^n(t)) \geq F_{x,y}(t)$. Using Lemma 4, we have $F_{x,y}(t) = 1$, i.e., $x = y$. The uniqueness of x follows from (10). □

Theorem 4. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}, \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(t) \geq \min\left\{F_{Sx,Su}(\varphi(t)), F_{Sy,Sv}(\varphi(t))\right\}; \tag{16}$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies CLR_S property.

Then A and S have a coupled coincidence point in X. Moreover, there exists a unique point $x \in X$ such that $x = A(x, x) = S(x)$.

Proof. Since A and S satisfy CLR_S property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = S(p), \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = S(q) \tag{17}$$

for some $x, y \in X$.

Step 1. We show that A and S have a coupled coincidence point.

From (16) and (17) it follows that

$$F_{S(x_n),A(p,q)}(t) = F_{A(x_n,y_n),A(p,q)}(t) \geq \min\left\{F_{S(x_n),S(p)}(\varphi(t)), F_{S(y_n),S(q)}(\varphi(t))\right\}. \tag{18}$$

Letting $n \to \infty$ in (18), we have $\lim_{n\to\infty} S(x_n) = A(p, q)$. Hence, $S(p) = A(p, q) = x$. Similarly, we can show that $S(q) = A(q, p) = y$.

Since the pair (A, S) is weakly compatible, it follows that $A(x, y) = S(x)$, $A(y, x) = S(y)$.

Step 2. We claim that $S(x) = y, S(y) = x$.

In fact, from (16) we have

$$F_{S(x_n),S(y)}(t) = F_{A(x_n,y_n),A(y,x)}(t) \geq \min\left\{F_{S(x_n),S(y)}(\varphi(t)), F_{S(y_n),S(x)}(\varphi(t))\right\}. \tag{19}$$

Similarly, we have

$$F_{S(x),S(y_n)}(t) \geq \min\left\{F_{S(x_n),S(y)}(\varphi(t)), F_{S(y_n),S(x)}(\varphi(t))\right\}. \tag{20}$$

Suppose that $M_n(t) = \min\left\{F_{S(x_n),S(y)}(t), F_{S(y_n),S(x)}(t)\right\}$. From (19) and (20) it follows that

$$M_n(t) \geq M_{n-1}(\varphi(t)) \geq \cdots \geq M_0(\varphi^n(t)).$$

Since $\lim_{n\to\infty} \varphi^n(t) = +\infty$, we have

$$M_0(\varphi^n(t)) = \min\left\{F_{S(x_0),S(y)}(t), F_{S(y_0),S(x)}(t)\right\} \to 1 \text{ as } n \to \infty.$$

This shows that $M_n(t) \to 1$ as $n \to \infty$, so

$$\lim_{n\to\infty} S(x_n) = S(y) \text{ and } \lim_{n\to\infty} S(y_n) = S(x).$$

Hence, $S(x) = y$ and $S(y) = x$.

Step 3. Finally, we prove that $x = y$.

By (16) we have

$$F_{x,y}(t) = F_{A(y,x),A(x,y)}(t) \geq \min\left\{F_{S(y),T(x)}(\varphi(t)), F_{S(x),T(y)}(\varphi(t))\right\} = F_{x,y}(\varphi(t)). \tag{21}$$

From (21), we have $F_{x,y}(t) \geq F_{x,y}(\varphi^n(t))$. Letting $n \to \infty$, we have $F_{x,y}(t) = 1$, i.e., $x = y$. Since the uniqueness of x follows from (16), the proof of Theorem 4 is completed. □

Now we give another generalization of Theorem 1.

Corollary 1. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$ and let $A : X \times X \to X, S : X \to X$ be two mappings satisfying the following conditions:*

(1) for all $x, y, u, v \in X$ and $t > 0$

$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Su}(t),\ F_{Sy,Sv}(t)\right)\right]^{1/2};$$

(2) the pair (A, S) is w-compatible;

(3) the pair (A, S) satisfies E.A. property.

If $S(X)$ is a closed subspace of X, then A and S have a unique common fixed point in X.

Proof. Since A and S satisfy E.A. property, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} S(x_n) = x, \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} S(y_n) = y$$

for some $x, y \in X$.

It follows from $S(X)$ being a closed subspace of X that $x = S(p)$, $y = S(q)$ for some $p, q \in X$ and then A and S satisfy CLR_S property. By Theorem 2, we get that A and S have a unique common fixed point in X. □

Corollary 2. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X, S : X \to X$ be two mappings satisfying the conditions of Corollary 1.*

Suppose that $A(X \times X) \subseteq S(X)$, if range of one of the maps A or S is a closed subspace of X, then A and S have a unique common fixed point in X.

Proof. It follows immediately from Corollary 1. □

Taking $S = I_X$ in Theorem 2, we obtain the following

Corollary 3. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for any $t > 0$. Let $A : X \times X \to X$ be a mapping satisfying the following condition, for all $x, y, u, v \in X$ and $t > 0$:*

(1)
$$F_{A(x,y),A(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{x,u}(t), F_{y,v}(t)\right)\right]^{1/2};$$

(2) there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$$\lim_{n\to\infty} A(x_n, y_n) = \lim_{n\to\infty} x_n = x, \quad \lim_{n\to\infty} A(y_n, x_n) = \lim_{n\to\infty} y_n = y$$
for some $x, y \in X$.

Then there exists a unique $z \in X$ such that $z = A(z, z)$.

Now, we prove Theorem 2, Theorem 3, Theorem 4 for four mappings satisfying CLR_{ST} property before proving our main theorems, we begin with the following observation.

Lemma 5. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the following conditions:*

(1) the pair (A, S) satisfies the CLR_S property (or the pair (B, T) satisfies the CLR_T property);

(2) $A(X \times X) \subseteq T(X)$ (or $B(X \times X) \subseteq S(X)$);

(3) $T(X)$ (or $S(X)$) is complete subspace of X;

(4) $B(x_n', y_n')$ converges for every sequences $\{x_n'\}$ and $\{y_n'\}$ in X whenever $T(x_n'), T(y_n')$ converges (or $A(x_n, y_n)$ converges for every sequences $\{x_n\}$ and $\{y_n\}$ in X whenever $S(x_n), S(y_n)$ converges);

(5) for all $x, y, u, v \in X$ and $t > 0$
$$F_{A(x,y),B(u,v)}(\varphi(t)) \geq \left[\Delta\left(F_{Sx,Tu}(t), F_{Sy,Tv}(t)\right)\right]^{1/2}. \tag{22}$$

Then (A, S) and (B, T) share the CLR_{ST} property.

Proof. Suppose the pair (A, S) satisfies the CLR_S property, then there exist $\{x_n\}$ and $\{y_n\}$ in X such that

$$\begin{aligned}
\lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} S(x_n) = a \in S(X), \\
\lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} S(y_n) = b \in S(X).
\end{aligned}$$

Since $A(X\times X)\subseteq T(X)$ (wherein $T(X)$ is complete), for each $\{x_n\}$, $\{y_n\}$ in X there correspond sequences $\{\acute{x}_n\}$ and $\{\acute{y}_n\}$ in X such that

$$A(x_n,y_n)=T\left(\acute{x}_n\right) \text{ and } A(y_n,x_n)=T\left(\acute{y}_n\right).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} A(x_n,y_n) &= \lim_{n\to\infty} T\left(\acute{x}_n\right)=a,\\ \lim_{n\to\infty} A(y_n,x_n) &= \lim_{n\to\infty} T\left(\acute{y}_n\right)=b,\end{aligned}$$

where $a,b\in S(X)\cap T(X)$. Now, we prove that $B\left(\acute{x}_n,\acute{y}_n\right)\to a$ and $B\left(\acute{y}_n,\acute{x}_n\right)\to b$.

Since $\sum_{n=1}^{\infty}\varphi^n(t)<+\infty$, we have $\lim_{n\to\infty}\varphi^n(t)=0$, and so there exists $n_0\in\mathbb{Z}^+$ such that $\varphi^{n_0}(t)<t$. Thus, from (22) we have

$$\begin{aligned}F_{A(x_n,y_n),B\left(\acute{x}_n,\acute{y}_n\right)}(t) &\geq F_{A(x_n,y_n),B\left(\acute{x}_n,\acute{y}_n\right)}\left(\varphi^{n_0}(t)\right)\\ &\geq \left[\Delta\left(F_{S(x_n),T\left(\acute{x}_n\right)}\left(\varphi^{n_0+1}(t)\right),F_{S(y_n),T\left(\acute{y}_n\right)}\left(\varphi^{n_0+1}(t)\right)\right)\right]^{1/2} \qquad (23)\\ &\geq \left[F_{S(x_n),T\left(\acute{x}_n\right)}\left(\varphi^{n_0+1}(t)\right)F_{S(y_n),T\left(\acute{y}_n\right)}\left(\varphi^{n_0+1}(t)\right)\right]^{1/2}.\end{aligned}$$

Letting $n\to\infty$ in (23), we get $\lim_{n\to\infty} B\left(\acute{x}_n,y_n^2\right)=a$. Similarly, we can show $\lim_{n\to\infty} B\left(\acute{y}_n,\acute{x}_n\right)=b$.

Thus, the pairs (A,S) and (B,T) share the CLR_{ST} property. □

Theorem 5. *Let (X,F,Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta\geq\Delta_p$. Let $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\})=\{0\}$ and $\sum_{n=1}^{\infty}\varphi^n(t)<+\infty$ for any $t>0$. Let $A:X\times X\to X$, $B:X\times X\to X$, $T:X\to X$ and $S:X\to X$ be four mappings satisfying the inequality (22) of Lemma 5.*

If the pairs (A,S) and (B,T) share the CLR_{ST} property, then (A,S) and (B,T) have a coincidence point each. Moreover A,B,S and T have a unique common fixed point if both the pairs (A,S) and (B,T) are weakly compatible.

Proof. Since both the pairs (A,S) and (B,T) share the CLR_{ST} property, there exist four sequences $\{x_n\},\{y_n\},\{\acute{x}_n\}$ and $\{\acute{y}_n\}$ in X such that:

$$\begin{aligned}\lim_{n\to\infty} A(x_n,y_n)=\lim_{n\to\infty} S(x_n)=\lim_{n\to\infty} T\left(\acute{x}_n\right)=\lim_{n\to\infty} B\left(\acute{x}_n,\acute{y}_n\right)=a,\\ \lim_{n\to\infty} A(y_n,x_n)=\lim_{n\to\infty} S(y_n)=\lim_{n\to\infty} T\left(\acute{y}_n\right)=\lim_{n\to\infty} B\left(\acute{y}_n,\acute{x}_n\right)=b,\end{aligned} \qquad (24)$$

where $a\in S(X)\cap T(X)$ and $b\in S(X)\cap T(X)$. It implies that there exist points $r,s,p,q\in X$ such that

$$S(r)=a,\ S(s)=b,\ T(p)=a \text{ and } T(q)=b.$$

Step 1. We show that $B(p,q)=T(p)$ and $B(q,p)=T(q)$. Since $\sum_{n=1}^{\infty}\varphi^n(t)<+\infty$, we have $\lim_{n\to\infty}\varphi^n(t)=0$ and so there exists $n_0\in\mathbb{Z}^+$ such that $\varphi^{n_0}(t)<t$. Thus, from (22) we have

$$\begin{aligned}F_{T\left(\acute{x}_n\right),B(p,q)}(t) &\geq F_{T\left(\acute{x}_n\right),B(p,q)}\left(\varphi^{n_0}(t)\right)=F_{A(x_n,y_n),B(p,q)}\left(\varphi^{n_0}(t)\right)\\ &\geq \left[\Delta\left(F_{S(x_n),T(p)}\left(\varphi^{n_0-1}(t)\right),F_{S(y_n),T(q)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2} \qquad (25)\\ &\geq \left[F_{S(x_n),T(p)}\left(\varphi^{n_0-1}(t)\right)F_{S(y_n),T(q)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2}.\end{aligned}$$

Letting $n \to \infty$ in (25), we have $\lim_{n\to\infty} T\left(\acute{x_n}\right) = B(p,q)$. By (24), $T(p) = B(p,q) = a$. Similarly, we can show that $T(q) = B(q,p) = b$.

Since the pair (B,T) is weakly compatible, so $T(p) = B(p,q) = a$ implies $T(a) = B(a,b)$, similarly $T(b) = B(b,a)$.

Now, we show that: $S(r) = A(r,s)$ and $S(s) = A(s,r)$.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n\to\infty} \varphi^n(t) = 0$ and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (22) we get

$$\begin{aligned} F_{A(r,s),S(x_n)}(t) &\geq F_{A(r,s),S(x_n)}\left(\varphi^{n_0}(t)\right) = F_{A(r,s),B\left(\acute{x_n},\acute{y_n}\right)}\left(\varphi^{n_0}(t)\right) \\ &\geq \left[\Delta\left(F_{S(r),T\left(\acute{x_n}\right)}\left(\varphi^{n_0-1}(t)\right), F_{S(s),T\left(\acute{y_n}\right)}\left(\varphi^{n_0-1}(t)\right)\right)\right]^{1/2} \\ &\geq \left[F_{S(r),T\left(\acute{x_n}\right)}\left(\varphi^{n_0-1}(t)\right) F_{S(s),T\left(\acute{y_n}\right)}\left(\varphi^{n_0-1}(t)\right)\right]^{1/2}. \end{aligned} \tag{26}$$

Letting $n \to \infty$ in (26), we have $\lim_{n\to\infty} S(x_n) = A(r,s)$. By (24), $S(r) = A(r,s) = a$. Similarly, we can show that $S(s) = A(s,r) = b$.

Since the pair (A,S) is weakly compatible, it follows that $A(a,b) = S(a)$, $A(b,a) = S(b)$.

Step 2. We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$.

In fact, from (22) we have

$$\begin{aligned} F_{T\left(\acute{y_n}\right),Ta}(\varphi(t)) = F_{A(y_n,x_n),B(a,b)}(\varphi(t)) &\geq \left[\Delta\left(F_{S(y_n),T(a)}(t), F_{S(x_n),T(b)}(t)\right)\right]^{1/2} \\ &\geq \left[F_{S(y_n),T(a)}(t)\, F_{S(x_n),T(b)}(t)\right]^{1/2}. \end{aligned} \tag{27}$$

Similarly, we have

$$F_{T\left(\acute{x_n}\right),Tb}(\varphi(t)) \geq \left[F_{S(x_n),T(b)}(t)\, F_{S(y_n),Ta}(t)\right]^{1/2}. \tag{28}$$

Suppose that $Q_n(t) = F_{S(x_n),T(b)}(t) F_{S(y_n),Ta}(t)$. By (27) and (28), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$, hence

$$Q_n(\varphi^n(t)) \geq Q_{n-1}\left(\varphi^{n-1}(t)\right) \geq \cdots \geq Q_0(t). \tag{29}$$

Furthermore, from (27)–(29) it follows that

$$F_{T\left(\acute{y_n}\right),Ta}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \quad \text{and} \quad F_{T\left(\acute{x_n}\right),Tb}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}. \tag{30}$$

It is evident that $[Q_0(t)]^{1/2} \in D^+$. Since $\lim_{n\to\infty} \varphi^n(t) = 0$, from (30) and Lemma 4 we have

$$\lim_{n\to\infty} T\left(\acute{y_n}\right) = Ta \;\text{ and }\; \lim_{n\to\infty} T\left(\acute{x_n}\right) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence $B(a,b) = b$ and $B(b,a) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence $A(a,b) = b$ and $A(b,a) = a$.

Step 3. Now we prove that $a = b$.

By (22) we have

$$F_{a,b}(\varphi(t)) = F_{A(b,a),B(a,b)}(\varphi(t)) \geq \left[\Delta\left(F_{S(b),T(a)}(t), F_{S(a),T(b)}(t)\right)\right]^{1/2} \geq F_{a,b}(t). \tag{31}$$

From (31), we have $F_{a,b}(\varphi^n(t)) \geq F_{a,b}(t)$. Using Lemma 4, we obtain $F_{a,b}(t) = 1$, i.e., $a = b$. The uniqueness of a follows from (22). So, the proof of Theorem 5 is finished. $\square$

Theorem 6. *Let (X, F, Δ) be a Menger metric space with Δ is a t-norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ be four mappings satisfying the condition (1)–(5) of Lemma 1.*

Then A, B, S and T have a unique common fixed point if both the pairs (A, S) and (B, T) are w-compatible.

Proof. In view of Lemma 5, both the pairs (A, S) and (B, T) enjoy the CLR_{ST} property, therefore there exist two sequences $\{x_n\}, \{y_n\}, \{x_n'\}$ and $\{y_n'\}$ in X such that:

$$\begin{aligned}
\lim_{n\to\infty} A(x_n, y_n) &= \lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T\left(x_n'\right) = \lim_{n\to\infty} B\left(x_n', y_n'\right) = a, \\
\lim_{n\to\infty} A(y_n, x_n) &= \lim_{n\to\infty} S(y_n) = \lim_{n\to\infty} T\left(y_n'\right) = \lim_{n\to\infty} B\left(y_n', x_n'\right) = b,
\end{aligned}$$

where $a \in S(X) \cap T(X)$ and $b \in S(X) \cap T(X)$.

The rest of the proof runs on the lines of the proof of Theorem 5. □

Similarly, we can prove Theorem 3 and Theorem 4 for four mappings using CLR_{ST} property.

Now, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results.

Example 1. *Let $X = \left[0, \frac{1}{2}\right) \cup \{1\}$ and $F_{x,y}(t) = \frac{t}{t+|x-y|}$ for all $x, y \in X$ and $t > 0$. Then (X, F, Δ) is a Menger metric space, but it is not complete.*

Obviously (X, F, Δ) is not complete. Define the mappings $A : X \times X \to X$, $B : X \times X \to X$, $T : X \to X$ and $S : X \to X$ by

$$A(x, y) = \begin{cases} 0 & \text{if } (x, y) = (1, 1), \\ \frac{x^2+y^2}{6} & \text{if } (x, y) \neq (1, 1), \end{cases}$$

$$B(x, y) = \begin{cases} \frac{1}{2} & \text{if } (x, y) = (1, 1), \\ \frac{x+y}{2} & \text{if } (x, y) \neq (1, 1), \end{cases}$$

$$S(x) = \begin{cases} \frac{1}{12} & \text{if } x = 1, \\ \frac{x^2}{3} & \text{if } x \neq 1, \end{cases}$$

$$T(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1, \\ x & \text{if } x \neq 1. \end{cases}$$

It is noted that $A(X \times X) = \left[0, \frac{1}{12}\right) \nsubseteq T(X) = \left[0, \frac{1}{2}\right]$, $B(X \times X) = \left[0, \frac{1}{2}\right] \nsubseteq S(X) = \left[0, \frac{1}{12}\right]$ and $T(X)$ and $S(X)$ are complete.

Next, we show that our results can be used for this case.

Let us prove that A, B, S and T satisfy the CLR_{ST} property. Consider the sequences $\{x_n\}$, $\{y_n\}$, $\{x_n'\}$ and $\{y_n'\}$ in X which are defined by

$$x_n = \frac{1}{2n},\ y_n = \frac{1}{3n},\ x_n' = \frac{1}{4n} \text{ and } y_n' = \frac{1}{5n},\ n = 1, 2, 3, \ldots$$

Since

$$\begin{aligned}\lim_{n\to\infty} A\left(x_n, y_n\right) &= \lim_{n\to\infty} S\left(x_n\right) = \lim_{n\to\infty} T\left(x_n'\right) = \lim_{n\to\infty} B\left(x_n', y_n'\right) = 0 \in S\left(X\right) \cap T\left(X\right),\\ \lim_{n\to\infty} A\left(y_n, x_n\right) &= \lim_{n\to\infty} S\left(y_n\right) = \lim_{n\to\infty} T\left(y_n'\right) = \lim_{n\to\infty} B\left(y_n', x_n'\right) = 0 \in S\left(X\right) \cap T\left(X\right).\end{aligned}$$

Thus A, B, S and T satisfy the CLR_{ST} property with these sequences.

Next, we will show that the pairs (A, S) and (B, T) are w-compatible.

It is obtained that

1. $A(x,y) = S(x)$ and $A(y,x) = S(y)$ if and only if $x = y = 0$, since $A(S(0), S(0)) = S(A(0,0))$, mappings A and S are w-compatible, and

2. $B(x,y) = T(x)$ and $B(y,x) = T(y)$ if and only if $x = y = 0$, since $B(T(0), T(0)) = T(B(0,0))$, mappings B and T are w-compatible.

Finally, we prove that for $x, y, u, v \in X$,

$$F_{A(x,y),B(u,v)}\left(\varphi\left(t\right)\right) \geq \left[\Delta\left(F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right)\right]^{1/2}.$$

Let $\varphi : (0, \infty) \to (0, \infty)$ by $\varphi\left(t\right) = \frac{1}{2}t$. Then $\lim\limits_{n\to+\infty} \varphi^n\left(t\right) = 0$ for any $t > 0$. For $x, y, u, v \in X$, we distinguish the following cases.

Case 1. *$(x,y) \neq (1,1)$ and $(u,v) \neq (1,1)$. In this case we have*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x^2+y^2}{6} - \frac{u+v}{2}\right|} = \frac{t}{t + \left|\left(\frac{x^2}{3} - u\right) + \left(\frac{y^2}{3} - v\right)\right|}\\ &\geq \frac{t}{t + \left|\frac{x^2}{3} - u\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 2. *$(x,y) \neq (1,1)$ and $(u,v) = (1,1)$.*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x^2+y^2}{6} - \frac{1}{2}\right|} = \frac{t}{t + \left|\frac{x^2+y^2}{3} - 1\right|}\\ &\geq \frac{t}{t + \left|\frac{x^2}{3} - \frac{1}{2}\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 3. *$(x,y) = (1,1)$ and $(u,v) \neq (1,1)$.*

$$\begin{aligned}F_{A(x,y),B(u,v)}\left(kt\right) &= \frac{\frac{t}{2}}{\frac{t}{2} + \left|\frac{x+y}{2}\right|} = \frac{t}{t + |x+y|}\\ &\geq \frac{t}{t + \left|x - \frac{1}{12}\right|} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.\end{aligned}$$

Case 4. *$(x,y) = (1,1)$ and $(u,v) = (1,1)$.*

$$F_{A(x,y),B(u,v)}\left(kt\right) = \frac{\frac{t}{2}}{\frac{t}{2} + \frac{1}{2}} = \frac{t}{t + \frac{1}{2}} \geq \min\left\{F_{Sx,Tu}\left(t\right), F_{Sy,Tv}\left(t\right)\right\}.$$

Hence, all the hypotheses of Theorem 5 hold. Clearly $(0,0)$ is the unique common coupled fixed point of A, B, S and T.

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Received 19.08.2016

Revised 01.12.2016

Бен Аоуа Л., Аліуче А. *Теореми про зчеплену нерухому точку для слабко сумісних відображень у сукупності з CLR властивістю в метричних просторах Менґера* // Карпатські матем. публ. — 2016. — Т.8, №2. — С. 195–210.

Проблеми зв'язної нерухомої точки привертають значну увагу в теперішній час. Мета цієї статті полягає у розширенні понять Е.А. властивості, *CLR* властивості та *JCLR* властивості для зв'язних відображень в метричному просторі Менґера і використанні цих понять для дослідження загальних результатів про зв'язну нерухому точку для чотирьох власних відображень. Наша робота узагальнює результати Цян-Хжонг Ксяо [24] та ін. Основний результат наведено з використанням відповідного прикладу.

Ключові слова і фрази: метричний простір Менґера, t-норма типу *H*, слабка відповідність зв'язної нерухомої точки, *CLR* властивість, Е.А. властивість, *JCLR* властивість.