



Extreme points of $\mathcal{L}_s(^2l_\infty)$ and $\mathcal{P}(^2l_\infty)$

Kim Sung Guen

For $n \geq 2$, we show that every extreme point of the unit ball of $\mathcal{L}_s(^2l_\infty)$ is extreme in $\mathcal{L}_s(^2l_\infty^{n+1})$, which answers the question in [Period. Math. Hungar. 2018, 77 (2), 274–290]. As a corollary we show that every extreme point of the unit ball of $\mathcal{L}_s(^2l_\infty^n)$ is extreme in $\mathcal{L}_s(^2l_\infty)$. We also show that every extreme point of the unit ball of $\mathcal{P}(^2l_\infty^2)$ is extreme in $\mathcal{P}(^2l_\infty)$. As a corollary we show that every extreme point of the unit ball of $\mathcal{P}(^2l_\infty^2)$ is extreme in $\mathcal{P}(^2l_\infty)$.

Key words and phrases: extreme point, symmetric bilinear form, 2-homogeneous polynomials on l_∞ .

Department of Mathematics, Kyungpook National University, 41566, Daegu, South Korea
E-mail: sgk317@knu.ac.kr

Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^* . An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y + z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is an $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. We denote by $\text{ext } B_E$ and $\text{exp } B_E$ the set of extreme points and the set of exposed points of B_E , respectively. A mapping $P : E \rightarrow \mathbb{R}$ is a continuous n -homogeneous polynomial if there exists a continuous n -linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \dots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n -homogeneous polynomials from E into \mathbb{R} endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(^n E)$ the Banach space of all continuous n -linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \dots, x_n)|$. $\mathcal{L}_s(^n E)$ denotes the closed subspace of all continuous symmetric n -linear forms on E . For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer the reader to [8].

Let us introduce the history of classification problems of the extreme points and the exposed points of the unit ball of continuous n -homogeneous polynomials on a Banach space. We let $l_p^n = \mathbb{R}^n$ for every $1 \leq p \leq \infty$ equipped with the l_p -norm. Y.S. Choi et al. [3–5] initiated and classified $\text{ext } B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2$. Y.S. Choi and S.G. Kim [7] classified $\text{exp } B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2, \infty$. B.C. Greco [12] classified $\text{ext } B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. In the paper [41], S.G. Kim et al. showed that if E is a separable real Hilbert space with $\dim(E) \geq 2$, then $\text{ext } B_{\mathcal{P}(^2E)} = \text{exp } B_{\mathcal{P}(^2E)}$. In [16], S.G. Kim classified $\text{exp } B_{\mathcal{P}(^2l_p^2)}$ for $1 \leq p \leq \infty$, and, in [18, 20], he characterized $\text{ext } B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with an octagonal norm

YΔK 517.982.3

2020 Mathematics Subject Classification: 46A22.

$\|(x, y)\|_w = \max \left\{ |x|, |y|, \frac{|x|+|y|}{1+w} \right\}$ for $0 < w < 1$. In [25], S.G. Kim classified $\exp B_{\mathcal{P}(2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(2d_*(1,w)^2)} \neq \text{ext } B_{\mathcal{P}(2d_*(1,w)^2)}$. Recently, in [30, 33], he classified $\text{ext } B_{\mathcal{P}(2\mathbb{R}_{h(\frac{1}{2})}^2)}$ and $\exp B_{\mathcal{P}(2\mathbb{R}_{h(\frac{1}{2})}^2)}$, where $\mathbb{R}_{h(\frac{1}{2})}^2 = \mathbb{R}^2$ with a hexagonal norm $\|(x, y)\|_{h(\frac{1}{2})} = \max \left\{ |y|, |x| + \frac{1}{2}|y| \right\}$.

Parallel to the classification problems of $\text{ext } B_{\mathcal{P}(nE)}$ and $\exp B_{\mathcal{P}(nE)}$, it seems to be very natural to study the classification problems of the extreme points and the exposed points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

In the works [17, 19, 21, 22, 24, 28, 29, 32, 34, 36, 37], S.G. Kim classified $\text{ext } B_{\mathcal{L}_s(2l_\infty^2)}$, $\exp B_{\mathcal{L}_s(2l_\infty^2)}$, $\text{ext } B_{\mathcal{L}_s(2d_*(1,w)^2)}$, $\text{ext } B_{\mathcal{L}(2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}(2d_*(1,w)^2)}$, $\text{ext } B_{\mathcal{L}_s(2l_\infty^3)}$, $\exp B_{\mathcal{L}_s(2l_\infty^3)}$, $\text{ext } B_{\mathcal{L}(2l_\infty^3)}$, $\exp B_{\mathcal{L}(2l_\infty^3)}$, $\text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$, $\exp B_{\mathcal{L}_s(nl_\infty^2)}$, $\text{ext } B_{\mathcal{L}(nl_\infty^2)}$, $\exp B_{\mathcal{L}(nl_\infty^2)}$ and $\text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$, characterized $\text{ext } B_{\mathcal{L}(2l_\infty^3)}$, $\exp B_{\mathcal{L}(2l_\infty^3)}$ and studied $\text{ext } B_{\mathcal{L}(2l_\infty^2)}$. He showed that $\text{ext } B_{\mathcal{L}_s(2l_\infty^2)} = \exp B_{\mathcal{L}_s(2l_\infty^2)}$, $\text{ext } B_{\mathcal{L}_s(2l_\infty^3)} = \exp B_{\mathcal{L}_s(2l_\infty^3)}$, $\text{ext } B_{\mathcal{L}_s(3l_\infty^2)} = \exp B_{\mathcal{L}_s(3l_\infty^2)}$, $\exp B_{\mathcal{L}(2l_\infty^2)} = \text{ext } B_{\mathcal{L}(2l_\infty^2)}$, $\exp B_{\mathcal{L}_s(2l_\infty^2)} = \text{ext } B_{\mathcal{L}_s(2l_\infty^2)}$, $|\text{ext } B_{\mathcal{L}(nl_\infty^2)}| = 2^{(2^n)}$, $|\text{ext } B_{\mathcal{L}_s(nl_\infty^2)}| = 2^{n+1}$, $\exp B_{\mathcal{L}(nl_\infty^2)} = \text{ext } B_{\mathcal{L}(nl_\infty^2)}$ and $\exp B_{\mathcal{L}_s(nl_\infty^2)} = \text{ext } B_{\mathcal{L}_s(nl_\infty^2)}$. In [2], M. Cavalcante et al. characterized $\text{ext } B_{\mathcal{L}(nl_\infty^m)}$. In [38], S.G. Kim classified extreme points and exposed points of the unit ball of the space of bilinear symmetric forms on the real Banach space of bilinear symmetric forms on l_∞^2 . It is shown that for this case, the set of extreme points is equal to the set of exposed points. In [39], he characterized $\text{ext } B_{\mathcal{L}(n\mathbb{R}_{\|\cdot\|}^m)}$ and $\text{ext } B_{\mathcal{L}_s(n\mathbb{R}_{\|\cdot\|}^m)}$, where $\mathbb{R}_{\|\cdot\|}^m$ is \mathbb{R}^m with a norm $\|\cdot\|$ such that $|\text{ext } B_{\mathbb{R}_{\|\cdot\|}^m}| = 2m$ for $m \geq 2$. Recently, S.G. Kim [40] characterized $\text{ext } B_{\mathcal{L}(nl_1)}$ and $\text{ext } B_{\mathcal{L}_s(nl_1)}$ for $n \geq 2$.

We refer the reader to [1, 6, 9–11, 13–15, 23, 26, 27, 31, 35–38, 42–49] and references therein for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

In this paper, for $n \geq 2$, we show that every extreme point of the unit ball of $\mathcal{L}_s(2l_\infty^n)$ is extreme in $\mathcal{L}_s(2l_\infty^{n+1})$, which answers the question in [32]. As a corollary we show that every extreme point of the unit ball of $\mathcal{L}_s(2l_\infty^n)$ is extreme in $\mathcal{L}_s(2l_\infty)$. We also show that every extreme point of the unit ball of $\mathcal{P}(2l_\infty^n)$ is extreme in $\mathcal{P}(2l_\infty)$. As a corollary we show that every extreme point of the unit ball of $\mathcal{P}(2l_\infty^2)$ is extreme in $\mathcal{P}(2l_\infty)$.

1 Results

Let $n, m \geq 2$ and $l_\infty^m = \mathbb{R}^m$ with the l_1 -norm. Set

$$\mathcal{W}_{n,m} := \left\{ \left((1, w_2^{(1)}, \dots, w_m^{(1)}), \dots, (1, w_2^{(n)}, \dots, w_m^{(n)}) \right) : w_j^{(k)} = \pm 1 \text{ for } 1 \leq k \leq n, 2 \leq j \leq m \right\}.$$

Note that $\mathcal{W}_{n,m}$ has $2^{(m-1)n}$ elements in $S_{l_\infty^m} \times \dots \times S_{l_\infty^m}$.

For $(a_{j_1}, \dots, a_{j_n}), (a_{i_1}, \dots, a_{i_n}) \in \mathcal{W}_{n,m}$, we define an equivalence relation \sim by

$$(a_{j_1}, \dots, a_{j_n}) \sim (a_{i_1}, \dots, a_{i_n})$$

if and only if $j_1 = i_{\phi(1)}, \dots, j_n = i_{\phi(n)}$ for some permutation ϕ on $\{1, \dots, n\}$. In $\mathcal{W}_{n,m}$,

$$\left[\left((1, w_2^{(1)}, \dots, w_m^{(1)}), \dots, (1, w_2^{(n)}, \dots, w_m^{(n)}) \right) \right]$$

denotes the equivalence class containing $\left((1, w_2^{(1)}, \dots, w_m^{(1)}), \dots, (1, w_2^{(n)}, \dots, w_m^{(n)}) \right)$.

Let

$$\mathcal{U}_{n,m} := \left\{ \left[\left((1, w_2^{(1)}, \dots, w_m^{(1)}), \dots, (1, w_2^{(n)}, \dots, w_m^{(n)}) \right) \right] : \right. \\ \left. \left((1, w_2^{(1)}, \dots, w_m^{(1)}), \dots, (1, w_2^{(n)}, \dots, w_m^{(n)}) \right) \in \mathcal{W}_{n,m} \right\}.$$

Lemma 1. *Let $n \geq 2$ and $Z_1, \dots, Z_{\frac{n(n+1)}{2}} \in \mathcal{U}_{2,n}$. Then there are $1 \leq i_1 < \dots < i_n \leq \frac{n(n+1)}{2}$ and $Z_{i_k} = \left[\left((1, x_2^{(i_k)}, \dots, x_n^{(i_k)}), (1, y_2^{(i_k)}, \dots, y_n^{(i_k)}) \right) \right]$ for $k = 1, \dots, n$, such that*

$$\left((1, x_2^{(i_1)}, \dots, x_n^{(i_1)}), \dots, (1, x_2^{(i_n)}, \dots, x_n^{(i_n)}) \right)$$

are linearly independent in \mathbb{R}^n or

$$\left((1, y_2^{(i_1)}, \dots, y_n^{(i_1)}), \dots, (1, y_2^{(i_n)}, \dots, y_n^{(i_n)}) \right)$$

are linearly independent in \mathbb{R}^n .

The following result answers the question in the front of Theorem 3.5 of [32].

Theorem 1. *The inclusion $\text{ext } B_{\mathcal{L}_s(2l_\infty^n)} \subset \text{ext } B_{\mathcal{L}_s(2l_\infty^{n+1})}$ holds for every $n \geq 2$.*

Proof. Let $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty^n)}$. Suppose that $T = \frac{1}{2}(S_1 + S_2)$ for some $S_1, S_2 \in \mathcal{L}_s(2l_\infty^{n+1})$ with $\|S_1\| = 1 = \|S_2\|$. Notice that, for $i = 1, 2$, we have $S_i \left((x_1, \dots, x_n, 0), (y_1, \dots, y_n, 0) \right) \in \mathcal{L}_s(2l_\infty^n)$, $\|S_i \left((x_1, \dots, x_n, 0), (y_1, \dots, y_n, 0) \right)\| \leq 1$ and

$$T = \frac{1}{2} \left(S_1 \left((x_1, \dots, x_n, 0), (y_1, \dots, y_n, 0) \right) + S_2 \left((x_1, \dots, x_n, 0), (y_1, \dots, y_n, 0) \right) \right).$$

Since $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty^n)}$, $T \left((x_1, \dots, x_n), (y_1, \dots, y_n) \right) = S_1 \left((x_1, \dots, x_n, 0), (y_1, \dots, y_n, 0) \right)$. Hence,

$$S_1 \left((x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1}) \right) = T \left((x_1, \dots, x_n), (y_1, \dots, y_n) \right) \\ + b_1(x_1 y_{n+1} + x_{n+1} y_1) + b_2(x_2 y_{n+1} + x_{n+1} y_2) + \dots \\ + b_n(x_n y_{n+1} + x_{n+1} y_n) + c x_{n+1} y_{n+1}$$

and

$$S_2 \left((x_1, \dots, x_{n+1}), (y_1, \dots, y_{n+1}) \right) = T \left((x_1, \dots, x_n), (y_1, \dots, y_n) \right) \\ - b_1(x_1 y_{n+1} + x_{n+1} y_1) + b_2(x_2 y_{n+1} + x_{n+1} y_2) + \dots \\ + b_n(x_n y_{n+1} + x_{n+1} y_n) + c x_{n+1} y_{n+1}.$$

We claim that $0 = b_1 = \dots = b_n = c$.

By [35, Theorem 3.4], there exist $Z_1, \dots, Z_{\frac{n(n+1)}{2}} \in \mathcal{U}_{2,n}$ such that $|T(Z_k)| = 1$ for $1 \leq k \leq \frac{n(n+1)}{2}$. Write

$$Z_k = \left[\left((1, x_2^{(k)}, \dots, x_n^{(k)}), (1, y_2^{(k)}, \dots, y_n^{(k)}) \right) \right] \quad \text{for } k = 1, \dots, \frac{n(n+1)}{2}.$$

It follows that

$$\begin{aligned}
 1 &\geq \max \left\{ \left| S_i \left((1, x_2^{(k)}, \dots, x_n^{(k)}, x_{n+1}), (1, y_2^{(k)}, \dots, y_n^{(k)}, y_{n+1}) \right) \right| : \right. \\
 &\quad \left. |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2 \right\} \\
 &= \max \left\{ |T(Z_k)| + \left| b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \dots \right. \right. \\
 &\quad \left. \left. + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1} \right| : \right. \\
 &\quad \left. |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2 \right\} \\
 &= \max \left\{ 1 + \left| b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \dots \right. \right. \\
 &\quad \left. \left. + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1} \right| : \right. \\
 &\quad \left. |x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}, i = 1, 2 \right\},
 \end{aligned}$$

which imply that, for all $|x_{n+1}| \leq 1, |y_{n+1}| \leq 1, 1 \leq k \leq \frac{n(n+1)}{2}$,

$$0 = b_1(y_{n+1} + x_{n+1}) + b_2(x_2^{(k)}y_{n+1} + x_{n+1}y_2^{(k)}) + \dots + b_n(x_n^{(k)}y_{n+1} + x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1},$$

$$0 = b_1(-y_{n+1} - x_{n+1}) + b_2(-x_2^{(k)}y_{n+1} - x_{n+1}y_2^{(k)}) + \dots + b_n(-x_n^{(k)}y_{n+1} - x_{n+1}y_n^{(k)}) + cx_{n+1}y_{n+1},$$

which shows that $c = 0$ and for $1 \leq k \leq \frac{n(n+1)}{2}$,

$$0 = b_1 + b_2x_2^{(k)} + \dots + b_nx_n^{(k)}, \tag{1}$$

$$0 = b_1 + b_2y_2^{(k)} + \dots + b_ny_n^{(k)}.$$

By Lemma 1, there exist Z_{i_1}, \dots, Z_{i_n} for $1 \leq i_1 < \dots < i_n \leq \frac{n(n+1)}{2}$ such that $(1, x_2^{(i_1)}, \dots, x_n^{(i_1)}), \dots, (1, x_2^{(i_n)}, \dots, x_n^{(i_n)})$ are linearly independent in \mathbb{R}^n or $(1, y_2^{(i_1)}, \dots, y_n^{(i_1)}), \dots, (1, y_2^{(i_n)}, \dots, y_n^{(i_n)})$ are linearly independent in \mathbb{R}^n .

Let A, B be the $n \times n$ matrices such that

$$[Row(A)]_j = (1, x_2^{(i_j)}, \dots, x_n^{(i_j)}), \quad [Row(B)]_j = (1, y_2^{(i_j)}, \dots, y_n^{(i_j)})$$

for $1 \leq j \leq n$. By (1), we have

$$A(b_1, \dots, b_n)^t = (0, \dots, 0)^t, \quad B(b_1, \dots, b_n)^t = (0, \dots, 0)^t.$$

If $(1, x_2^{(i_1)}, \dots, x_n^{(i_1)}), \dots, (1, x_2^{(i_n)}, \dots, x_n^{(i_n)})$ are linearly independent in \mathbb{R}^n , then A is invertible and so $0 = b_1 = \dots = b_n$. Similarly, if $(1, y_2^{(i_1)}, \dots, y_n^{(i_1)}), \dots, (1, y_2^{(i_n)}, \dots, y_n^{(i_n)})$ are linearly independent in \mathbb{R}^n , then B is invertible and so $0 = b_1 = \dots = b_n$.

Hence, $S_1 = T = S_2$. Therefore, $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty^{n+1})}$. □

Theorem 2. *The inclusion $\bigcup_{n=1}^\infty \text{ext } B_{\mathcal{L}_s(2l_\infty^n)} \subset \text{ext } B_{\mathcal{L}_s(2l_\infty)}$ holds.*

Proof. Let $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty^n)}$ for some $n \geq 1$. Suppose that $T = \frac{1}{2}(S_1 + S_2)$ for some $S_1, S_2 \in \mathcal{L}_s(2l_\infty)$ with $\|S_1\| = 1 = \|S_2\|$. Write

$$T\left((x_1, \dots, x_n), (y_1, \dots, y_n)\right) = \sum_{1 \leq i, j \leq n} a_{ij}x_iy_j$$

and

$$S_k\left((x_1, \dots, x_n, \dots), (y_1, \dots, y_n, \dots)\right) = \sum_{i,j=1}^{\infty} b_{ij}^{(k)} x_i y_j \quad \text{for } k = 1, 2,$$

where $a_{ij}, b_{ij}^{(k)} \in \mathbb{R}$.

Claim. $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ if $i > n$ or $j > n$.

Let $m = \max\{i, j\} > n$ and $T_k := S_k|_{l_\infty^m \times l_\infty^m} \in \mathcal{L}_s(2l_\infty^m)$ for $k = 1, 2$. By Theorem 1, $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty^m)}$. Since $\|T_k\| \leq 1$ for all $k = 1, 2$ and $T = \frac{1}{2}(T_1 + T_2)$, $T = T_1 = T_2$. Hence, $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ and $a_{ls} = b_{ls}^{(1)} = b_{ls}^{(2)}$ for every $1 \leq l, s \leq n$. Hence, $T = S_1 = S_2$. Therefore, $T \in \text{ext } B_{\mathcal{L}_s(2l_\infty)}$. \square

Question. Is it true that $\bigcup_{n=1}^{\infty} \text{ext } B_{\mathcal{L}_s(2l_\infty^n)} = \text{ext } B_{\mathcal{L}_s(2l_\infty)}$?

Let $n \in \mathbb{N}$ with $n \geq 2$ and $P \in \mathcal{P}(2l_\infty^n)$. Write, for $(x_1, \dots, x_n) \in l_\infty^n$,

$$P(x_1, \dots, x_n) = \sum_{1 \leq i \leq n} a_i x_i^2 + \sum_{1 \leq j < k \leq n} b_{jk} x_j x_k$$

for some $a_i, b_{jk} \in \mathbb{R}$.

In [5], it was shown that

$$\text{ext } B_{\mathcal{P}(2l_\infty^n)} = \left\{ \pm x_1^2, \pm x_2^2, \pm \left(ax_1^2 - ax_2^2 \pm 2\sqrt{a(1-a)}x_1x_2 \right) : \frac{1}{2} \leq a \leq 1 \right\}.$$

Theorem 3. The inclusion $\text{ext } B_{\mathcal{P}(2l_\infty^n)} \subset \text{ext } B_{\mathcal{P}(2l_\infty)}$ holds for every $n \geq 3$.

Proof. **Claim 1.** $P(x_1, \dots, x_n) = ax_1^2 - ax_2^2 + 2\sqrt{a(1-a)}x_1x_2 \in \text{ext } B_{\mathcal{P}(2l_\infty^n)}$ for $\frac{1}{2} \leq a \leq 1$.

Use induction on n .

Let $n = 3$. Suppose that $P = \frac{1}{2}(Q_1 + Q_2)$ for some $Q_1, Q_2 \in \mathcal{P}(2l_\infty^3)$ with $\|Q_1\| = 1 = \|Q_2\|$. Write

$$Q_1(x, y, z) = (a + c_1)x^2 - (a + c_2)y^2 + \left(2\sqrt{a(1-a)} + c_3\right)xy + d_1xz + d_2yz + d_3z^2$$

and

$$Q_2(x, y, z) = (a - c_1)x^2 - (a - c_2)y^2 + \left(2\sqrt{a(1-a)} - c_3\right)xy - d_1xz - d_2yz - d_3z^2$$

for some $c_j, d_j \in \mathbb{R}$ for $j = 1, 2, 3$.

Notice that $Q_j(x, y, 0) \in \mathcal{P}(2l_\infty^2)$, $\|Q_j(x, y, 0)\| = 1$, $P = \frac{1}{2}(Q_1(x, y, 0) + Q_2(x, y, 0))$ for $j = 1, 2$. Since $P(x, y) \in \text{ext } B_{\mathcal{P}(2l_\infty^2)}$, $P(x, y) = Q_1(x, y, 0)$, which shows that $c_j = 0$ for $j = 1, 2, 3$. Hence,

$$Q_1(x, y, z) = P(x, y) + d_1xz + d_2yz + d_3z^2 \quad \text{and} \quad Q_2(x, y, z) = P(x, y) - d_1xz - d_2yz - d_3z^2.$$

It follows that

$$\begin{aligned} 1 &\geq \max \left\{ \left| Q_j\left(1, \sqrt{\frac{1-a}{a}}, z\right) \right|, \left| Q_j\left(\sqrt{\frac{1-a}{a}}, -1, z\right) \right| : |z| \leq 1, j = 1, 2 \right\} \\ &= \max \left\{ 1 + \left| d_1z + d_2\sqrt{\frac{1-a}{a}}z + d_3z^2 \right|, 1 + \left| d_1\sqrt{\frac{1-a}{a}}z - d_2z + d_3z^2 \right| \right\}, \end{aligned}$$

which imply that, for all $|z| \leq 1$,

$$0 = d_1z + d_2\sqrt{\frac{1-a}{a}}z + d_3z^2, \quad 0 = d_1\sqrt{\frac{1-a}{a}}z - d_2z + d_3z^2.$$

Therefore, $d_j = 0$ for $j = 1, 2, 3$. Hence, $P(x, y) \in \text{ext } B_{\mathcal{P}(2l_\infty^3)}$ for $\frac{1}{2} \leq a \leq 1$.

Suppose that for $n = k$, $P(x, y) = ax^2 - ay^2 + 2\sqrt{a(1-a)}xy \in \text{ext } B_{\mathcal{P}(2l_\infty^k)}$ for $\frac{1}{2} \leq a \leq 1$. We will show that $P(x, y) \in \text{ext } B_{\mathcal{P}(2l_\infty^{k+1})}$ for $\frac{1}{2} \leq a \leq 1$. Suppose that $P = \frac{1}{2}(R_1 + R_2)$ for some $R_1, R_2 \in \mathcal{P}(2l_\infty^{k+1})$ with $\|R_1\| = 1 = \|R_2\|$. By the above argument, we may assume that

$$R_1(x, y, z_1, \dots, z_{k-1}) = P(x, y) + Axz_{k-1} + Byz_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1}$$

and

$$R_2(x, y, z_1, \dots, z_{k-1}) = P(x, y) - \left(Axz_{k-1} + Byz_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1} \right)$$

for some $A, B, C, D_j \in \mathbb{R}, 1 \leq j \leq k - 2$. It follows that

$$\begin{aligned} 1 &\geq \max \left\{ \left| R_j \left(1, \sqrt{\frac{1-a}{a}}, z_1, \dots, z_{k-1} \right) \right|, \left| Q_j \left(\sqrt{\frac{1-a}{a}}, -1, z_1, \dots, z_{k-1} \right) \right| : \right. \\ &\quad \left. |z_i| \leq 1, i = 1, \dots, k - 1, j = 1, 2 \right\} \\ &= \max \left\{ 1 + \left| Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1} \right|, \right. \\ &\quad \left. 1 + \left| A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1} \right| \right\}, \end{aligned}$$

which imply that, for all $|z_i| \leq 1, i = 1, \dots, k - 1$, we have

$$\begin{aligned} 0 &= Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1}, \\ 0 &= A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2 + \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1}. \end{aligned}$$

If $z_j = 0$ for $1 \leq j \leq k - 2$, then

$$0 = Az_{k-1} + B\sqrt{\frac{1-a}{a}}z_{k-1} + Cz_{k-1}^2, \quad 0 = A\sqrt{\frac{1-a}{a}}z_{k-1} - Bz_{k-1} + Cz_{k-1}^2,$$

which shows that $A = B = C = 0$. If $z_{k-1} = 1$, then

$$0 = \sum_{1 \leq j \leq k-2} D_jz_jz_{k-1} = \sum_{1 \leq j \leq k-2} D_jz_j \quad \text{for } |z_j| \leq 1, j = 1, \dots, k - 2,$$

which implies that $D_j = 0$ for $j = 1, \dots, k - 2$. Hence, $P(x, y) \in \text{ext } B_{\mathcal{P}(2l_\infty^{k+1})}$ for $\frac{1}{2} \leq a \leq 1$. We have shown Claim 1. Similarly, $-P \in \text{ext } B_{\mathcal{P}(2l_\infty^n)}$.

Claim 2. $P(x_1, \dots, x_n) = x_1^2 \in \text{ext } B_{\mathcal{P}(2l_\infty^n)}$.

Suppose that $P = \frac{1}{2}(Q_1 + Q_2)$ for some $Q_1, Q_2 \in \mathcal{P}(2l_\infty^n)$ with $\|Q_1\| = 1 = \|Q_2\|$. Since $1 = P(e_1) = \frac{1}{2}(Q_1(e_1) + Q_2(e_1))$, we have $1 = Q_1(e_1) = Q_2(e_1)$. Hence,

$$Q_1(x_1, \dots, x_n) = x_1^2 + \sum_{2 \leq i \leq n} a_i x_i^2 + \sum_{1 \leq j < k \leq n} b_{jk} x_j x_k$$

and

$$Q_2(x_1, \dots, x_n) = x_1^2 - \sum_{2 \leq i \leq n} a_i x_i^2 - \sum_{1 \leq j < k \leq n} b_{jk} x_j x_k$$

for some $a_i, b_{jk} \in \mathbb{R}$.

We will show that $a_i = 0 = b_{jk}$ for every $2 \leq i \leq n, 1 \leq j < k \leq n$. Let $2 \leq j \leq n$ be fixed. It follows that, for $i = 1, 2, |t| \leq 1$,

$$1 \geq |Q_i(e_1 + te_j)| = 1 + |a_j t^2 + b_{1j} t|,$$

which shows that $a_j t^2 + b_{1j} t = 0$ for $|t| \leq 1$, hence $a_j = b_{1j} = 0$ for $2 \leq j \leq n$. Let $2 \leq k < j \leq n$ be fixed. It follows that

$$1 \geq |Q_i(e_1 + e_k + e_j)| = 1 + |b_{kj}|,$$

which shows that $b_{kj} = 0$ for $2 \leq k < j \leq n$. We have shown Claim 2. Similarly, $-x_1^2, \pm x_2^2 \in \text{ext } B_{\mathcal{P}(2l_\infty^m)}$. Therefore, we complete the proof. \square

Corollary 1. *The inclusion $\text{ext } B_{\mathcal{P}(2l_\infty^m)} \subset \text{ext } B_{\mathcal{P}(2l_\infty)}$ holds.*

Proof. Let $P \in \text{ext } B_{\mathcal{P}(2l_\infty^m)}$. Suppose that $P = \frac{1}{2}(Q_1 + Q_2)$ for some $Q_1, Q_2 \in \mathcal{P}(2l_\infty)$ with $\|Q_1\| = 1 = \|Q_2\|$. Write

$$P((x_1, x_2)) = \sum_{i,j=1}^2 a_{ij} x_i x_j \quad \text{and} \quad Q_k((x_1, \dots, x_n, \dots)) = \sum_{i,j=1}^\infty b_{ij}^{(k)} x_i x_j \quad \text{for } k = 1, 2,$$

where $a_{ij}, b_{ij}^{(k)} \in \mathbb{R}$.

Claim. $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ if $i > 2$ or $j > 2$.

Let $m = \max\{i, j\} > 2$ and $R_k := Q_k|_{l_\infty^m} \in \mathcal{P}(2l_\infty^m)$ for $k = 1, 2$. By Theorem 3, $P \in \text{ext } B_{\mathcal{P}(2l_\infty^m)}$. Since $\|R_k\| \leq 1$ for all $k = 1, 2$ and $P = \frac{1}{2}(R_1 + R_2)$, $P = R_1 = R_2$. Hence, $b_{ij}^{(1)} = 0 = b_{ij}^{(2)}$ and $a_{ls} = b_{ls}^{(1)} = b_{ls}^{(2)}$ for every $1 \leq l, s \leq n$. Hence, $P = Q_1 = Q_2$. Therefore, $P \in \text{ext } B_{\mathcal{P}(2l_\infty)}$. \square

References

- [1] Aron R.M., Klimek M. *Supremum norms for quadratic polynomials*. Arch. Math. (Basel) 2001, **76** (1), 73–80. doi:10.1007/s000130050544
- [2] Cavalcante M., Pellegrino D., Teixeira E. *Geometry of multilinear forms*. Commun. Contemp. Math. 2020, **22** (2), article ID 1950011. doi:10.1142/S0219199719500111
- [3] Choi Y.S., Kim S.G., Ki H. *Extreme polynomials and multilinear forms on l_1* . J. Math. Anal. Appl. 1998, **228** (2), 467–482. doi:10.1006/jmaa.1998.6161
- [4] Choi Y.S., Kim S.G. *The unit ball of $\mathcal{P}(2l_2^2)$* . Arch. Math. (Basel) 1998, **71** (6), 472–480. doi:10.1007/s000130050292
- [5] Choi Y.S., Kim S.G. *Extreme polynomials on c_0* . Indian J. Pure Appl. Math. 1998, **29**, 983–989.
- [6] Choi Y.S., Kim S.G. *Smooth points of the unit ball of the space $\mathcal{P}(2l_1^2)$* . Results Math. 1999, **36** (1–2), 26–33. doi:10.1007/BF03322099
- [7] Choi Y.S., Kim S.G. *Exposed points of the unit balls of the spaces $\mathcal{P}(2l_p^2)$ ($p = 1, 2, \infty$)*. Indian J. Pure Appl. Math. 2004, **35**, 37–41.
- [8] Dineen S. *Complex Analysis on Infinite Dimensional Spaces*. Springer-Verlag, London, 1999.

- [9] Gámez-Merino J., Muñoz-Fernández G., Sánchez V., Seoane-Sepúlveda J. *Inequalities for polynomials on the unit square via the Krein–Milman Theorem*. J. Convex Anal. 2013, **20** (1), 125–142.
- [10] Grecu B.C. *Geometry of three-homogeneous polynomials on real Hilbert spaces*. J. Math. Anal. Appl. 2000, **246** (1), 217–229. doi:10.1006/jmaa.2000.6783
- [11] Grecu B.C. *Smooth 2-homogeneous polynomials on Hilbert spaces*. Arch. Math. (Basel) 2001, **76** (6), 445–454. doi:10.1007/PL00000456
- [12] Grecu B.C. *Geometry of 2-homogeneous polynomials on l_p spaces, $1 < p < \infty$* . J. Math. Anal. Appl. 2002, **273** (2), 262–282. doi:10.1016/S0022-247X(02)00217-2
- [13] Grecu B.C. *Extreme 2-homogeneous polynomials on Hilbert spaces*. Quaest. Math. 2002, **25** (4), 421–435. doi:10.2989/160736000209486027
- [14] Grecu B.C. *Geometry of homogeneous polynomials on two-dimensional real Hilbert spaces*. J. Math. Anal. Appl. 2004, **293** (2), 578–588. doi:10.1016/j.jmaa.2004.01.020
- [15] Grecu B.C., Muñoz-Fernández G., Seoane-Sepúlveda J. *The unit ball of the complex $P(^3H)$* . Math. Z. 2009, **263**, 775–785. doi:10.1007/s00209-008-0438-y
- [16] Kim S.G. *Exposed 2-homogeneous polynomials on L_p^2 , $1 \leq p \leq \infty$* . Math. Proc. R. Ir. Acad. 2007, **107A** (2), 123–129.
- [17] Kim S.G. *The unit ball of $\mathcal{L}_s(^2l_\infty^2)$* . Extracta Math. 2009, **24** (1), 17–29.
- [18] Kim S.G. *The unit ball of $\mathcal{P}(^2d_*(1, w)^2)$* . Math. Proc. R. Ir. Acad. 2011, **111A** (2), 77–92. doi:10.3318/pria.2011.111.1.9
- [19] Kim S.G. *The unit ball of $\mathcal{L}_s(^2d_*(1, w)^2)$* . Kyungpook Math. J. 2013, **53** (2), 295–306.
- [20] Kim S.G. *Smooth polynomials of $\mathcal{P}(^2d_*(1, w)^2)$* . Math. Proc. R. Ir. Acad. 2013, **113A** (1), 45–58.
- [21] Kim S.G. *Extreme bilinear forms of $\mathcal{L}(^2d_*(1, w)^2)$* . Kyungpook Math. J. 2013, **53** (4), 625–638.
- [22] Kim S.G. *Exposed symmetric bilinear forms of $\mathcal{L}_s(^2d_*(1, w)^2)$* . Kyungpook Math. J. 2014, **54** (3), 341–347.
- [23] Kim S.G. *Polarization and unconditional constants of $\mathcal{P}(^2d_*(1, w)^2)$* . Commun. Korean Math. Soc. 2014, **29** (3), 421–428. doi:10.4134/ckms.2014.29.3.421
- [24] Kim S.G. *Exposed bilinear forms of $\mathcal{L}(^2d_*(1, w)^2)$* . Kyungpook Math. J. 2015, **55** (1), 119–126.
- [25] Kim S.G. *Exposed 2-homogeneous polynomials on the two-dimensional real predual of Lorentz sequence space*. Mediterr. J. Math. 2016, **13** (5), 2827–2839. doi:10.1007/s00009-015-0658-4
- [26] Kim S.G. *The unit ball of $\mathcal{L}(^2\mathbb{R}_{h(w)}^2)$* . Bull. Korean Math. Soc. 2017, **54** (2), 417–428. doi:10.4134/bkms.b150851
- [27] Kim S.G. *Extremal problems for $\mathcal{L}_s(^2\mathbb{R}_{h(w)}^2)$* . Kyungpook Math. J. 2017, **57** (2), 223–232.
- [28] Kim S.G. *The unit ball of $\mathcal{L}_s(^2l_\infty^3)$* . Comment. Math. 2017, **57** (1), 1–7. doi:10.14708/cm.v57i1.1230
- [29] Kim S.G. *The geometry of $\mathcal{L}_s(^3l_\infty^2)$* . Commun. Korean Math. Soc. 2017, **32** (4), 991–997. doi:10.4134/ckms.c170016
- [30] Kim S.G. *Extreme 2-homogeneous polynomials on the plane with a hexagonal norm and applications to the polarization and unconditional constants*. Studia Sci. Math. Hungar. 2017, **54** (3), 362–393. doi:10.1556/012.2017.54.3.1371
- [31] Kim S.G. *The geometry of $\mathcal{L}(^3l_\infty^2)$ and optimal constants in the Bohnenblust–Hille inequality for multilinear forms and polynomials*. Extracta Math. 2018, **33** (1), 51–66. doi:10.17398/2605-5686.33.1.51
- [32] Kim S.G. *Extreme bilinear forms on \mathbb{R}^n with the supremum norm*. Period. Math. Hungar. 2018, **77** (2), 274–290. doi:10.1007/s10998-018-0246-z
- [33] Kim S.G. *Exposed polynomials of $\mathcal{P}(^2\mathbb{R}_{h(\frac{1}{2})}^2)$* . Extracta Math. 2018, **33** (2), 127–143. doi:10.17398/2605-5686.33.2.127
- [34] Kim S.G. *The unit ball of the space of bilinear forms on \mathbb{R}^3 with the supremum norm*. Commun. Korean Math. Soc. 2019, **34** (2), 487–494. doi:10.4134/ckms.c180111

- [35] Kim S.G. *Smooth points of $\mathcal{L}_s(^n l_\infty^2)$* . Bull. Korean Math. Soc. 2020, **57** (2), 443–447. doi:10.4134/bkms.b190311
- [36] Kim S.G. *Extreme points of the space $\mathcal{L}(^2 l_\infty)$* . Commun. Korean Math. Soc. 2020, **35** (3), 799–807. doi:10.4134/CKMS.c190300
- [37] Kim S.G. *Extreme and exposed points of $\mathcal{L}(^n l_\infty^2)$ and $\mathcal{L}_s(^n l_\infty^2)$* . Extracta Math. 2020, **35** (2), 127–135. doi:10.17398/2605-5686.35.2.127
- [38] Kim S.G. *Extreme and exposed symmetric bilinear forms on the space $\mathcal{L}_s(^2 l_\infty^2)$* . Carpathian Math. Publ. 2020, **12** (2), 340–352. doi:10.15330/cmp.12.2.340-352
- [39] Kim S.G. *Geometry of multilinear forms on \mathbb{R}^m with a certain norm*. Acta Sci. Math. (Szeged) 2021, **87** (1–2), 233–245. doi:10.14232/actasm-020-824-2
- [40] Kim S.G. *Geometry of multilinear forms on l_1* . Acta Comment. Univ. Tartu. Math. 2021, **25** (1), 57–66. In print.
- [41] Kim S.G., Lee S.H. *Exposed 2-homogeneous polynomials on Hilbert spaces*. Proc. Amer. Math. Soc. 2003, **131** (2), 449–453. doi:10.1090/S0002-9939-02-06544-9
- [42] Krein M.G., Milman D.P. *On extreme points of regular convex sets*. Studia Math. 1940, **9**, 133–137.
- [43] Milev L., Naidenov N. *Strictly definite extreme points of the unit ball in a polynomial space*. C. R. Acad. Bulgare Sci. 2008, **61** (11), 1393–1400.
- [44] Milev L., Naidenov N. *Semidefinite extreme points of the unit ball in a polynomial space*. J. Math. Anal. Appl. 2013, **405** (2), 631–641. doi:10.1016/j.jmaa.2013.04.026
- [45] Muñoz-Fernández G., Pellegrino D., Seoane-Sepúlveda J., Weber A. *Supremum norms for 2-homogeneous polynomials on circle sectors*. J. Convex Anal. 2014, **21** (3), 745–764.
- [46] Muñoz-Fernández G., Révész S.G., Seoane-Sepúlveda J. *Geometry of homogeneous polynomials on non symmetric convex bodies*. Math. Scand. 2009, **105** (1), 147–160. doi:10.7146/math.scand.a-15111
- [47] Muñoz-Fernández G., Seoane-Sepúlveda J. *Geometry of Banach spaces of trinomials*. J. Math. Anal. Appl. 2008, **340** (2), 1069–1087. doi:10.1016/j.jmaa.2007.09.010
- [48] Neuwirth S. *The maximum modulus of a trigonometric trinomial*. J. Anal. Math. 2008, **104** (1), 371–396. doi:10.1007/s11854-008-0028-2
- [49] Ryan R.A., Turett B. *Geometry of spaces of polynomials*. J. Math. Anal. Appl. 1998, **221** (2), 698–711. doi:10.1006/jmaa.1998.5942

Received 23.09.2020

Кім С.Г. *Екстремальні точки просторів $\mathcal{L}_s(^2 l_\infty)$ та $\mathcal{P}(^2 l_\infty)$* // Карпатські матем. публ. — 2021. — Т.13, №2. — С. 289–297.

Ми доводимо, що для $n \geq 2$ кожна екстремальна точка одиничної кулі простору $\mathcal{L}_s(^2 l_\infty)$ є екстремальною в $\mathcal{L}_s(^2 l_\infty^{n+1})$. Це дає відповідь на питання, поставлене в [Period. Math. Hungar. 2018, **77** (2), 274–290]. Як наслідок, ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{L}_s(^2 l_\infty^n)$ є екстремальною в $\mathcal{L}_s(^2 l_\infty)$. Також ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{P}(^2 l_\infty^2)$ є екстремальною в $\mathcal{P}(^2 l_\infty^n)$. Як наслідок, ми показуємо, що кожна екстремальна точка одиничної кулі простору $\mathcal{P}(^2 l_\infty^2)$ є екстремальною в $\mathcal{P}(^2 l_\infty)$.

Ключові слова і фрази: екстремальна точка, симетрична білінійна форма, двоходнорідні поліноми на просторі l_∞ .