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New fixed point theorems for orthogonal F_m-contractions in incomplete m-metric spaces

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In this paper, we introduce the concept of orthogonal m-metric spaces and by using F_m-contraction in orthogonal m-metric spaces, we give the concept of orthogonal F_m-contraction (briefly, $\perp_{F_m}$-contraction) and investigate fixed point results for such mappings. Many existing results in the literature appear to be special case of results proved in this paper. An example to support our main results is also mentioned.

Key words and phrases: unique fixed point, orthogonal, complete, $\perp_{F_m}$-contraction, incomplete m-metric space.

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Introduction

Fixed point theory is one of very important tools for proving the existence and uniqueness of the solutions to various mathematical models like integral and partial differential equations, variational inequalities, optimization and approximation theory, etc. It has gained a considerable importance in the recent past after the famous Banach contraction mapping principle [5]. Since then, there have been many results related to mapping satisfying various types of contractive inequalities, we refer the reader to [6, 8, 15, 16, 20, 21] and references therein. In recent years, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order, see [7,10–13,18,19]. Recently, M.E. Gordji et al. [9] coined an exciting notion of the orthogonal sets after which, orthogonal metric spaces was introduced. The concepts of sequence, continuity and completeness were redefined for this space. Further, they gave an extension of Banach fixed point theorem on this newly described shape and also, applied their theorem to show the existence of a solution for a differential equation which can not be applied by the Banach's fixed point theorem. Many authors generalized Banach contractive condition using some control functions, see [14,22]. In 2012, D. Wardowski [23] introduced a new kind of contractions, called F-contractions, and proved some fixed point results using the family of F-contractions. Recently, H. Baghani et al. [4] introduced the notion of orthogonal F-contraction mapping and established some fixed point results for such mappings.

On the other hand, in 1994, S.G. Matthews [15] introduced the notion of partial metric space

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and proved the contraction principle of Banach in this new framework. Next, many fixed point theorems in partial metric spaces have been given by several researchers. In 2014, M. Asadi et al. [3] extended the concept of partial metric space to an m-metric space, and showed that their definition is a real generalization of partial metric by presenting some examples. In 2018, N. Mlaiki [17] introduced the notion of F_m-contractive and F_m-expanding mappings in m-metric space, where he proved that self mappings on a complete m-metric spaces which are F_m-contractive have a unique fixed point, also see [1,2].

In this paper, we apply the F-contraction in orthogonal m-metric spaces and introduced $\perp_{F_m}$-contraction and investigate the fixed point results for such operators. We give an example to explain the theory presented in the paper.

1 Preliminaries

We recall the following definitions and results, which will be useful to understand the paper.

Definition 1 ([23]). *Let Ω be the set of all functions $F : (0, \infty) \to \mathbb{R}$ with the following properties:*

(F_1) F is strictly increasing;

(F_2) for every sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers,

$$\lim_{n\to\infty} x_n = 0 \quad \textit{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty;$$

(F_3) there exists $h \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^h F(\alpha) = 0$.

Let $F_i : (0, \infty) \to \mathbb{R}$, where $i \in \{1, 2, 3, 4\}$, be defined by

$$F_1(x) = \ln(x), \qquad F_2(x) = \ln(x) + x, \qquad F_3(x) = \frac{-1}{\sqrt{x}}, \qquad F_4(x) = \ln(x^2 + x).$$

Then each F_i belong to Ω.

Notation ([3]). $m_{x,y} = \min\{m(x, x), m(y, y)\}$; $M_{x,y} = \max\{m(x, x), m(y, y)\}$.

Definition 2 ([3]). *Let X be a nonempty set. If the function $m : X \times X \to \mathbb{R}^+$, for all $x, y, z \in X$, satisfies the following conditions:*

1) $m(x, x) = m(y, y) = m(x, y)$ if and only if $x = y$,

2) $m_{x,y} \le m(x, y)$,

3) $m(x, y) = m(y, x)$,

4) $m(x, y) - m_{x,y} \le (m(x, z) - m_{x,z}) + (m(z, y) - m_{z,y})$,

then the pair (X, m) is called an m-metric space.

Example 1 ([3]). *Let $X = [0, \infty)$ and $m(x, y) = \frac{x+y}{2}$ on X. Then (X, m) is an m-metric space.*

Example 2 ([3]). *Let m be an m-metric. Put*

1) $m^z(x,y) = m(x,y) - 2m_{x,y} + M_{x,y}$,

2) $m^s(x,y) = m(x,y) - m_{x,y}$ *if* $x \neq y$, *and* $m^s(x,y) = 0$ *if* $x = y$.

Then m^z and m^s are ordinary metrics.

As mentioned in [3], each m-metric on X generates a T_0 topology τ_m on X. The set $\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{x,y} + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$, forms a basis of τ_m.

Definition 3 ([9]). *Let X be a nonempty set and $\perp$ be a binary relation defined on $X \times X$, then $(X, \perp)$ is said to be orthogonal set or O-set, if*

$$\exists\, x_0 : \forall\, y \in X \;\; y \perp x_0 \;\text{ or }\; \forall\, y \in X \;\; x_0 \perp y.$$

The element x_0 is called an orthogonal element. An orthogonal set may have more than one orthogonal elements.

Definition 4 ([9]). *Let $(X, \perp)$ be an orthogonal set. Any two elements $x, y \in X$ are said to be orthogonally related if $x \perp y$.*

Definition 5 ([9]). *Let $(X, \perp)$ be an orthogonal set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if*

$$\forall\, n \;\; x_n \perp x_{n+1} \quad \text{or} \quad \forall\, n \;\; x_{n+1} \perp x_n.$$

2 Main Results

We start this section with the following definitions.

Definition 6. *Let $(X, \perp, m)$ be an orthogonal m-metric space, i.e. $(X, \perp)$ is an orthogonal set and (X, m) is an m-metric space. Then,*

1) *an O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X converges to a point $x \in X$ if and only if*

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0;$$

2) *an O-sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is said to be m-Cauchy O-sequence if and only if*

$$\lim_{n,m\to\infty} (m(x_n, x_m) - m_{x_n,x_m}) \text{ and } \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m})$$

exist (and are finite);

3) *an orthogonal m-metric space X is said to be O-complete if every m-Cauchy O-sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in X$ with respect to τ_m such that*

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n,x}) = 0 \text{ and } \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0.$$

It is easy to see that every complete m-metric space is O-complete and the converse is not true in general. In the next example, X is O-complete and is not complete m-metric space.

Example 3. *Let $X = [0,1)$ and suppose that*

$$x \perp y \iff x \leq y \leq \frac{1}{2} \text{ or } x = 0.$$

Then $(X, \perp)$ is an O-set. Clearly, X with $m(x,y) = \frac{x+y}{2}$ is not complete m-metric space, but it is O-complete. In fact, if $\{x_n\}_{n\in\mathbb{N}}$ is an arbitrary m-Cauchy O-sequence in X, then there exists a monotonic subsequence $\{x_{n_k}\}$ of $\{x_n\}$ for which $x_{n_k} \leq \frac{1}{2}$ for all $n \geq 1$. It follows that $\{x_n\}_{n\in\mathbb{N}}$ converges to a point $x \in [0, \frac{1}{2}] \subseteq X$. Hence, $\{x_n\}_{n\in\mathbb{N}}$ is convergent.

Lemma 1. *If $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ are two O-sequences such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in an orthogonal m-metric space $(X, \perp, m)$, then*

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n,y_n}) = m(x,y) - m_{x,y}.$$

Lemma 2. *If $\{x_n\}_{n\in\mathbb{N}}$ is an O-sequence such that $x_n \to x$ as $n \to \infty$ in an orthogonal m-metric space $(X, \perp, m)$, then*

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n,y}) = m(x,y) - m_{x,y}.$$

Definition 7. *Let $(X, \perp, m)$ be an orthogonal m-metric space. A mapping $T : X \to X$ is called $\perp$-preserving, if $Tx \perp Ty$ whenever $x \perp y$.*

Definition 8. *Let $(X, \perp, m)$ be an orthogonal m-metric space and $F \in \Omega$. A self-mapping T on X is called $\perp_{F_m}$-contraction, if there exists $\tau > 0$ such that*

$$\tau + F(m(Tx_1, Tx_2)) \leq F(m(x_1, x_2)),$$

for all $x_1, x_2 \in X$ with $x_1 \perp x_2$ and $m(Tx_1, Tx_2) > 0$.

Example 4. *Let $X = [0,1)$ and $m : X \times X \to \mathbb{R}^+$ be defined by $m(x,y) = \frac{x+y}{2}$. Define $x \perp y$, if $xy \leq x$ or $xy \leq y$, for all $x, y \in X$. Let $F : (0,\infty) \to \mathbb{R}$ be defined by $F(x) = \ln(x)$ and $T : X \to X$ be defined by*

$$Tx = \begin{cases} \frac{x}{10}, & x \in \mathbb{Q} \cap X, \\ 0, & x \in \mathbb{Q}^c \cap X. \end{cases}$$

Then it can be easily shown that T is an $\perp_{F_m}$-contraction on X with $\tau = 2$.

Lemma 3. *Let $(X, \perp, m)$ be an orthogonal m-metric space and a self mapping T be $\perp$-preserving and $\perp_{F_m}$-contraction. Consider an O-sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by $x_{n+1} = Tx_n$. If $x_n \to u^*$ as $n \to \infty$, then $Tx_n \to Tu^*$ as $n \to \infty$.*

Proof. First, note that if $m(Tx_n, Tu^*) = 0$, then $m_{Tx_n,Tu^*} = 0$ and due to the fact that $m_{Tx_n,Tu^*} \leq m(Tx_n, Tu^*)$, which implies that $m(Tx_n, Tu^*) - m_{Tx_n,Tu^*} \to 0$ and hence $Tx_n \to Tu^*$ as $n \to \infty$.

So, we may assume that $m(Tx_n, Tu^*) > 0$. By the $\perp_{F_m}$-contraction condition of T we conclude that $m(Tx_n, Tu^*) < m(x_n, u^*)$. Then, we have the following cases.

Case I. If $m(u^*, u^*) \leq m(x_n, x_n)$, by $\perp_{F_m}$-contractive property of T, it is easy to see that $m(x_n, x_n) \to 0$, which implies that $m(u^*, u^*) = 0$ and since $m(Tu^*, Tu^*) \leq m(u^*, u^*) = 0$, we conclude that $m(Tu^*, Tu^*) = m(u^*, u^*) = 0$. On the other hand, due to $m(x_n, u^*) \to 0$, we have $m(Tx_n, Tu^*) \leq m(x_n, u^*) \to 0$. Hence, $m(Tx_n, Tu^*) - m_{Tx_n,Tu^*} \to 0$ and thus $Tx_n \to Tu^*$.

Case II. If $m(u^*, u^*) \geq m(x_n, x_n)$ and once again by the $\perp_{F_m}$-contractive property of T, it is easy to see that $m(x_n, x_n) \to 0$, which implies that $m_{x_n,u^*} \to 0$ and so $m(x_n, u^*) \to 0$. Since $m(Tx_n, Tu^*) \leq m(x_n, u^*) \to 0$, we conclude that $m(Tx_n, Tu^*) - m_{Tx_n,Tu^*} \to 0$ and thus $Tx_n \to Tu^*$ as desired. □

Theorem 1. *Let $(X, \perp, m)$ be an orthogonal complete m-metric space (not necessarily complete) and $T : X \to X$ be a $\perp$-preserving, $\perp_{F_m}$-contraction, then T has a unique fixed point u^* in X. Moreover, for every $x_0 \in X$, the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ is convergent to u^*.*

Proof. For the uniqueness of fixed point, suppose that there exist two orthogonally related elements x, y belonging to X such that $x = Tx$ and $y = Ty$ with $x \neq y$. If $m(Tx, Ty) = 0$, without loss of generality, suppose that $m_{x,y} = m(x, x)$, then

$$m(Tx, Ty) = 0 = m(x, x).$$

Now, if $m(y, y) = 0$, then $x = y$. So, assume that $m(y, y) > 0$. By using contractive condition, we have

$$F(m(y, y)) = F(m(Ty, Ty)) \leq F(m(y, y)) - \tau < F(m(y, y)),$$

which is a contradiction. Hence, $m(y, y) = 0$ and so $x = y$.

Now, we may assume that $m(x, y) > 0$. By using the fact that T is an $\perp_{F_m}$-contraction, we deduce that

$$F(m(x, y)) = F(m(Tx, Ty)) \leq F(m(x, y)) - \tau < F(m(x, y)),$$

which leads to a contradiction. Thus, if T has a fixed point then it is unique.

Now, by the definition of orthogonality, there exists an orthogonal element $x_0 \in X$ such that $\forall y \in X \;\; x_0 \perp y$ or $\forall y \in X \; y \perp x_0$. It follows that $x_0 \perp Tx_0$ or $Tx_0 \perp x_0$. Let us define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n = T^{n+1}x_0$ for all $n \in \mathbb{N}$. From the property of $\perp$-preserving of T, we can easily check that $\{x_n\}$ is an O-sequence, i.e. $\forall n \in \mathbb{N}\; x_n \perp x_{n+1}$ or $\forall n \in \mathbb{N}\; x_{n+1} \perp x_n$. If there exists a natural number i such that $x_{i+1} = x_i$, then x_i is a fixed point of T.

Now, assume that $m(x_n, x_n) = 0$ for some n. We want to show that in this case $m(x_m, x_m) = 0$ for all $m > n$. So, assume that $m(x_n, x_n) = 0$ and $m(x_{n+1}, x_{n+1}) \neq 0$, by the $\perp_{F_m}$-contractive property of T, we obtain

$$F(m(x_{n+1}, x_{n+1})) = F(m(Tx_n, Tx_n)) \leq F(m(x_n, x_n)) - \tau < F(m(x_n, x_n)).$$

Since F is increasing function, we have $m(x_{n+1}, x_{n+1}) \leq m(x_n, x_n) = 0$. Hence, by induction on n, we get if $m(x_n, x_n) = 0$, then $m(x_m, x_m) = 0$ for all $m > n$.

Also, note that if $m > n$, then we have $m_{x_n,x_m} = m(x_m, x_m)$, to see this, assume that $m_{x_n,x_m} = m(x_n, x_n)$. If $m(x_n, x_n) = 0$, then by the above claim, we obtain $m(x_m, x_m) = 0$. If $m(x_n, x_n) > 0$, then $m(x_m, x_m) > 0$ for all $m > n$. Thus,

$$\begin{aligned} F(m(x_m, x_m)) = F(m(Tx_{m-1}, Tx_{m-1})) &\leq F(m(x_{m-1}, x_{m-1})) - \tau \leq \ldots \\ &\leq F(m(x_n, x_n)) - (m-n)\tau < F(m(x_n, x_n)), \end{aligned}$$

but F is an increasing function. Therefore, if $m > n$, we have $m_{x_n,x_m} = m(x_m, x_m)$.

Now suppose that $m(x_{n+1}, x_n) = 0$ for some n. This implies that $m_{x_n,x_{n+1}} = 0$. Also, we know that $m_{x_n,x_{n+1}} = m(x_{n+1}, x_{n+1}) = 0$.

So, by the above argument we have $m(x_{n+2}, x_{n+2}) = 0$. Thus, we have two cases: either $m(x_{n+1}, x_{n+2}) = 0$, in this case $x_{n+1} = x_{n+2}$, that is x_{n+1} is the fixed point, or $m(x_{n+1}, x_{n+2}) > 0$, again by $\perp_{F_m}$-contractive property of T, we have

$$F(m(x_{n+1}, x_{n+2})) = F(m(Tx_n, Tx_{n+1})) \leq F(m(x_n, x_{n+1})) - \tau < F(m(x_n, x_{n+1})) = F(0),$$

which is a contradiction. Hence, we can suppose that $m(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Let $\gamma_n = m(x_n, x_{n+1})$. Then

$$F(\gamma_n) \le F(\gamma_{n-1}) - \tau \le F(\gamma_{n-2}) - 2\tau \le \cdots \le F(\gamma_0) - n\tau, \tag{1}$$

for $n \ge 1$. Taking limit as $n \to \infty$, we get that $\lim_{n\to\infty} F(\gamma_n) = -\infty$, and hence $\lim_{n\to\infty} \gamma_n = 0$ by (F_2). Now from (F_3), there exists $h \in (0,1)$ so that $\lim_{n\to\infty} \gamma_n^h F(\gamma_n) = 0$. From (1), we have

$$\gamma_n^h\, F(\gamma_n) - \gamma_n^h\, F(\gamma_0) \le \gamma_n^h\, (F(\gamma_0) - n\tau) - \gamma_n^h\, F(\gamma_0) = -\gamma_n^h\, n\tau \le 0.$$

Hence, $\lim_{n\to\infty} n\gamma_n^h = 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that $n\gamma_n^h \le 1$ for all $n > n_0$, and so $\gamma_n \le \frac{1}{n^{\frac{1}{h}}}$ for all $n > n_0$. Now, we prove that the O-sequence $\{x_n\}_{n\in\mathbb{N}}$ is an m-Cauchy. Take $n, m \in \mathbb{N}$ with $m > n > n_0$. First, notice the following fact about triangular inequality of m-metric spaces

$$(m(x,y) - m_{x,y}) \le (m(x,z) - m_{x,z}) + (m(z,y) - m_{z,y}) \le m(x,z) + m(z,y)$$

for all $x, y, z \in X$. Thus, it is clear that

$$m(x_n, x_m) - m_{x_n,x_m} \le \gamma_n + \gamma_{n+1} + \ldots + \gamma_m < \sum_{i=n}^{\infty} \gamma_i \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}$ converges, it implies that $m(x_n, x_m) - m_{x_n,x_m}$ converges as $m, n \to \infty$.

Now, if $M_{x_n,x_m} = 0$, then $m_{x_n,x_m} = 0$ which implies that $M_{x_n,x_m} - m_{x_n,x_m} = 0$. So, we may assume that $M_{x_n,x_m} > 0$, this implies that $m(x_n, x_n) > 0$.

Now, let $\eta_n = m(x_n, x_n)$. Then

$$F(\eta_n) \le F(\eta_{n-1}) - \tau \le F(\eta_{n-2}) - 2\tau \le \cdots \le F(\eta_0) - n\tau \tag{2}$$

for $n \ge 1$. On taking limit as $n \to \infty$, we get that $\lim_{n\to\infty} F(\eta_n) = -\infty$ and so $\lim_{n\to\infty} \eta_n = 0$ by (F_2). Then from (F_3), there exists $h \in (0,1)$ so that $\lim_{n\to\infty} \eta_n^h\, F(\eta_n) = 0$, and by using (2) we obtain

$$\eta_n^h\, F(\eta_n) - \eta_n^h\, F(\eta_0) \le \eta_n^h\, (F(\eta_0) - n\tau) - \eta_n^h\, F(\eta_0) \le -\eta_n^h\, n\tau \le 0.$$

Letting $n \to \infty$ in the above inequality, we get $\lim_{n\to\infty} n\eta_n^h = 0$. Thus, there exists $n_1 \in \mathbb{N}$ such that $n\eta_n^h \le 1$ for all $n > n_1$. Consequently, we have $\eta_n \le \dfrac{1}{n^{\frac{1}{h}}}$ for all $n > n_1$. Therefore, we obtain

$$m(x_n, x_n) - m(x_m, x_m) \le \eta_n + \eta_{n+1} + \ldots + \eta_m < \sum_{i=n}^{\infty} \eta_i \le \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{h}}}.$$

Since the series $\sum_{i=n}^{\infty} \frac{1}{i^{1/h}}$ is convergent, we conclude that $m(x_n, x_n) - m(x_m, x_m)$ converges as $m, n \to \infty$, which implies that $M_{x_n,x_m} - m_{x_n,x_m}$ converges as desired. Therefore $\{x_n\}$ is an m-Cauchy O-sequence in X. Since $(X, \perp, m)$ is an O-complete m-metric space, $\{x_n\}$ converges to some $u^* \in X$.

Since $m(x_n, x_{n+1}) > 0$, by $\perp_{F_m}$-contractive property of T, we conclude that $m(x_n, Tx_n) \to 0$ and $m(Tu^*, Tu^*) < m(u^*, u^*)$. Now, using the fact that $m_{x_n,Tx_n} \to 0$, and by Lemmas 1 and 2, we conclude that

$$m(u^*, Tu^*) = m_{u^*,Tu^*} = m(Tu^*, Tu^*).$$

Again by Lemmas 1, 2 and 3, and $x_n = Tx_{n-1} \to u^*$, we obtain

$$0 = \lim_{n\to\infty}(m(x_n, Tx_n) - m_{x_n,Tx_n}) = \lim_{n\to\infty}(m(x_n, x_{n-1}) - m_{x_n,Tx_n}) = m(u^*, u^*) - m_{u^*,Tu^*}.$$

Therefore, $m(u^*, u^*) = m_{u^*,Tu^*}$. Hence, $m(u^*, u^*) = m_{u^*,Tu^*} = m(Tu^*, Tu^*)$, that is, $Tu^* = u^*$. □

Example 5. *Let* $X = [1, 10)$ *and* $m(x, y) = \frac{x+y}{2}$ *for all* $x, y \in X$. *Define*

$$x \perp y \iff x \leq y \leq 5 \ \text{ or } \ x = 1$$

for all $x, y \in X$. *First, note that* $(X, \perp, m)$ *is an O-complete (not complete)* m*-metric space. Now, consider the function* $F : (0, \infty) \to \mathbb{R}$ *defined by* $F(x) = \ln(x)$.

Notice that $F \in \Omega$. *Next, let* $T : X \to X$ *such that* $T(x) = \frac{x+1}{2}$ *for all* $x \in X$. *Thus* T *is* $\perp$*-preserving. Also, since* $x, y \in [1,\ 10)$, $x + y > 2$ *for all* $x, y \in X$. *Hence,*

$$m(x, y) - m(Tx, Ty) = \frac{x+y}{2} - \frac{x+y+2}{2} = \frac{x+y-2}{4} > 0.$$

Also, we have $m(x, y) > 0$ *for all* $x, y \in X$ *and given the fact that* F *is increasing function, we conclude that* T *is an* $\perp_{F_m}$*-contraction. Therefore, by Theorem 1,* T *has a unique fixed point in* X*, which is* 1.

References

[1] Altun I., Sahin H., Türkoğlu D. *Caristi-type fixed point theorems and some generalizations on M-metric space*. Bull. Malays. Math. Sci. Soc. 2020, **43**, 2647–2657. doi:10.1007/s40840-019-00823-8

[2] Sahin H., Altun I., Türkoğlu D. *Two fixed point results for multivalued F-contractions on M-metric spaces*. Rev. R. Acad. Cienc. Exactas Fís. Nat. (Esp.) 2019, **113**, 1839–1849. doi:10.1007/s13398-018-0585-x

[3] Asadi M., Karapinar E., Salimi P. *New extension of p-metric spaces with some fixed-point results on M-metric spaces*. J. Inequal. Appl. 2014, **18**. doi:10.1186/1029-242X-2014-18

[4] Baghani H., Gordji M.E., Ramezani M. *Orthogonal sets: The axiom of choice and proof of a fixed point theorem*. J. Fixed Point Theory Appl. 2016, **18**, 465–477. doi:10.1007/s11784-016-0297-9

[5] Banach S. *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*. Fund. Math. 1922, **3** (1), 133–181.

[6] Berinde V. *Approximating fixed points of weak φ-contractions using the Picard iteration*. Fixed Point Theory 2003, **4** (2), 131–142.

[7] Bhaskar T.G., Lakshmikantham V. *Fixed point theorems in partially ordered metric spaces and applications*. Nonlinear Anal. 2006, **65** (7), 1379–1393. doi:10.1016/j.na.2005.10.017

[8] Gordji M.E., Baghani H., Kim G.H. *Common fixed point theorems for (ψ, φ)-weak nonlinear contraction in partially ordered sets*. Fixed Point Theory Appl. 2012, **62**. doi:10.1186/1687-1812-2012-62

[9] Gordji M.E., Rameani M., De La Sen M., Cho Y.J. *On orthogonal sets and Banach fixed point theorem*. Fixed Point Theory 2017, **18** (2), 569–578. doi:10.24193/fpt-ro.2017.2.45

[10] Işık H., Radenović S. *A new version of coupled fixed point results in ordered metric spaces with applications*. Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. 2017, **79** (2), 131–138.

[11] Işık H., Türkoğlu D. *Fixed point theorems for weakly contractive mappings in partially ordered metric-like spaces*. Fixed Point Theory Appl. 2013, **51**. doi:10.1186/1687-1812-2013-51

[12] Işık H., Türkoğlu D. *Some fixed point theorems in ordered partial metric spaces*. J. Inequal. Spec. Funct. 2013, **4** (2), 13–18.

[13] Lakshmikantham V., Ćirić L. *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*. Nonlinear Anal. 2009, **70** (12), 4341–4349.

[14] Matkowski J. *Fixed point theorems for mappings with a contractive iterate at a point*. Proc. Amer. Math. Soc. 1977, **62** (2), 344–348.

[15] Matthews S.G. *Partial metric topology*. Annals of the New York Academy of Sciences 1994, **728** (1), 183–197. doi:10.1111/j.1749-6632.1994.tb44144.x

[16] Mitrović Z., Işık H., Radenović S. *The new results in extended b-metric spaces and applications*. Int. J. Nonlinear Anal. Appl. 2020, **11** (1), 473–482. doi:10.22075/ijnaa.2019.18239.1998

[17] Mlaiki N. *F_m-contractive and F_m-expanding mappings in M-metric spaces*. J. Math. Comp. Sci. 2018, **18** (1), 262–271. doi:10.22436/jmcs.018.03.02

[18] Nieto J.J., Rodríguez-López R. *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*. Order 2005, **22** (3), 223–239. doi:10.1007/s11083-005-9018-5

[19] Ran A.C.M., Reurings M.C.B. *A fixed point theorem in partially ordered sets and some applications to matrix equations*. Proc. Amer. Math. Soc. 2004, **132** (5), 1435–1443. doi:10.1090/S0002-9939-03-07220-4

[20] Rhoades B.E. *Some theorems on weakly contractive maps*. Nonlinear Anal. 2001, **47** (4), 2683–2693. doi:10.1016/S0362-546X(01)00388-1

[21] Salazar L.A., Reich S. *A remark on weakly contractive mappings*. J. Nonlinear Convex. Anal. 2015, **16** (4), 767–773.

[22] Suzuki T. *A generalized Banach contraction principle that characterizes metric completeness*. Proc. Amer. Math. Soc. 2008, **136**, 1861–1869. doi:10.1090/S0002-9939-07-09055-7

[23] Wardowski D. *Fixed point theory of a new type of contractive mappings in complete metric spaces*. Fixed Point Theory Appl. 2012, **94**. doi:10.1186/1687-1812-2012-94

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Мехмуд М., Ішік Г., Уддін Ф., Шоаїб А. *Нові теореми про нерухому точку для ортогональних F_m-стисків в неповних m-метричних просторах* // Карпатські матем. публ. — 2021. — Т.13, №2. — C. 405–412.

У цій статті ми вводимо поняття ортогональних m-метричних просторів і, використовуючи F_m-стиск у ортогональних m-метричних просторах, ми даємо поняття ортогонального F_m-стиску (скорочено $\perp_{F_m}$-стиск) і досліджуємо результати про нерухому точку для таких відображень. Багато існуючих в літературі результатів є окремими випадками результатів, доведених у цій статті. Також наведено приклад, що ілюструє наші основні результати.

Ключові слова і фрази: єдина нерухома точка, ортогональний, повний, $\perp_{F_m}$-стиск, неповний m-метричний простір.