ISSN 2075-9827 **e-ISSN 2313-0210**
Carpathian Math. Publ. 2019, **11** (2), 321–334
doi:10.15330/cmp.11.2.321-334

http://www.journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2019, Т.11, №2, С.321–334

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APPROXIMATION OF THE CLASSES $W^r_{\beta,\infty}$ BY THREE-HARMONIC POISSON INTEGRALS

In the paper, we solve one extremal problem of the theory of approximation of functional classes by linear methods. Namely, questions are investigated concerning the approximation of classes of differentiable functions by λ-methods of summation for their Fourier series, that are defined by the set $\Lambda = \{\lambda_\delta(\cdot)\}$ of continuous on $[0, \infty)$ functions depending on a real parameter δ. The Kolmogorov-Nikol'skii problem is considered, that is one of the special problems among the extremal problems of the theory of approximation. That is, the problem of finding of asymptotic equalities for the quantity $\mathcal{E}(\mathfrak{N}; U_\delta)_X = \sup\limits_{f\in\mathfrak{N}} \|f(\cdot) - U_\delta(f;\cdot;\Lambda)\|_X$, where X is a normalized space, $\mathfrak{N} \subseteq X$ is a given function class, $U_\delta(f;x;\Lambda)$ is a specific method of summation of the Fourier series. In particular, in the paper we investigate approximative properties of the three-harmonic Poisson integrals on the Weyl-Nagy classes. The asymptotic formulas are obtained for the upper bounds of deviations of the three-harmonic Poisson integrals from functions from the classes $W^r_{\beta,\infty}$. These formulas provide a solution of the corresponding Kolmogorov-Nikol'skii problem. Methods of investigation for such extremal problems of the theory of approximation arised and got their development owing to the papers of A.N. Kolmogorov, S.M. Nikol'skii, S.B. Stechkin, N.P. Korneichuk, V.K. Dzyadyk, A.I. Stepanets and others. But these methods are used for the approximations by linear methods defined by triangular matrices. In this paper we modified the mentioned above methods in order to use them while dealing with the summation methods defined by a set of functions of a natural argument.

Key words and phrases: Kolmogorov-Nikol'skii problem, three-harmonic Poisson integral, Weyl-Nagy classes.

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1 INTRODUCTION

Let L be a space of 2π-periodic summable on a period functions f equipped with the norm $\|f\|_L = \int\limits_{-\pi}^{\pi} |f(t)|dt$; C be a space of 2π-periodic continuous functions f in which the norm is set by means of the equality $\|f\|_C = \max\limits_t |f(t)|$; L_∞ be a space of 2π-periodic measurable essentially bounded functions f with the norm $\|f\|_\infty = \operatorname{ess\,sup}\limits_t |f(t)|$.

Assume that $f \in L$ and $S[f] = \frac{a_0}{2} + \sum\limits_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$ is the corresponding Fourier series. Let, further, $r > 0$ and $\beta \in \mathbb{R}$. If the series

$$\sum_{k=1}^{\infty} k^r \left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

УДК 517.5
2010 *Mathematics Subject Classification:* 42A05, 41A60.

is the Fourier series of a summable function φ, then we call the function φ a (r,β)-derivative of f in the Weyl–Nagy sense and denote it by f^r_β (see, e.g., [14], p. 130). A set of functions for which this condition is satisfied is denoted by W^r_β. If $f \in W^r_\beta$ and, besides, $\|f^r_\beta(\cdot)\|_\infty \leq 1$, then f belongs to the class $W^r_{\beta,\infty}$.

Let $f \in L$, $\delta > 0$. Functions of the following form

$$P_1(\delta;f;x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} e^{-\frac{k}{\delta}} \left(a_k \cos kx + b_k \sin kx\right),$$

$$P_2(\delta;f;x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(1 + \frac{k}{2}(1 - e^{-\frac{2}{\delta}})\Big) e^{-\frac{k}{\delta}} \left(a_k \cos kx + b_k \sin kx\right),$$

$$P_3(\delta;f;x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \Big(1 + \frac{1}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}})k + \frac{1}{8}(1 - e^{-\frac{2}{\delta}})^2 k^2\Big) e^{-\frac{k}{\delta}} \big(a_k \cos kx + b_k \sin kx\big),$$

are called the Poisson integral [10], the biharmonic Poisson integral [16] and the three-harmonic Poisson integral [2] of the function f, respectively.

The paper is devoted to investigation of asymptotic behavior as $\delta \to \infty$ of the quantity

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \sup_{f \in W^r_{\beta,\infty}} \|f(\cdot) - P_3(\delta;f;\cdot)\|_C. \quad (1)$$

If the function $\varphi(\delta)$ is found in an explicit form, such that $\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \varphi(\delta) + o(\varphi(\delta))$ as $\delta \to \infty$, then according to Stepanets [14, p. 198] we say that the Kolmogorov-Nikol'skii problem is solved for the class $W^r_{\beta,\infty}$ and the three-harmonic Poisson integral in the uniform metric.

The Kolmogorov-Nikol'skii problem for the Poisson integral on classes of differentiable functions have been solved in [7,9,12,15,18,19]. The papers [5,11,20] are devoted to an investigation of analogous problem for the biharmonic Poisson integral. Asymptotic properties of the three-harmonic Poisson integrals were considered in [2], [17]. Nevertheless, the Kolmogorov-Nikol'skii problem have not been solved for the three-harmonic Poisson integral on the classes $W^r_{\beta,\infty}$. Therefore a question arose of finding asymptotic equalities for the quantities (1).

2 ASYMPTOTIC EQUALITIES FOR UPPER BOUNDS OF DEVIATIONS OF THREE-HARMONIC POISSON INTEGRALS FROM FUNCTIONS FROM THE CLASS $W^r_{\beta,\infty}$.

For the three-harmonic Poisson integral, analogous to the relation (6) from [8], let us rewrite a sum function $\tau(u)$ in the following form

$$\tau(u) = \begin{cases} \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) \delta^r, & 0 \leq u \leq \frac{1}{\delta}, \\ \left(1 - \left(1 + \gamma u + \theta u^2\right) e^{-u}\right) u^{-r}, & u \geq \frac{1}{\delta}, \end{cases} \quad (2)$$

where $\gamma = \gamma(\delta) = \frac{1}{4}(3 - e^{-\frac{2}{\delta}})(1 - e^{-\frac{2}{\delta}})\delta$, $\theta = \theta(\delta) = \frac{1}{8}(1 - e^{-\frac{2}{\delta}})^2\delta^2$, $\delta > 0$.

The following statement is true.

Theorem 1. *Let $0 < r \leq 3$. Then the asymptotic equality holds as $\delta \to \infty$*

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \frac{1}{\delta^r} A(\tau) + O\left(\frac{1}{\delta^3} + \frac{1}{\delta^{r+1}}\right),$$

where the quantity $A(\tau)$ is defined by

$$A(\tau) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\Big|\int\limits_{0}^{\infty}\tau(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du\Big|dt \tag{3}$$

and the estimate

$$A(\tau) = \begin{cases} O(1), & 0 < r < 3, \\ O(\ln\delta), & r = 3, \end{cases} \tag{4}$$

takes place.

Proof. To conduct the proof let us use theorem A from [1]. We now check if its conditions are fulfilled. For that reason let us show a summability of the Fourier transform $\widehat{\tau}_\beta(t)$ of function $\tau(u)$ of the form

$$\widehat{\tau}_\beta(t) = \frac{1}{\pi}\int\limits_{0}^{\infty}\tau(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du, \tag{5}$$

i.e., a convergence of integral $A(\tau)$ of the form (3). According to theorem 1 from [1], for proving a convergence of the integral (3) it is necessary and sufficient to show that the following integrals are convergent

$$\int\limits_{0}^{\frac12} u\,|d\tau'(u)|,\quad \int\limits_{\frac12}^{\infty}|u-1|\,|d\tau'(u)|,\quad \int\limits_{0}^{\infty}\frac{|\tau(u)|}{u}du,\quad \int\limits_{0}^{1}\frac{|\tau(1-u)-\tau(1+u)|}{u}du. \tag{6}$$

As while investigating the first integral of (10) from [6] let us estimate the first integral of (6) on each segment $\left[0,\frac{1}{\delta}\right]$ and $\left[\frac{1}{\delta},\frac12\right]$ (assume, that $\delta > 3$). Taking into account that $\tau''(u) \ge 0$ if $u \in \left[0,\frac{1}{\delta}\right]$, $\delta > 3$, and the inequalities

$$e^{-u} \le 1,\quad e^{-u} \le 1-u+\frac{u^2}{2},\quad u \ge 0, \tag{7}$$

we get

$$\int\limits_{0}^{\frac{1}{\delta}} u\,|d\tau'(u)| = \big(u\tau'(u)-\tau(u)\big)\Big|_0^{\frac{1}{\delta}} \le \delta^r\left(\frac{1}{\delta^2}\Big(\frac12-\theta\Big)+\frac{1}{\delta^3}\Big(\frac{\gamma}{2}+\theta\Big)\right).$$

In view of estimates $\frac12-\theta \le \frac{1}{\delta}$, $\frac{\gamma}{2}+\theta \le \frac32$, we obtain

$$\int\limits_{0}^{\frac{1}{\delta}} u\,|d\tau'(u)| = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta\to\infty. \tag{8}$$

Let further $u \in \left[\frac{1}{\delta},\frac12\right]$. We set

$$\begin{aligned} \tau_1(u) &= \left(1-(1+\gamma u+\theta u^2)e^{-u}-\frac{4}{3\delta^2}u-\frac{1}{\delta}u^2-\frac16 u^3\right)u^{-r}, \\ \tau_2(u) &= \frac{4}{3\delta^2}u^{1-r}+\frac{1}{\delta}u^{2-r}+\frac16 u^{3-r}, \end{aligned} \tag{9}$$

then $\tau(u) = \tau_1(u) + \tau_2(u)$ and

$$\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau'(u)\right| \le \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau_1'(u)\right| + \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u\left|d\tau_2'(u)\right|. \tag{10}$$

To estimate the first integral from the right-hand side of inequality (10), we first investigate the following function

$$\widetilde{\mu}(u) = 1 - (1+\gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3. \tag{11}$$

Taking into account, that

$$\widetilde{\mu}'(u) = (1+\gamma+\theta u^2)e^{-u} - (\gamma+2\theta u)e^{-u} - \frac{4}{3\delta^2} - \frac{2}{\delta}u - \frac{1}{2}u^2,$$

$$\widetilde{\mu}''(u) = -(1+\gamma+\theta u^2)e^{-u} - 2(\gamma+2\theta u)e^{-u} - 2\theta e^{-u} - \frac{2}{\delta} - u,$$

$$\widetilde{\mu}(0) = 0,\ \widetilde{\mu}'(0) = 1 - \gamma - \frac{4}{3\delta^2} < 0,$$

we can show that if $u \ge 0$, then

$$\widetilde{\mu}(u) \le 0,\ \ \widetilde{\mu}'(u) < 0,\ \ \widetilde{\mu}''(u) < 0. \tag{12}$$

In view of (12) and the inequalities (7) and

$$e^{-u} \le 1 - u + \frac{u^2}{2} - \frac{u^3}{6} + \frac{u^4}{24},\quad e^{-u} \ge 1 - u + \frac{u^2}{2} - \frac{u^3}{6},\quad e^{-u} \ge 1-u,\quad u \ge 0,$$

we have

$$|\widetilde{\mu}(u)| \le u\Big(\gamma - 1 + \frac{4}{3\delta^2}\Big) + u^2\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + u^3\Big(\frac{\gamma}{2} - \theta\Big) + u^4\Big(\frac{1}{24} + \frac{\theta}{2}\Big),$$

$$\left|\widetilde{\mu}'(u)\right| \le \Big(\gamma - 1 + \frac{4}{3\delta^2}\Big) + 2u\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + 3u^2\Big(\frac{\gamma}{2} - \theta\Big) + u^3\Big(\frac{1}{6} + 2\theta\Big),$$

$$\left|\widetilde{\mu}''(u)\right| \le 2\Big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\Big) + 6u\Big(\frac{\gamma}{2} - \theta\Big) + u^2\Big(\frac{1}{2} + 6\theta\Big).$$

Further, using the estimates

$$\gamma - 1 + \frac{4}{3\delta^2} \le \frac{3}{\delta^3},\ \frac{1}{2} - \gamma + \theta + \frac{1}{\delta} \le \frac{3}{\delta^2},\ \frac{\gamma}{2} - \theta \le \frac{2}{\delta},\ \frac{1}{24} + \frac{\theta}{2} \le 1,\ \frac{1}{2} + 6\theta \le 3,\ \frac{1}{6} + 2\theta \le 2,$$

we obtain

$$\begin{gathered}|\widetilde{\mu}(u)| \le \frac{3}{\delta^3}u + \frac{3}{\delta^2}u^2 + \frac{2}{\delta}u^3 + u^4,\ \left|\widetilde{\mu}'(u)\right| \le \frac{3}{\delta^3} + \frac{6}{\delta^2}u + \frac{6}{\delta}u^2 + 2u^3,\\ \left|\widetilde{\mu}''(u)\right| \le \frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2.\end{gathered} \tag{13}$$

Taking into account (9), (11) and relation (13), in the case $u \geq \frac{1}{\delta}$ we get

$$\begin{aligned} \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\tau_1'(u)| \leq \int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{6}{\delta^2} + \frac{12}{\delta}u + 3u^2\Big)u^{1-r}du + r\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{6}{\delta^3} + \frac{12}{\delta^2}u + \frac{12}{\delta}u^2 + 4u^3\Big)u^{-r}du \\ + r(r+1)\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} \Big(\frac{3}{\delta^3}u + \frac{3}{\delta^2}u^2 + \frac{2}{\delta}u^3 + u^4\Big)u^{-r-1}du \leq K_1. \end{aligned} \tag{14}$$

One can easily verify that the estimate

$$\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\tau_2'(u)| = O(1) \quad \text{as } \delta \to \infty \tag{15}$$

is true. Combining (14) and (15), we have

$$\int\limits_{0}^{\frac{1}{2}} u|d\tau'(u)| = O(1) \quad \text{as } \delta \to \infty. \tag{16}$$

Now we move to an estimation of the second integral from (6). If $u \geq \frac{1}{\delta}$ from a representation of function $\tau(u)$ of the form (2) we obtain

$$\begin{aligned} \tau''(u) = e^{-u}\big((2\gamma - 2\theta - 1) + u(4\theta - \gamma) - \theta u^2)\big)u^{-r} - 2re^{-u}\big((1-\gamma) + u(\gamma - 2\theta) \\ + \theta u^2\big)u^{-r-1} + r(r+1)(1 - (1 + \gamma u + \theta u^2)e^{-u})u^{-r-2}. \end{aligned} \tag{17}$$

The relaton (17) yields

$$\begin{aligned} \int\limits_{\frac{1}{2}}^{\infty} |u-1||d\tau'(u)| \leq \int\limits_{\frac{1}{2}}^{\infty} u|d\tau'(u)| \leq \int\limits_{\frac{1}{2}}^{\infty} e^{-u}\big((2\gamma - 2\theta - 1) + u(4\theta - \gamma) - \theta u^2)\big)u^{1-r}du \\ + 2r\int\limits_{\frac{1}{2}}^{\infty} e^{-u}\big((1-\gamma) + u(\gamma - 2\theta) + \theta u^2\big)u^{-r}du + r(r+1)\int\limits_{\frac{1}{2}}^{\infty} (1 - (1 + \gamma u + \theta u^2))e^{-u}u^{-r-1}du. \end{aligned} \tag{18}$$

Further, taking into account the following estimates for $u \geq 0$

$$\begin{aligned} &1 - (1 + \gamma u + \theta u^2)e^{-u} \leq 1, \\ &ue^{-u}((1-\gamma) + u(\gamma - 2\theta) + \theta u^2) \leq 2, \\ &(2\gamma - 2\theta - 1) + u(4\theta - \gamma) - \theta u^2 \leq 8, \end{aligned} \tag{19}$$

from (18) we have

$$\int\limits_{\frac{1}{2}}^{\infty} |u-1||d\tau'(u)| = O(1) \quad \text{as } \delta \to \infty. \tag{20}$$

Let us estimate the third integral from (6) on each segment $[0,\frac{1}{\delta}]$, $[\frac{1}{\delta},1]$ and $[1,\infty)$. In view of (2) and the inequality

$$1-e^{-u}-\gamma ue^{-u}-\theta u^2e^{-u}\le\frac{2}{\delta^2}u+\frac{2}{\delta}u^2+u^3,\quad u\ge 0, \tag{21}$$

we get

$$\int\limits_0^{\frac{1}{\delta}}\frac{|\tau(u)|}{u}du=\delta^r\int\limits_0^{\frac{1}{\delta}}(1-e^{-u}-\gamma ue^{-u}-\theta u^2e^{-u})\frac{du}{u}\le\delta^r\int\limits_0^{\frac{1}{\delta}}\left(\frac{2}{\delta^2}+\frac{2}{\delta}u+u^2\right)du\le\frac{K_1}{\delta^{3-r}}. \tag{22}$$

From relations (2), (11), (13) we obtain

$$\left|\int\limits_{\frac{1}{\delta}}^{1}\frac{\tau(u)}{u}du-\frac{4}{3\delta^2}\int\limits_{\frac{1}{\delta}}^{1}u^{-r}du-\frac{1}{\delta}\int\limits_{\frac{1}{\delta}}^{1}u^{1-r}du-\frac{1}{6}\int\limits_{\frac{1}{\delta}}^{1}u^{2-r}du\right|$$

$$\le\int\limits_{\frac{1}{\delta}}^{1}\frac{|\widetilde{\mu}(u)|}{u}du\le\int\limits_{\frac{1}{\delta}}^{1}\left(\frac{3}{\delta^3}+\frac{3}{\delta^2}u+\frac{2}{\delta}u^2+u^3\right)u^{-r-1}du\le\begin{cases}K_1, & r<3,\\ K_2\ln\delta, & r=3.\end{cases}$$

Therefore

$$\int\limits_{\frac{1}{\delta}}^{1}\frac{|\tau(u)|}{u}du=\frac{4}{3\delta^2}\int\limits_{\frac{1}{\delta}}^{1}u^{-r}du+\frac{1}{\delta}\int\limits_{\frac{1}{\delta}}^{1}u^{1-r}du+\frac{1}{6}\int\limits_{\frac{1}{\delta}}^{1}u^{2-r}du+\begin{cases}O(1), & r<3,\\ O(\ln\delta), & r=3,\end{cases}$$
$$=\begin{cases}O(1), & r<3,\\ O(\ln\delta), & r=3,\end{cases}\quad\text{as }\delta\to\infty. \tag{23}$$

Taking into account the formula (2) and the first inequality from (19), we get

$$\int\limits_1^{\infty}\frac{\tau(u)}{u}du=\int\limits_1^{\infty}(1-(1+\gamma u+\theta u^2)e^{-u})u^{-r-1}du\le\int\limits_1^{\infty}u^{-r-1}du=\frac{1}{r}. \tag{24}$$

From (22)–(24) the estimate follows

$$\int\limits_0^{\infty}\frac{|\tau(u)|}{u}du=\begin{cases}O(1), & r<3,\\ O(\ln\delta), & r=3,\end{cases}\quad\text{as }\delta\to\infty. \tag{25}$$

Now we estimate the fourth integral from (6). Similarly as to obtain the formula (39) form [3], we can get the equalities

$$\int\limits_0^1\frac{|\tau(1-u)-\tau(1+u)|}{u}du=\int\limits_0^1\frac{|\lambda(1-u)-\lambda(1+u)|}{u}du+O\left(H(\tau)\right), \tag{26}$$

where $H(\tau)$ is defined by equality

$$H(\tau)=|\tau(0)|+|\tau(1)|+\int\limits_0^{\frac{1}{2}}u\left|d\tau'(u)\right|+\int\limits_{\frac{1}{2}}^{\infty}|u-1|\left|d\tau'(u)\right|, \tag{27}$$

and $\lambda(u) = (1+\gamma u+\theta u^2)e^{-u}$. Taking into account, that $\int\limits_0^1 |\lambda(1-u)-\lambda(1+u)|\frac{du}{u} = O(1)$ and using the estimates (16), (20), we have

$$\int\limits_0^1 \frac{|\tau(1-u)-\tau(1+u)|}{u}du = O(1),\ \delta\to\infty. \tag{28}$$

Therefore, in view of theorem 1 from [1], integral $A(\tau)$ of the form (3) is convergent. Using inequalities (2.14) and (2.15) from [1] and the formulas (16), (20), (25) and (28) we obtain the estimate (4).

Hence, we proved that for the function $\tau(u)$ defined by (2) the conditions of theorem A from [1] are fulfilled. Then, as $\delta\to\infty$, the equality

$$\mathcal{E}(W^r_{\beta,\infty}; P_3(\delta))_C = \frac{1}{\delta^r}A(\tau) + O\Big(\frac{1}{\delta^r}a(\tau)\Big) \tag{29}$$

holds, where

$$a(\tau) = \int\limits_{|t|\geq\frac{\delta\pi}{2}} \big|\widehat{\tau}_\beta(t)\big|dt. \tag{30}$$

Let us estimate the integral (30). First, we represent a transform $\widehat{\tau}_\beta(t)$ in the form

$$\widehat{\tau}_\beta(t) = \frac{1}{\pi}\Big(\int\limits_0^{\frac{1}{\delta}} + \int\limits_{\frac{1}{\delta}}^{\infty}\Big)\tau(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)\,du. \tag{31}$$

Integrating both integrals from the right-hand side of the equality (31) twice by parts and taking into account that $\tau(0)=0$ and $\lim\limits_{u\to\infty}\tau(u) = \lim\limits_{u\to\infty}\tau'(u) = 0$, we have

$$\widehat{\tau}_\beta(t) = -\frac{1}{\pi t^2}\Bigg((1-\gamma)\delta^r\cos\frac{\beta\pi}{2} - r\delta^{r+1}\Big(1-(1+\frac{\gamma}{\delta}+\frac{\theta}{\delta^2})e^{-\frac{1}{\delta}}\Big)\cos\Big(\frac{t}{\delta}+\frac{\beta\pi}{2}\Big)$$
$$+\int\limits_0^{\frac{1}{\delta}}\tau''(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du + \int\limits_{\frac{1}{\delta}}^{\infty}\tau''(u)\cos\Big(ut+\frac{\beta\pi}{2}\Big)du\Bigg).$$

Further, in view of inequalities (21) and $1-\gamma\leq\frac{2}{\delta^2}$, we obtain

$$\big|\widehat{\tau}_\beta(t)\big| \leq \frac{K_1}{t^2\delta^{2-r}} + \frac{1}{\pi t^2}\Big(\int\limits_0^{\frac{1}{\delta}} + \int\limits_{\frac{1}{\delta}}^{1} + \int\limits_1^{\infty}\Big)|\tau''(u)|du. \tag{32}$$

Taking into account that $\tau''(u)\geq 0$ if $u\in[0,\frac{1}{\delta}]$ ($\delta>3$) and using inequalities $\gamma-2\theta\leq\frac{3}{\delta}$, $\theta\leq\frac{1}{2}$, we get

$$\int\limits_0^{\frac{1}{\delta}}|\tau''(u)|du = \delta^r e^{-\frac{1}{\delta}}\Big((1-\gamma)+\frac{\gamma-2\theta}{\delta}+\frac{\theta}{\delta^2}\Big) - \delta^r(1-\gamma) \leq \frac{K_2}{\delta^{2-r}}. \tag{33}$$

Let $u \in [\frac{1}{\delta}, 1]$. Repeating the argumentations used to estimate the first integral from (6) on the segment $[\frac{1}{\delta}, \frac{1}{2}]$, we can easily verify that the estimate

$$\int\limits_{\frac{1}{\delta}}^{1} |\tau''(u)| du = O\left(1 + \frac{1}{\delta^{2-r}}\right), \ \delta \to \infty, \tag{34}$$

holds.

Consider now $u \in [1, \infty)$. Taking into account the relation (17), we get

$$\int\limits_{1}^{\infty} |\tau''(u)| du \leq \int\limits_{1}^{\infty} e^{-u} u^{-r}((2\gamma - 2\theta - 1) + u(4\theta - \gamma) - \theta u^2) du$$

$$+2r \int\limits_{1}^{\infty} e^{-u} u^{-r-1}((1-\gamma) + u(\gamma - 2\theta) + \theta u^2) du$$

$$+r(r+1) \int\limits_{1}^{\infty} (1 - (1 + \gamma u + \theta u^2) e^{-u}) u^{-r-2} du.$$

In view of the first and the third inequalities from (19) and the inequality

$$e^{-u}((1-\gamma) + u(\gamma - 2\theta) + \theta u^2) \leq 2, \ \ u \geq 1,$$

the last relation yields

$$\int\limits_{1}^{\infty} |\tau''(u)| du \leq K_3. \tag{35}$$

Combining formulas (32)–(35), we obtain

$$\left|\widehat{\tau}_{\beta}(t)\right| = O\left(1 + \frac{1}{\delta^{2-r}}\right) \frac{1}{t^2}.$$

Therefore,

$$a(\tau) = \int\limits_{|t| \geq \frac{\delta\pi}{2}} |\widehat{\tau}_{\beta}(t)| dt = O\left(\frac{1}{\delta} + \frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty. \tag{36}$$

From the relations (29) and (36) the equality follows. Theorem 1 is proved.

□

Theorem 2. *If $r > 3$ the following asymptotic equality holds as $\delta \to \infty$*

$$\mathcal{E}\left(W^r_{\beta,\infty}; P_3(\delta)\right)_C = \frac{1}{\delta^3} \sup_{f \in W^r_{\beta,\infty}} \left\| \frac{4}{3} f_0^{(1)}(\cdot) + f_0^{(2)}(\cdot) + \frac{1}{6} f_0^{(3)}(\cdot) \right\|_C + O(\Upsilon(\delta; r)), \tag{37}$$

where $f_0^{(r)}$, $r = 1, 2, 3$, are (r, β)-derivatives in the Weyl-Nagy sense for $\beta = 0$, and

$$\Upsilon(\delta; r) = \begin{cases} \frac{1}{\delta^r}, & 3 < r < 4, \\ \frac{\ln \delta}{\delta^4}, & r = 4, \\ \frac{1}{\delta^4}, & r > 4. \end{cases}$$

Proof. As in the paper [4], let us represent function $\tau(u)$ defined by the relation (2) in the form $\tau(u) = \varphi(u) + \mu(u)$, where

$$\varphi(u) = \begin{cases} (\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3)\delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3\right)u^{-r}, & u \ge \frac{1}{\delta}, \end{cases} \quad (38)$$

$$\mu(u) = \begin{cases} \left(1 - (1+\gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3\right)\delta^r, & 0 \le u \le \frac{1}{\delta}, \\ \left(1 - (1+\gamma u + \theta u^2)e^{-u} - \frac{4}{3\delta^2}u - \frac{1}{\delta}u^2 - \frac{1}{6}u^3\right)u^{-r}, & u \ge \frac{1}{\delta}. \end{cases} \quad (39)$$

Now we show a convergence of the integrals $A(\varphi)$ and $A(\mu)$ of the form (3).

To prove a convergence of the integral $A(\varphi)$, in view of theorem 1 from [1], let us show a convergence of the integrals

$$\int_0^{\frac{1}{2}} u\left|d\varphi'(u)\right|, \quad \int_{\frac{1}{2}}^{\infty} |u-1|\left|d\varphi'(u)\right|, \quad \int_0^{\infty} \frac{|\varphi(u)|}{u}du, \quad \int_0^{1} \frac{|\varphi(1-u)-\varphi(1+u)|}{u}du \quad (40)$$

and find their upper estimates.

From (38) we get that for $u \in \left[0, \frac{1}{\delta}\right], \delta > 2$,

$$\int_0^{\frac{1}{\delta}} u|d\varphi'(u)| = \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{2}{\delta}u + u^2\right) du \le \frac{K_1}{\delta^{3-r}}. \quad (41)$$

Since $\int_{\frac{1}{\delta}}^{\frac{1}{2}} u|d\varphi'(u)| \le \int_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$ and $\int_{\frac{1}{2}}^{\infty} |u-1||d\varphi'(u)| \le \int_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$, then it is sufficient to get an estimate of the integral $\int_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|$. If $u \ge \frac{1}{\delta}$ we have

$$\int_{\frac{1}{\delta}}^{\infty} u|d\varphi'(u)|du \le \int_{\frac{1}{\delta}}^{\infty} \left(\frac{2}{\delta} + u\right) u^{-r+1}du + 2r\int_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2} + \frac{2}{\delta}u + \frac{1}{2}u^2\right) u^{-r}du$$
$$+ r(r+1)\int_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2}u + \frac{1}{\delta}u^2 + \frac{1}{6}u^3\right) u^{-r-1}du \le \frac{K_2}{\delta^{3-r}}. \quad (42)$$

Combining (41) and (42), we get

$$\int_0^{\frac{1}{2}} u|d\varphi'(u)| = O\left(\frac{1}{\delta^{3-r}}\right), \int_{\frac{1}{2}}^{\infty} |u-1||d\varphi'(u)| = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty. \quad (43)$$

From (38) we easily derive that

$$\int_0^{\frac{1}{\delta}} \frac{|\varphi(u)|}{u}du = \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{4}{3\delta^2} + \frac{1}{\delta}u + \frac{1}{6}u^2\right) du \le \frac{K_3}{\delta^{3-r}},$$

$$\int\limits_{\frac{1}{\delta}}^{\infty} \frac{|\varphi(u)|}{u} du = \int\limits_{\frac{1}{\delta}}^{\infty} \left(\frac{4}{3\delta^2} + \frac{1}{\delta} u + \frac{1}{6} u^2 \right) u^{-r} du \leq \frac{K_4}{\delta^{3-r}}.$$

Hence,

$$\int\limits_{0}^{\infty} \frac{|\varphi(u)|}{u} du = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty.$$

Analogous to (26), the formula

$$\int\limits_{0}^{1} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = \int\limits_{0}^{1} \frac{|\lambda(1-u) - \lambda(1+u)|}{u} du + O(H(\varphi)) \tag{44}$$

is true, where $\lambda(u) = 1 - \frac{4}{3\delta^2} u - \frac{1}{\delta} u^2 - \frac{1}{6} u^3$, and $H(\varphi)$ is defined by formula (27). In view of the relation $\int\limits_0^1 \frac{|\lambda(1-u)-\lambda(1+u)|}{u} du = O(1)$ and (43), from (44) we have

$$\int\limits_{0}^{1} \frac{|\varphi(1-u) - \varphi(1+u)|}{u} du = O\left(\frac{1}{\delta^{3-r}}\right) \quad \text{as } \delta \to \infty.$$

Therefore, all integrals from (40) are convergent. Further, applying Theorem 1 from the paper [1] we conclude that the integral $A(\varphi)$ converges and the estimate

$$A(\varphi) = O\Big(\frac{1}{\delta^{3-r}}\Big) \quad \text{as } \delta \to \infty$$

holds.

Now we prove a convergence of the integral $A(\mu)$. For this reason, according to Theorem 1 from [1], let us show a convergence of the integrals

$$\int\limits_{0}^{\frac{1}{2}} u \left| d\mu'(u) \right|, \quad \int\limits_{\frac{1}{2}}^{\infty} |u-1| \left| d\mu'(u) \right|, \quad \int\limits_{0}^{\infty} \frac{|\mu(u)|}{u} du, \quad \int\limits_{0}^{1} \frac{|\mu(1-u) - \mu(1+u)|}{u} du. \tag{45}$$

Repeating the argumentations used to estimate the first integral of (24) from [4], we divide the segment $[0, \frac{1}{2}]$ into two parts: $[0, \frac{1}{\delta}]$ and $[\frac{1}{\delta}, \frac{1}{2}]$, $\delta > 2$. From the representation (39) of function $\mu(u)$, for $u \in [0, \frac{1}{\delta}]$ we have $\mu''(u) = \widetilde{\mu}''(u)\delta^r$, where $\widetilde{\mu}(u)$ is defined by equality (11). Then, taking into account the third inequality from (13), we get

$$\int\limits_{0}^{\frac{1}{\delta}} u |d\mu'(u)| \leq \delta^r \int\limits_{0}^{\frac{1}{\delta}} \left(\frac{6}{\delta^2} u + \frac{12}{\delta} u^2 + 3u^3 \right) du = \frac{K_1}{\delta^{4-r}}. \tag{46}$$

Analogous to (14), we obtain

$$\int\limits_{\frac{1}{\delta}}^{\frac{1}{2}} u |d\mu'(u)| = \begin{cases} O(1), & 3 < r < 4, \\ O(\ln \delta), & r = 4, \\ O\Big(\frac{1}{\delta^{4-r}}\Big), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{47}$$

Combining (46) and (47) we get the estimate

$$\int_0^{\frac{1}{2}} u|d\mu'(u)| = \begin{cases} O(1), & 3 < r < 4, \\ O\big(\ln \delta\big), & r = 4, \\ O\Big(\frac{1}{\delta^{4-r}}\Big), & r > 4, \end{cases} \qquad \text{as } \delta \to \infty. \tag{48}$$

Let us move to an estimation of the second integral from (45). In view of (39), for $u \geq \frac{1}{\delta}$ holds

$$|\mu''(u)| \leq \frac{r(r+1)|\widetilde{\mu}(u)|}{u^{r+2}} + \frac{2r|\widetilde{\mu}'(u)|}{u^{r+1}} + \frac{|\widetilde{\mu}''(u)|}{u^r}. \tag{49}$$

To make further estimations, we take into account inequalities (12) and

$$e^{-u} \leq 1, \quad e^{-u} \leq 1 - u + \frac{u^2}{2}, \quad e^{-u} \geq 1 - u, \quad u \geq 0,$$

and, hence, get

$$|\widetilde{\mu}(u)| \leq u\big(-1 + \gamma + \frac{4}{3\delta^2}\big) + u^2\big(\frac{1}{2} - \gamma + \theta + \frac{1}{\delta}\big) + u^3\big(\frac{\gamma}{2} + \frac{1}{6}\big),$$

$$|\widetilde{\mu}'(u)| \leq \big(-1 + \gamma + \frac{4}{3\delta^2}\big) + u\big(1 - 2\gamma + 2\theta + \frac{2}{\delta}\big) + u^2\big(\frac{3}{2}\gamma + \theta + \frac{1}{2}\big),$$

$$|\widetilde{\mu}''(u)| \leq \big(1 - 2\gamma + 2\theta + \frac{2}{\delta}\big) + u\big(3\gamma + 1\big) + \big(\theta u^2 + 4\theta u\big)e^{-u}.$$

Then, using estimates

$$-1 + \gamma + \frac{4}{3\delta^2} \leq \frac{2}{\delta^2}, \ \frac{1}{2} - \gamma + \theta + \frac{1}{\delta} \leq \frac{2}{\delta}, \ \frac{\gamma}{2} + \frac{1}{6} \leq 1, \ \frac{3}{2}\gamma + \theta + \frac{1}{2} \leq 4,$$

$$3\gamma + 1 \leq 6, \ \big(4\theta u + \theta u^2\big)e^{-u} \leq 2u, \ u \geq 0,$$

we obtain

$$|\widetilde{\mu}(u)| \leq \frac{2}{\delta^2}u + \frac{2}{\delta}u^2 + u^3, \ |\widetilde{\mu}'(u)| \leq \frac{2}{\delta^2} + \frac{4}{\delta}u + 4u^2, \ |\widetilde{\mu}''(u)| \leq \frac{4}{\delta} + 8u, \ u \geq 0. \tag{50}$$

In view of (49), (50), we have

$$\begin{aligned} \int_{\frac{1}{2}}^{\infty} |u-1||d\mu'(u)| \leq \int_{\frac{1}{2}}^{\infty} u|d\mu'(u)| &\leq r(r+1) \int_{\frac{1}{2}}^{\infty} \left(\frac{2}{\delta^2}u + \frac{2}{\delta}u^2 + u^3\right) u^{-r-1}du \\ + 2r \int_{\frac{1}{2}}^{\infty} \left(\frac{2}{\delta^2} + \frac{4}{\delta}u + 4u^2\right) u^{-r}du &+ \int_{\frac{1}{2}}^{\infty} \left(\frac{4}{\delta} + 8u\right) u^{-r+1}du \leq K_1, \ \ r > 3. \end{aligned} \tag{51}$$

Let us estimate the third integral from (45). We devide the segment $[0, \infty)$ into three parts: $[0, \frac{1}{\delta}]$, $[\frac{1}{\delta}, 1]$, $[1, \infty)$. From formula (11) using the first inequality from (13) and (50) we obtain

$$\int_0^{\frac{1}{\delta}} \frac{|\mu(u)|}{u}du = \delta^r \int_0^{\frac{1}{\delta}} |\widetilde{\mu}(u)|\frac{du}{u} \leq \delta^r \int_0^{\frac{1}{\delta}} \left(\frac{3}{\delta^3} + \frac{3}{\delta^2}u + \frac{2}{\delta}u^2 + u^3\right) du \leq \frac{K_1}{\delta^{4-r}};$$

$$\int\limits_{\frac{1}{\delta}}^{1} \frac{|\mu(u)|}{u} du \leq \int\limits_{\frac{1}{\delta}}^{1} \left(\frac{3}{\delta^3} + \frac{3}{\delta^2} u + \frac{2}{\delta} u^2 + u^3 \right) u^{-r} du \leq \begin{cases} K_2, & 3 < r < 4, \\ K_3 \ln \delta, & r = 4, \\ \frac{K_4}{\delta^{4-r}}, & r > 4, \end{cases}$$

$$\int\limits_{1}^{\infty} \frac{|\mu(u)|}{u} du \leq \int\limits_{1}^{\infty} \left(\frac{2}{\delta^2} + \frac{2}{\delta} u + u^2 \right) u^{-r} du \leq K_5, \quad r > 3.$$

Combining last relations, we have

$$\int\limits_{0}^{\infty} \frac{|\mu(u)|}{u} du = \begin{cases} O(1), & 3 < r < 4, \\ O(\ln \delta), & r = 4, \\ O(\frac{1}{\delta^{4-r}}), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{52}$$

To estimate the fourth integral from (45) we use the formula

$$\int\limits_{0}^{1} |\mu(1-u) - \mu(1+u)| \frac{du}{u} = \int\limits_{0}^{1} |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} + O(H(\mu)), \tag{53}$$

where $\lambda(u) = e^{-u}(1 + \gamma u + \theta u^2) + \frac{4}{3\delta^2} u + \frac{1}{\delta} u^2 + \frac{1}{6} u^3$, and $H(\mu)$ is defined by formula (27).

In view of $\int\limits_{0}^{1} |\lambda(1-u) - \lambda(1+u)| \frac{du}{u} = O(1)$, using relations (48) and (51), from (53) we get

$$\int\limits_{0}^{1} |\mu(1-u) - \mu(1+u)| \frac{du}{u} = \begin{cases} O(1), & 3 < r < 4, \\ O(\ln \delta), & r = 4, \\ O(\frac{1}{\delta^{4-r}}), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{54}$$

Hence, taking into account Theorem 1 from [1], according to formulas (48), (51), (52) and (54) we can verify that the integral $A(\mu)$ is convergent and the following estimate holds

$$A(\mu) = \begin{cases} O(1), & 3 < r < 4, \\ O(\ln \delta), & r = 4, \\ O(\frac{1}{\delta^{4-r}}), & r > 4, \end{cases} \quad \text{as } \delta \to \infty. \tag{55}$$

In view of the fact, that the Fourier transform $\widehat{\tau}_\beta(t)$ of the form (5) is summable on a whole real axis, for an arbitrary function $f \in W^r_{\beta,\infty}$ and $x \in \mathbb{R}$ the equality

$$f(x) - P_3(\delta; f; x) = \delta^{-r} \int\limits_{-\infty}^{\infty} f^r_\beta \left(x + \frac{t}{\delta} \right) \widehat{\tau}_\beta(t) dt, \quad \delta > 0, \tag{56}$$

is true.

Using (56), (38), (39), for the quantity (1) we get

$$\begin{aligned} \mathcal{E}\left(W^r_{\beta,\infty}; P_3(\delta) \right)_C &= \sup_{f \in W^r_{\beta,\infty}} \left\| \delta^{-r} \int\limits_{-\infty}^{\infty} f^r_\beta \left(x + \frac{t}{\delta} \right) \widehat{\tau}_\beta(t) dt \right\|_C \\ &= \sup_{f \in W^r_{\beta,\infty}} \left\| \delta^{-r} \int\limits_{-\infty}^{\infty} f^r_\beta \left(x + \frac{t}{\delta} \right) (\widehat{\varphi}_\beta(t) + \widehat{\mu}_\beta(t)) dt \right\|_C \\ &= \sup_{f \in W^r_{\beta,\infty}} \left\| \delta^{-r} \int\limits_{-\infty}^{\infty} f^r_\beta \left(x + \frac{t}{\delta} \right) \widehat{\varphi}_\beta(t) dt \right\|_C + O\left(\delta^{-r} A(\mu) \right). \end{aligned} \tag{57}$$

It is easy to show, that the Fourier series of a continuous function

$$f_\varphi(x) = \int\limits_{-\infty}^{\infty} f^r_\beta\left(x + \frac{t}{\delta}\right)\widehat{\varphi}_\beta(t)dt$$

takes the form

$$S\left[f_\varphi\right] = \sum_{k=1}^{\infty} \varphi\big(\frac{k}{\delta}\big)k^r\big(a_k(f)\cos kx + b_k(f)\sin kx\big), \tag{58}$$

(see speculations used in proving Theorem 1.3.1 from the paper of A.I. Stepanets [13], p. 54). Due to (58), taking into account (38), we obtain the equality

$$S\left[f_\varphi\right] = \frac{1}{\delta^{3-r}}\sum_{k=1}^{\infty}\left(\frac{4}{3}k + k^2 + \frac{1}{6}k^3\right)(a_k(f)\cos kx + b_k(f)\sin kx).$$

On the other hand,

$$S\left[\frac{4}{3}f_0^{(1)}(x) + f_0^{(2)}(x) + \frac{1}{6}f_0^{(3)}(x)\right] = \frac{1}{\delta^{3-r}}\sum_{k=1}^{\infty}\left(\frac{4}{3}k + k^2 + \frac{1}{6}k^3\right)(a_k(f)\cos kx + b_k(f)\sin kx).$$

In view of (58), we get for all $x \in \mathbb{R}$

$$\int\limits_{-\infty}^{\infty} f^r_\beta\left(x + \frac{t}{\delta}\right)\widehat{\varphi}_\beta(t)dt = \frac{1}{\delta^{3-r}}\left(\frac{4}{3}f_0^{(1)}(x) + f_0^{(2)}(x) + \frac{1}{6}f_0^{(3)}(x)\right). \tag{59}$$

Therefore, from (57), in view of formulas (55) and (59), we get the equality (37). Theorem 2 is proved. □

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Received 01.07.2013

Грабова У.З., Кальчук І.В. *Наближення класів* $W^{r}_{\beta,\infty}$ *тригармонійними інтегралами Пуассона* // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 321–334.

Робота присвячена розв'язанню однієї з екстремальних задач теорії наближення функціональних класів лінійними методами, а саме дослідженню питань про наближення класів диференційовних функцій λ-методами підсумовування їх рядів Фур'є, заданими сукупністю $\Lambda = \{\lambda_{\delta}(\cdot)\}$ неперервних на $[0, \infty)$ функцій, залежних від дійсного параметра δ. Розглянуто задачу Колмогорова-Нікольського, що займає особливе місце серед екстремальних задач теорії наближення, тобто задачу про знаходження асимптотичних рівностей для величини $\mathcal{E}(\mathfrak{N}; U_{\delta})_X = \sup\limits_{f \in \mathfrak{N}} \|f(\cdot) - U_{\delta}(f; \cdot; \Lambda)\|_X$, де X — нормований простір, $\mathfrak{N} \subseteq X$ — заданий клас функцій, $U_{\delta}(f; x; \Lambda)$ — конкретний метод підсумовування рядів Фур'є. Зокрема, в роботі досліджуються апроксимативні властивості тригармонійних інтегралів Пуассона на класах Вейля-Надя. Отримано асимптотичні формули для верхніх граней відхилень тригармонійних інтегралів Пуассона від функцій з класів $W^{r}_{\beta,\infty}$, які забезпечують розв'язок відповідної задачі Колмогорова-Нікольського. Методи дослідження екстремальних задач наближення такого типу виникли і отримали свій розвиток завдяки роботам А.М. Колмогорова, С.М. Нікольського, С.Б. Стечкина, М.П. Корнейчука, В.К. Дзядика, О.І. Степанця та інших, але вони використовуються для наближень лінійними методами підсумовування, що задаються трикутними числовими матрицями. В даній же роботі згадані методи модифіковано для методів підсумовування, що задаються множиною функцій натурального аргументу.

Ключові слова і фрази: задача Колмогорова-Нікольського, тригармонійний інтеграл Пуассона, класи Вейля-Надя.