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Some results concerning localization property of generalized Herz, Herz-type Besov spaces and Herz-type Triebel-Lizorkin spaces

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In this paper, based on generalized Herz-type function spaces $\dot{K}_q^p(\theta)$ were introduced by Y. Komori and K. Matsuoka in 2009, we define Herz-type Besov spaces $\dot{K}_q^p B_\beta^s(\theta)$ and Herz-type Triebel-Lizorkin spaces $\dot{K}_q^p F_\beta^s(\theta)$, which cover the Besov spaces and the Triebel-Lizorkin spaces in the homogeneous case, where $\theta = \{\theta(k)\}_{k\in\mathbb{Z}}$ is a sequence of non-negative numbers such that

$$C^{-1}2^{\delta(k-j)} \leq \frac{\theta(k)}{\theta(j)} \leq C2^{\alpha(k-j)}, \quad k > j,$$

for some $C \geq 1$ (α and δ are numbers in $\mathbb{R}$).

Further, under the condition mentioned above on θ, we prove that $\dot{K}_q^p(\theta)$ and $\dot{K}_q^p B_\beta^s(\theta)$ are localizable in the ℓ_q-norm for $p = q$, and $\dot{K}_q^p F_\beta^s(\theta)$ is localizable in the ℓ_q-norm, i.e. there exists $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying $\sum_{k\in\mathbb{Z}^n} \varphi(x-k) = 1$, for any $x \in \mathbb{R}^n$, such that

$$\|f|E\| \approx \Big(\sum_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k)\cdot f|E\|^q\Big)^{1/q}.$$

Results presented in this paper improve and generalize some known corresponding results in some function spaces.

Key words and phrases: generalized Herz space, Herz-type Besov space, Herz-type Triebel-Lizorkin space, localization property.

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1 Introduction and preliminaries

As usual, $\mathbb{R}^n$ is the n-dimensional real Euclidean space, $\mathbb{N}$ is the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers.

For any $u > 0$, $k \in \mathbb{Z}$ we set $C(u) = \{x \in \mathbb{R}^n : u/2 < |x| \leq u\}$ and $C_k = C(2^k)$. For $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B(x,r)$ the open ball in $\mathbb{R}^n$ with center x and radius r. Let χ_k, for $k \in \mathbb{Z}$, denote the characteristic function of the set C_k.

As usual, $L^p(\mathbb{R}^n)$ for $0 < p \leq \infty$ stands for the Lebesgue spaces on $\mathbb{R}^n$ normed by (quasi-normed for $p < 1$)

$$\|f|L^p(\mathbb{R}^n)\| = \|f\|_p = \Big(\int_{\mathbb{R}^n} |f(x)|^p\,dx\Big)^{1/p} < \infty, \quad 0 < p < \infty,$$

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and

$$\|f|L^{\infty}(\mathbb{R}^n)\| = \|f\|_{\infty} = \operatorname*{ess\text{-}sup}_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

By ℓ_q, $0 < q \le \infty$, we denote the space of all (complex) sequences $\{a_k\}_{k\in\mathbb{Z}}$ equipped with the quasi-norm

$$\left\|\{a_k\}_{k\in\mathbb{Z}}\,|\ell_q\right\| = \left(\sum_{k=-\infty}^{\infty} |a_k|^q\right)^{1/q}$$

(with the usual modification if $q = \infty$).

Given two quasi-Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous. We use c as a generic positive constant, i.e. a constant whose value may change from appearance to appearance.

By $\mathcal{S}(\mathbb{R}^n)$ we denote the Schwartz space of all complex-valued, infinitely differentiable and rapidly decreasing functions on $\mathbb{R}^n$. The topology in the complete locally convex space $\mathcal{S}(\mathbb{R}^n)$ is generated by the norms

$$p_N(\varphi) = \sup_{x\in\mathbb{R}^n} (1+|x|)^N \sum_{|\alpha|\le N} |D^{\alpha}\varphi(x)|, \quad N = 1, 2, 3, \dots .$$

By $\mathcal{S}'(\mathbb{R}^n)$ we denote the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)dx.$$

Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

It is well known that the localization property was first introduced by G. Bourdaud (see [1]) and under some conditions he proved that Besov spaces is localizable in the ℓ_p norm. N. Ferahtia and S. Allaoui (see [4]) generalized the Bourdaud theorem of a localization property of Besov spaces $B^s_{p,q}$ on the ℓ_r space, where $r \in [1, +\infty]$.

Recently, the localization property of some function spaces have attracted great attention (see [11, 13, 20]).

In this paper, we define Herz-type Besov spaces $\dot{K}^p_q B^s_{\beta}(\theta)$ and Herz-type Triebel-Lizorkin spaces $\dot{K}^p_q F^s_{\beta}(\theta)$ which covers Besov spaces and Triebel-Lizorkin spaces in the homogeneous case. Notice that these spaces based on generalized Herz-type function spaces $\dot{K}^p_q(\theta)$ were introduced by Y. Komori and K. Matsuoka in [8]. After this, we treat and discuss the localization property of these spaces and then we compare our results with existing ones.

2 Function spaces

We start by recalling the definition and some properties of the generalized Herz spaces.

Definition 1. *Let $\alpha, \delta \in \mathbb{R}$. A sequence of numbers $\theta = \{\theta(k)\}_{k\in\mathbb{Z}}$ belongs to the class $\mathcal{A}(\alpha, \delta)$ if and only if*

(i) $\theta(k) > 0$ for all $k \in \mathbb{Z}$;

(ii) there exists a constant $C \ge 1$ such that

$$C^{-1}2^{\delta(k-j)} \le \frac{\theta(k)}{\theta(j)} \le C2^{\alpha(k-j)} \tag{1}$$

for $k > j$.

The size condition (1) in the above definition can be satisfied by many sequences of numbers such as:

$$\theta = \left\{2^{\mu k}\right\}_{k\in\mathbb{Z}} \in \mathcal{A}(\alpha,\delta) \text{ for } \delta \le \mu \le \alpha, \qquad \theta = \left\{2^{\lambda k}(1+\max(0,k\ln 2))\right\}_{k\in\mathbb{Z}} \in \mathcal{A}(\lambda,\lambda+1).$$

Definition 2. *Let $\theta \in \mathcal{A}(\alpha,\delta)$ and $0 < p, q \le \infty$. The generalized Herz space $\dot{K}_q^p(\theta)$ is defined by*

$$\dot{K}_q^p(\theta) := \{f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f|\dot{K}_q^p(\theta)\| < \infty\},$$

where

$$\|f|\dot{K}_q^p(\theta)\| = \Big(\sum_{k=-\infty}^{\infty} \theta^p(k)\, \|f\chi_k|L^q\|^p\Big)^{1/p},$$

with the usual modifications made when $p=\infty$ and/or $q=\infty$.

The spaces $\dot{K}_p^q(\theta)$ were first defined by Y. Komori and K. Matsuoka [8] and under the condition above, the authors studied the boundedness of singular integral operators and fractional integral operators on these spaces.

The Definition 2 coincide with the classical definition of Herz spaces for the case of the particular function, i.e

$$\dot{K}_q^p(\theta) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n) \quad \text{if} \quad \theta \in \mathcal{A}(\alpha,\alpha).$$

The spaces $\dot{K}_q^p(\theta)$ are quasi-Banach spaces and if $\min(p,q) \ge 1$ then $\dot{K}_q^p(\theta)$ are Banach spaces. If $\theta \in \mathcal{A}(0,0)$ and $0 < p = q \le \infty$ then $\dot{K}_p^p(\theta)$ coincides with the Lebesque spaces $L^p(\mathbb{R}^n)$. A detailed discussion of the properties of $\dot{K}_q^p(\theta)$, where $\theta \in \mathcal{A}(\alpha,\alpha)$, may be found in the papers [6,7,9,10], and references therein.

Next, we present the Fourier analytical definition of Herz-type Besov spaces $\dot{K}_q^p B_\beta^s(\theta)$ and Herz-type Triebel-Lizorkin spaces $\dot{K}_q^p F_\beta^s(\theta)$ and recall their basic properties. We first need the concept of a smooth dyadic resolution of unity.

Definition 3. *Let Ψ be a function in $\mathcal{S}(\mathbb{R}^n)$ satisfying $\Psi(x)=1$ for $|x|\le 1$ and $\Psi(x)=0$ for $|x| \ge \frac{3}{2}$. We put $\varphi_0(x)=\Psi(x)$, $\varphi_1(x)=\Psi(x/2)-\Psi(x)$ and*

$$\varphi_j(x) = \varphi_1(2^{-j+1}x) \quad \textit{for} \quad j = 2,3,\dots.$$

Then we have $\operatorname{supp}\varphi_j \subset \left\{x\in\mathbb{R}^n : 2^{j-1} \le |x| \le 3\cdot 2^{j-1}\right\}$, *$\varphi_j(x)=1$ for $3\cdot 2^{j-2} \le |x| \le 2^j$ and $\sum_{j=0}^\infty \varphi_j(x) = 1$ for all $x\in\mathbb{R}^n$. The system of functions $\left\{\varphi_j\right\}_{j\in\mathbb{N}_0}$ is called a smooth dyadic resolution of unity. We define the convolution operators Δ_j as follows*

$$\Delta_j f = \mathcal{F}^{-1}\varphi_j * f, \quad j\in\mathbb{N}, \quad \textit{and} \quad \Delta_0 f = \mathcal{F}^{-1}\Psi * f, \quad f \in \mathcal{S}'(\mathbb{R}^n).$$

Thus we obtain the Littlewood-Paley decomposition $f = \sum_{j=0}^\infty \Delta_j f$ of all $f\in\mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

We are now ready to state the definitions of Herz-type Besov and Triebel-Lizorkin spaces.

Definition 4. (i) *Let $\theta\in\mathcal{A}(\alpha,\delta)$, $s\in\mathbb{R}$, and $0<p,q,\beta\le\infty$. The generalized Herz-type Besov space $\dot{K}_q^p B_\beta^s(\theta)$ is the collection of all $f\in\mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f|\dot{K}_q^p B_\beta^s(\theta)\| = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \|\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{1/\beta} < \infty,$$

with the obvious modification if $\beta = \infty$.

(ii) *Let $\theta \in \mathcal{A}(\alpha,\delta)$, $s \in \mathbb{R}$, $0 < p, q < \infty$ and $0 < \beta \leq \infty$. The generalized Herz-type Triebel-Lizorkin space $\dot{K}^p_q F^s_\beta(\theta)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f|\dot{K}^p_q F^s_\beta(\theta)\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} |\Delta_j f|^\beta \Big)^{1/\beta} |\dot{K}^p_q(\theta) \Big\| < \infty,$$

with the obvious modification if $\beta = \infty$.

Observing that, if $\theta \in \mathcal{A}(\alpha,\alpha)$ then $\dot{K}^p_q B^s_\beta(\theta) = \dot{K}^{\alpha,p}_q B^s_\beta(\mathbb{R}^n)$ (resp., $\dot{K}^p_q F^s_\beta(\theta) = \dot{K}^{\alpha,p}_q F^s_\beta$) are the classical Herz-type Besov spaces (resp., the classical Herz-type Triebel-Lizorkin spaces). The spaces $\dot{K}^p_q B^s_\beta(\theta)$ and $\dot{K}^p_q F^s_\beta(\theta)$ are quasi-Banach spaces and if $p, q, \beta \geq 1$, then both $\dot{K}^p_q B^s_\beta(\theta)$ and $\dot{K}^p_q F^s_\beta(\theta)$ are Banach spaces. Further results, concerning, for instance, lifting properties, Fourier multiplier and local means characterizations can be found in [17–19].

Now we give the definitions of the spaces $B^s_{p,\beta}$ and $F^s_{p,\beta}$.

Definition 5. (i) *Let $s \in \mathbb{R}$ and $0 < p, \beta \leq \infty$. The Besov space $B^s_{p,\beta}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f|B^s_{p,\beta}(\mathbb{R}^n)\| = \Big(\sum_{j=0}^{\infty} 2^{js\beta} \|\Delta_j f|L^p(\mathbb{R}^n)\|^\beta \Big)^{1/\beta} < \infty.$$

(ii) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < \beta \leq \infty$. The Triebel-Lizorkin space $F^s_{p,\beta}(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f|F^s_{p,\beta}(\mathbb{R}^n)\| = \Big\| \Big(\sum_{j=0}^{\infty} 2^{js\beta} |\Delta_j f|^\beta \Big)^{1/\beta} |L^p(\mathbb{R}^n) \Big\| < \infty.$$

The theory of the spaces $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,\beta}(\mathbb{R}^n)$ has been developed in detail in [14–16], but has a longer history already including many contributors; we do not want to discuss this here. In particular, with $p = q = \infty$, $s > 0$, one recovers Hölder-Zygmund spaces $\mathcal{C}^s = B^s_{\infty,\infty}$, cf. [14, Theorem 2.5.12]. Clearly, for $\theta \in \mathcal{A}(0,0)$, $s \in \mathbb{R}$, $0 < p \leq \infty$ ($0 < p < \infty$ for the $\dot{K}^p_p F^s_\beta(\theta)$ spaces) and $0 < \beta \leq \infty$,

$$\dot{K}^p_p B^s_\beta(\theta) = B^s_{p,\beta}(\mathbb{R}^n) \quad \text{and} \quad \dot{K}^p_p F^s_\beta(\theta) = F^s_{p,\beta}(\mathbb{R}^n).$$

For the proof of the localization property of Herz-type Besov and Herz-type Triebel-Lizorkin spaces, we need the following proposition, see [2, Proposition 3.5].

Proposition 1. *Let $\theta \in \mathcal{A}(\alpha,\delta)$, $s \in \mathbb{R}$ and $1 \leq p, q, \beta \leq \infty$ such that $-n/q < \alpha, \delta < n(1-1/q)$. For all $\gamma, \rho > 1$, there exists $c > 0$ such that for any sequence $\{g_l\}_{l\in\mathbb{N}_0}$ of functions, where*

$$\operatorname{supp}\mathcal{F}g_0 \subset \{\xi : |\xi| \leq \rho\} \qquad \text{and} \qquad \operatorname{supp}\mathcal{F}g_l \subset \Big\{\xi : \gamma^{-1}2^l \leq |\xi| \leq \gamma 2^l\Big\},$$

we have

$$\Big\| \sum_{l=0}^{\infty} g_l |\dot{K}^p_q B^s_\beta(\theta) \Big\| \leq c \Big(\sum_{l=0}^{\infty} 2^{ls\beta} \|g_l|\dot{K}^p_q(\theta)\|^\beta \Big)^{1/\beta}$$

and

$$\Big\| \sum_{l=0}^{\infty} g_l |\dot{K}^p_q F^s_\beta(\theta) \Big\| \leq c \Big\| \Big(\sum_{l=0}^{\infty} 2^{ls\beta} |g_l|^\beta \Big)^{1/\beta} |\dot{K}^p_q(\theta) \Big\|, \quad 1 \leq p,\ q < \infty.$$

Before the proof of Proposition 1, we need some technical lemmas. The following assertion is the $\dot{K}_q^p(\theta)$-version of lemma by J. Franke [5].

Lemma 1. *Let $\theta \in \mathcal{A}(\alpha,\delta)$, $1 < p,\ q \leq \infty$ and $\gamma,\ \rho > 1$. For any sequence $\{g_l\}_{l\in\mathbb{N}_0} \subset \mathcal{S}'(\mathbb{R}^n) \cap \dot{K}_q^p(\theta)$ with*

$$\text{supp}\mathcal{F}g_0 \subset \{\xi : |\xi| \leq \rho\} \qquad \textit{and} \qquad \text{supp}\mathcal{F}g_l \subset \left\{\xi : \gamma^{-1}2^l \leq |\xi| \leq \gamma 2^l\right\},$$

we have

$$\|\Delta_j g_l|\dot{K}_q^p(\theta)\| \leq c\|g_l|\dot{K}_q^p(\theta)\|.$$

The constant $c > 0$ is independent of j and l.

For the proof of this lemma, we can repeat arguments similar to the ones used in the proof of Lemma 3.3 in [2].

Lemma 2. *Let $0 < b < 1$ and $0 < q \leq \infty$. Let $\{\varepsilon_j\}_{j\in\mathbb{Z}}$ be a sequence of real positive numbers in ℓ_q. Then there exists a constant $c > 0$ depending only on b and q such that*

$$\Big\|\Big\{\sum_{j=-\infty}^{k} b^{(k-j)}\varepsilon_j\Big\}_{k\in\mathbb{Z}}|\ell_q\Big\| + \Big\|\Big\{\sum_{j=k}^{\infty} b^{(j-k)}\varepsilon_j\Big\}_{k\in\mathbb{Z}}|\ell_q\Big\| \leq c\Big\|\{\varepsilon_k\}_{k\in\mathbb{Z}}|\ell_q\Big\|.$$

The proof of Lemma 2 is immediate by using Young's inequality in ℓ_q.

Proof of Proposition 1. By similarity we prove only the Herz-type Besov case. We observe that there existe $H_1 = [\log_2 2\gamma]$ and $H_2 = [\log_2((3\gamma)/2)]$ in $\mathbb{N}$ such that

$$\Delta_j g_l = 0 \quad \text{if} \quad l \geq j + H_2 \quad \text{or} \quad l \leq j + H_1.$$

Observe that

$$\Delta_j\Big(\sum_{l=0}^{\infty} g_l\Big) = \sum_{l=j-H_1}^{j+H_2} \Delta_j g_l.$$

Therefore,

$$\Big\|\sum_{l=0}^{\infty} g_l|\dot{K}_q^p B_\beta^s(\theta)\Big\| \leq \Big(\sum_{j=0}^{\infty} 2^{js\beta}\Big(\sum_{l=j-H_1}^{j+H_2}\|\Delta_j g_l|\dot{K}_q^p(\theta)\|\Big)^\beta\Big)^{1/\beta}.$$

Now, according to sign of s and by Lemma 1, we separate the cases.

1. The case $s > 0$. We obtain

$$\sum_{l=j-H_1}^{j+H_2} 2^{sj}\|\Delta_j g_l|\dot{K}_q^p(\theta)\| \leq c\sum_{l=j}^{\infty} 2^{sj}2^{-sl}\Big(2^{sl}\|\Delta_{j+H_1}g_l|\dot{K}_q^p(\theta)\|\Big) \leq c\,2^{sj}\sum_{l=j}^{\infty}2^{-sl}\Big(2^{sl}\|g_l|\dot{K}_q^p(\theta)\|\Big).$$

2. The case $s < 0$. Similarly

$$\sum_{l=j-H_1}^{j+H_2} 2^{sj}\|\Delta_j g_l|\dot{K}_q^p(\theta)\| \leq c\sum_{l=0}^{j} 2^{sj}2^{-sl}\Big(2^{sl}\|\Delta_{j-H_2}g_l|\dot{K}_q^p(\theta)\|\Big) \leq c\,2^{js}\sum_{l=0}^{j}2^{-sj}\Big(2^{sl}\|g_l|\dot{K}_q^p(\theta)\|\Big).$$

3. If $s = 0$, we immediately get

$$\sum_{l=j-H_1}^{j+H_2}\|\Delta_j g_l|\dot{K}_q^p(\theta)\| \leq c\sum_{l=j-H_1}^{j+H_2} 1\,\|g_l|\dot{K}_q^p(\theta)\| \leq c\Big(\sum_{l=j-H_1}^{j+H_2} 1\Big)^{1/\beta'}\Big(\sum_{l=j-H_1}^{j+H_2}\big(\|g_l|\dot{K}_q^p(\theta)\|\big)^\beta\Big)^{1/\beta}.$$

Finally, we apply ℓ_β-norm and Lemma 2, and obtain

$$\Big\|\sum_{l=0}^{\infty} g_l|\dot{K}_q^p B_\beta^s(\theta)\Big\| \leq c\Big(\sum_{l=0}^{\infty} 2^{ls\beta}\|g_l|\dot{K}_q^p(\theta)\|^\beta\Big)^{1/\beta}.$$

□

3 Localization of Herz-type Besov spaces

In this section, we present three results concerning the localization property of generalized Herz, Herz-type Besov and Herz-type Triebel-Lizorkin spaces on the ℓ_r spaces.

We first need the concept of a localization spaces. Let E be a Banach space of distributions. We associate on the space E the following hypothesis.

(1) Translation invariance: if τ_k denotes the operator given by $\tau_k f(t) = f(t-k)$, then τ_k is an isometry of E.

(2) Localization invariance: for all $f \in E$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have that $\varphi \cdot f \in E$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. The notion of localized is defined by $f_x = \tau_x \varphi \cdot f$, it follows immediately from the hypothesis (1) and (2) that the family $(f_x)_{x\in\mathbb{R}^n}$ is bounded in E. We consider the set A as the class of all the functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying

$$\operatorname{supp}\varphi \subset B(0,R) \quad \text{with} \quad R > \sqrt{n},$$

and

$$\sum_{k\in\mathbb{Z}^n} \varphi(x-k) = 1, \quad \forall x \in \mathbb{R}^n \setminus \{0\}. \tag{2}$$

Definition 6. *Let E be a Banach space of distributions, E is localizable in the ℓ_r norm, $1 \le r \le \infty$, if there exist $\varphi \in A$ and a constant $c \ge 1$ such that*

$$\frac{1}{c}\|f|E\| \le \|f|(E)_{\ell_r}\| = \Big(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi \cdot f|E\|^r\Big)^{1/r} \le c\,\|f|E\|.$$

Remark 1. *Let $f \in (E)_{\ell_r}$. If $g \in \mathcal{S}$ such that $g(x) \neq 0$ for $x \in \operatorname{supp}\varphi$. Then the following expression*

$$\Big(\sum_{k\in\mathbb{Z}^n} \|\tau_k g \cdot f|E\|^r\Big)^{1/r}$$

defines an equivalent norm in $(E)_{\ell_r}$, cf. [1, Proposition 5, p. 156].

The following result play a fundamental role in the proof of Theorem 3.

Lemma 3. *Let $\theta \in \mathcal{A}(\alpha,\delta)$ and $1 \le p,\, q \le \infty$. There exists a constant $c > 0$ such that the inequality*

$$\Big\|\sum_{k\in\mathbb{Z}^n} f_k|\dot{K}^p_q(\theta)\Big\| \le c\Big(\sum_{k\in\mathbb{Z}^n}\Big\|f_k|\dot{K}^p_q(\theta)\Big\|^p\Big)^{1/p}$$

holds, for all $R > 1$ and for all family $\{f_k\}_{k\in\mathbb{Z}^n}$ from $\mathcal{S}'$ with $\operatorname{supp} f_k$ contained in the ball $|x-k| \le R$.

Proof. For any sequence $\{f_k\}_{k\in\mathbb{Z}^n}$, let us write $\sum_{k\in\mathbb{Z}^n} f_k$ as in (2), namely

$$\sum_{k\in\mathbb{Z}^n} f_k = \sum_{k\in\mathbb{Z}^n} \varphi(x-k)\, f_k = \Lambda_\varphi\left(\{f_k\}_{k\in\mathbb{Z}^n}\right),$$

where $\varphi \in \mathcal{D}(\mathbb{R}^n)$ is chosen such that $\varphi = 1$ on the ball $|x| \le R$. We want to show that Λ_φ is bounded operator from $\ell_1(\dot{K}^p_1(\theta))$ to $\dot{K}^p_1(\theta)$ and from $\ell_\infty(\dot{K}^p_\infty(\theta))$ to $\dot{K}^p_\infty(\theta)$.

Consider $\left\|\Lambda_\varphi\left(\{f_k\}_{k\in\mathbb{Z}^n}\right)|\dot{K}_1^p(\theta)\right\|$ first. By Minkowski's inequality, Hölder's inequality and since $p \geq 1$, we have

$$\begin{aligned}
\left\|\Lambda_\varphi\left(\{f_k\}_{k\in\mathbb{Z}^n}\right)|\dot{K}_1^p(\theta)\right\| &= \Big(\sum_{l=-\infty}^{\infty}\theta^p(l)\Big\|\sum_{k\in\mathbb{Z}^n}\varphi\left(\cdot-k\right)f_k\cdot\chi_l|L^1\Big\|^p\Big)^{1/p} \\
&\leq \Big\|\left\|\theta(l)\varphi\left(\cdot-k\right)f_k\cdot\chi_l|L^1\right\||\ell_p\left(\ell_1\right)\Big\| \\
&\leq \left\|\varphi|L^\infty\right\|\Big\|\left\|\theta(l)f_k\cdot\chi_l|L^1\right\||\ell_p\left(\ell_1\right)\Big\| \\
&\leq \Big\|\left\|\theta(l)f_k\cdot\chi_l|L^1\right\||\ell_1\left(\ell_p\right)\Big\| \\
&\leq \sum_{k\in\mathbb{Z}^n}\left\|f_k|\dot{K}_1^p(\theta)\right\| = \Big\|\left\|\left(f_k\right)_{k\in\mathbb{Z}^n}|\dot{K}_1^p(\theta)\right\||\ell_1\Big\|.
\end{aligned}$$

We now consider $\left\|\Lambda_\varphi\left(\{f_k\}_{k\in\mathbb{Z}^n}\right)|\dot{K}_\infty^p(\theta)\right\|$. By Hölder's inequality, we have

$$\begin{aligned}
\left\|\Lambda_\varphi\left(\{f_k\}_{k\in\mathbb{Z}^n}\right)|\dot{K}_\infty^p(\theta)\right\| &= \Big(\sum_{l=-\infty}^{\infty}\theta^p(l)\Big\|\sum_{k\in\mathbb{Z}^n}\varphi\left(\cdot-k\right)f_k\cdot\chi_l|L^\infty\Big\|^p\Big)^{1/p} \\
&\leq \Big(\sum_{l=-\infty}^{\infty}\theta^p(l)\Big\|\sup_{k\in\mathbb{Z}^n}f_k\cdot\chi_l|L^\infty\Big\|^p\Big\|\sum_{k\in\mathbb{Z}^n}\varphi\left(\cdot-k\right)|L^\infty\Big\|^p\Big)^{1/p} \\
&\leq \Big\|\left\|\left(f_k\right)_{k\in\mathbb{Z}^n}|\dot{K}_\infty^p(\theta)\right\||\ell_\infty\Big\|.
\end{aligned}$$

Since $\frac{1}{p} \in (0,1)$, by the complex interpolation theory established in [3, Theorem 4.1] and [12, p. 121/4]), we have

$$[\ell_1(\dot{K}_1^1(\theta)),\ell_\infty(\dot{K}_\infty^\infty(\theta))]_{\frac{1}{p}} = \ell_p([\dot{K}_1^1(\theta),\dot{K}_\infty^\infty(\theta)]_{\frac{1}{p}}) = \ell_p(\dot{K}_p^p(\theta)).$$

This finishes the proof of the lemma. □

The following result gives the localization property of generalized Herz spaces on the ℓ_r spaces.

Theorem 1. *Let $\theta \in \mathcal{A}(\alpha,\delta)$, $1 < p,q \leq \infty$. Then*

(i) $\dot{K}_q^p(\theta) \hookrightarrow (\dot{K}_q^p(\theta))_{\ell_r}$ *for* $r \geq \max(p,q)$,

(ii) $(\dot{K}_q^p(\theta))_{\ell_r} \hookrightarrow \dot{K}_q^p(\theta)$ *for* $r \leq \min(p,q)$.

In particular, $\dot{K}_q^q(\theta)$ space is localizable in the ℓ_q-norm.

Proof. (i) We must show that

$$\left\|f|(\dot{K}_q^p(\theta))_{\ell_r}\right\| \leq c\left\|f|\dot{K}_q^p(\theta)\right\|$$

for all $f \in \dot{K}_q^p(\theta)$. We have

$$\left\|\tau_k\varphi\cdot f|\dot{K}_q^p(\theta)\right\| = \Big\|\sum_{l=-\infty}^{\infty}\tau_k\varphi\cdot f\cdot\chi_l|\dot{K}_q^p(\theta)\Big\| = \Big(\sum_{k=-\infty}^{\infty}\theta^p(k)\Big\|\sum_{l=-\infty}^{\infty}\tau_k\varphi\cdot f\cdot\chi_l\cdot\chi_k|L^q\Big\|^p\Big)^{1/p}.$$

We observe that $\chi_l \cdot \chi_k \neq 0$, when $|l-k|<1$, which means $l=k$. We have

$$\left\|\tau_k\varphi \cdot f|\dot{K}_q^p(\theta)\right\| \leq \Big(\sum_{l=-\infty}^{\infty} \theta^p(l)\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^p\Big)^{1/p},$$

this implies

$$\left\|f|(\dot{K}_q^p(\theta))_{\ell_r}\right\| = \Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f|\dot{K}_q^p(\theta)\right\|^r\Big)^{1/r} \leq \Big(\sum_{k\in\mathbb{Z}^n}\Big(\sum_{l=-\infty}^{\infty} \theta^p(l)\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^p\Big)^{r/p}\Big)^{1/r}.$$

Since $r \geq \max(p,q)$ implies $r \geq p$, from Minkowski inequality we have

$$\left\|f|(\dot{K}_q^p(\theta))_{\ell_r}\right\| \leq \Big(\sum_{l=-\infty}^{+\infty} \theta^p(l)\Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^r\Big)^{p/r}\Big)^{1/p}. \tag{3}$$

Since $r \geq \max(p,q) \geq q$, by the embedding $\ell_q \hookrightarrow \ell_r$ and the localization of L^q in ℓ^q spaces we obtain

$$\Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^r\Big)^{1/r} \leq \Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^q\Big)^{1/q} \leq c\left\|f \cdot \chi_l|L^q\right\|,$$

the right hand side inequality of (3) is bounded by

$$\left\|f|(\dot{K}_q^p(\theta))_{\ell_r}\right\| \leq c\Big(\sum_{l=-\infty}^{\infty} \theta^p(l)\left\|f \cdot \chi_l|L^q\right\|^p\Big)^{1/p} = c\left\|f|\dot{K}_q^p(\theta)\right\|.$$

(ii) By the localization of L^q spaces in the ℓ_q norm (with $r \leq \min(p,q) \leq q$), we have

$$\begin{aligned}\left\|f \cdot \chi_l|L^q\right\| = \Big\|\sum_{k\in\mathbb{Z}^n}\tau_k\varphi \cdot f \cdot \chi_l|L^q\Big\| &\leq c\Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^q\Big)^{1/q} \\ &\leq c\Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^r\Big)^{1/r},\end{aligned}$$

this implies that

$$\left\|f|\dot{K}_q^p(\theta)\right\| = \Big(\sum_{l=-\infty}^{\infty} \theta^p(l)\left\|f \cdot \chi_l|L^q\right\|^p\Big)^{1/p} \leq c\Big(\sum_{l=-\infty}^{+\infty} \theta^p(l)\Big(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^r\Big)^{p/r}\Big)^{1/p}.$$

Since $r \leq \min(p,q)$, it holds that $r \leq p$, then from Minkowski inequality we have

$$\left\|f|\dot{K}_q^p(\theta)\right\| \leq c\Big(\sum_{k\in\mathbb{Z}^n}\Big(\sum_{\ell=-\infty}^{\infty} 2^{l\alpha p}\left\|\tau_k\varphi \cdot f \cdot \chi_l|L^q\right\|^p\Big)^{r/p}\Big)^{1/r} = \left\|f|(\dot{K}_q^p(\theta))_{\ell_r}\right\|.$$

This finishes the proof of the theorem. □

Remark 2. *We would like to mention if $\theta \in \mathcal{A}(0,0)$ and $p=q$, then the statements corresponding to Theorem 1 present the localization of Lebesgue spaces L^q on the ℓ_q spaces.*

Motivated by [1, 4, 13], we give the localization property of generalized Herz-type Besov and Herz-type Triebel-Lizorkin spaces on the ℓ_r spaces.

Theorem 2. *Let $\theta \in \mathcal{A}(\alpha,\delta), s \in \mathbb{R}, 1 \le p,q,\beta \le \infty$ such that $-n/q < \alpha,\delta < n(1-1/q)$. Then*

(i) $\dot{K}_q^p B_\beta^s(\theta) \hookrightarrow (\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}$ *for* $r \ge \max(p,q,\beta)$,

(ii) $(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r} \hookrightarrow \dot{K}_q^p B_\beta^s(\theta)$ *for* $r \le \min(p,q,\beta)$.

In particular, $\dot{K}_q^q B_q^s(\theta)$ space is localizable in the ℓ_q-norm.

Proof. Our proofs use partially some techniques already used in [4], where Besov spaces case was studied.

First, we prove (i). We must show that

$$\|f|(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}\| \le c\|f|\dot{K}_q^p B_\beta^s(\theta)\|$$

for all $f \in \dot{K}_q^p B_\beta^s(\theta)$. By Proposition 1, we have

$$\|\tau_k\varphi \cdot f|\dot{K}_q^p B_\beta^s(\theta)\| = \Big\|\sum_{j=0}^{\infty}\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p B_\beta^s(\theta)\Big\| \le \Big(\sum_{j=0}^{\infty}2^{js\beta}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{1/\beta},$$

this implies that,

$$\|f|(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}\| = \Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot f|\dot{K}_q^p B_\beta^s(\theta)\|^r\Big)^{1/r} \le \Big(\sum_{k\in\mathbb{Z}^n}\Big(\sum_{j=0}^{\infty}2^{js\beta}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{r/\beta}\Big)^{1/r}.$$

Since $r \ge \max(p,q,\beta)$ implies $r \ge \beta$, from Minkowski inequality we have

$$\|f|(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}\| \le \Big(\sum_{j=0}^{\infty}2^{js\beta}\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^r\Big)^{\beta/r}\Big)^{1/\beta}. \tag{4}$$

Since $\dot{K}_q^p(\theta) \hookrightarrow (\dot{K}_q^p(\theta))_{\ell_r}$, i.e.

$$\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^r\Big)^{1/r} \le c\|\Delta_j f|\dot{K}_q^p(\theta)\|,$$

the right hand side inequality of (4) is bounded by

$$\|f|(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}\| \le \Big(\sum_{j=0}^{\infty}2^{js\beta}\|\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{1/\beta} = \|f|\dot{K}_q^p B_\beta^s(\theta)\|.$$

(ii) By the localization of Herz spaces in the ℓ_r-norm (with $r \le \min(p,q)$, see Theorem 2 (ii)), we have

$$\|\Delta_j f|\dot{K}_q^p(\theta)\| = \Big\|\sum_{k\in\mathbb{Z}^n}\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\Big\| \le c\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^r\Big)^{1/r},$$

this implies that

$$\|f|\dot{K}_q^p B_\beta^s(\theta)\| = \Big(\sum_{j=0}^{\infty}2^{js\beta}\|\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{1/\beta} \le c\Big(\sum_{j=0}^{\infty}2^{js\beta}\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^r\Big)^{\beta/r}\Big)^{1/\beta}.$$

Since $r \le \min(p,q,\beta)$, it holds that $r \le q$, then from Minkowski inequality we have

$$\begin{aligned}\|f|\dot{K}_q^p B_\beta^s(\theta)\| &= \Big(\sum_{j=0}^{\infty}2^{js\beta}\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^r\Big)^{\beta/r}\Big)^{1/\beta}\\ &\le \Big(\sum_{k\in\mathbb{Z}^n}\Big(\sum_{j=0}^{\infty}2^{js\beta}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p(\theta)\|^\beta\Big)^{r/\beta}\Big)^{1/r} = \|f|(\dot{K}_q^p B_\beta^s(\theta))_{\ell_r}\|.\end{aligned}$$

This finishes the proof of the theorem. □

Remark 3. *We would like to mention if $\theta \in \mathcal{A}(0,0)$ and $p = q$, then the statements corresponding to Theorem 2 can be found in Theorem 2 of [4].*

Theorem 3. *Let $\theta \in \mathcal{A}(\alpha,\delta), s \in \mathbb{R}, 1 \leq p,q,\beta \leq \infty$ such that $-n/q < \alpha,\delta < n(1-1/q)$. Then*

$$\dot{K}_q^p F_\beta^s(\theta) = (\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}.$$

Proof. First, we prove $(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q} \hookrightarrow \dot{K}_q^p F_\beta^s(\theta)$. We must show that

$$\|f|\dot{K}_q^p F_\beta^s(\theta)\| \leq c\|f|(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}\|$$

for all $f \in (\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}$. We have

$$\begin{aligned}\|f|\dot{K}_q^p F_\beta^s(\theta)\| = \Big\|\Big(\sum_{j=0}^{\infty} 2^{js\beta}|\Delta_j f|^\beta\Big)^{1/\beta}|\dot{K}_q^p(\theta)\Big\| &\leq \Big\|\Big(\sum_{j=0}^{\infty}|\sum_{k\in\mathbb{Z}^n} 2^{js}\tau_k\varphi\cdot\Delta_j f|^\beta\Big)^{1/\beta}|\dot{K}_q^p(\theta)\Big\| \\ &\leq \Big\|\Big\|2^{js}\tau_k\varphi\cdot\Delta_j f|\ell_\beta(\ell_1)\Big\|\,|\dot{K}_q^p(\theta)\Big\|.\end{aligned}$$

Since $\beta \geq 1$, by Minkowski inequality, Lemma 3 and Remark 1, we have

$$\begin{aligned}\|f|\dot{K}_q^p F_\beta^s(\theta)\| &\leq c\Big\|\Big\|2^{js}\tau_k\varphi\cdot\Delta_j f|\ell_1(\ell_\beta)\Big\|\,|\dot{K}_q^p(\theta)\Big\| \\ &= c\Big\|\sum_{k\in\mathbb{Z}^n}\Big(\sum_{j=0}^{\infty}|2^{js}\tau_k\varphi\cdot\Delta_j f|^\beta\Big)^{1/\beta}|\dot{K}_q^p(\theta)\Big\| \\ &\leq c\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p F_\beta^s(\theta)\| \leq c\Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot f|\dot{K}_q^p F_\beta^s(\theta)\|^q\Big)^{1/q} \\ &\leq c\|f|(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}\|.\end{aligned}$$

Second, we prove $\dot{K}_q^p F_\beta^s(\theta) \hookrightarrow (\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}$. We must show that

$$\|f|(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}\| \leq c\|f|\dot{K}_q^p F_\beta^s(\theta)\|$$

for all $f \in \dot{K}_q^p F_\beta^s(\theta)$. We have

$$\|f|(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}\| = \Big(\sum_{k\in\mathbb{Z}^n}\|\tau_k\varphi\cdot f|\dot{K}_q^p F_\beta^s(\theta)\|^q\Big)^{1/q} = \Big(\sum_{k\in\mathbb{Z}^n}\Big\|\sum_{j=0}^{+\infty}\tau_k\varphi\cdot\Delta_j f|\dot{K}_q^p F_\beta^s(\theta)\Big\|^q\Big)^{1/q}.$$

By Proposition 1, we obtain

$$\|f|(\dot{K}_q^p F_\beta^s(\theta))_{\ell_q}\| \leq c\Big(\sum_{k\in\mathbb{Z}^n}\Big\|\Big(\sum_{j=0}^{\infty}|2^{js}\tau_k\varphi\cdot\Delta_j f|^\beta\Big)^{1/\beta}|\dot{K}_q^p(\theta)\Big\|^q\Big)^{1/q}.$$

By Theorem 2 (i) with $r = q$, the right hand side inequality of the last inequality is bounded by

$$c\Big\|\Big(\sum_{j=0}^{\infty}|2^{js}\tau_k\varphi\cdot\Delta_j f|^\beta\Big)^{1/\beta}|\dot{K}_q^p(\theta)\Big\| = c\|f|\dot{K}_q^p F_\beta^s(\theta)\|.$$

This finishes the proof of the Theorem. □

Remark 4. *We would like to mention if $\theta \in \mathcal{A}(0,0)$ and $p = q$, then the statements corresponding to Theorem 3 can be found in Theorem 3 of [4].*

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Джеріу А., Гераїз Р. *Деякі результати стосовно властивості локалізації узагальнених просторів Герца, просторів Бєсова типу Герца і просторів Трібеля-Лізоркіна типу Герца* // Карпатські матем. публ. — 2021. — Т.13, №1. — C. 217–228.

У цій статті, використовуючи узагальнені функційні простори типу Герца $\dot{K}_q^p(\theta)$, що були введені Й. Коморі та К. Мацуока у 2009 році, ми визначаємо простори Бєсова типу Герца $\dot{K}_q^p B_\beta^s(\theta)$ і простори Трібеля-Лізоркіна типу Герца $\dot{K}_q^p F_\beta^s(\theta)$, які узагальнюють простори Бєсова і простори Трібеля-Лізоркіна в однорідному випадку, де $\theta = \{\theta(k)\}_{k\in\mathbb{Z}}$ — така послідовність невід'ємних чисел, що

$$C^{-1}2^{\delta(k-j)} \leq \frac{\theta(k)}{\theta(j)} \leq C2^{\alpha(k-j)}, \quad k > j,$$

для деякого $C \geq 1$ (α і δ — дійсні числа).

При зазначених вище умовах на θ ми доводимо, що $\dot{K}_q^p(\theta)$ і $\dot{K}_q^p B_\beta^s(\theta)$ є локалізовні у ℓ_q-нормі при $p = q$, $\dot{K}_q^p F_\beta^s(\theta)$ є локалізовні у ℓ_q-нормі, тобто існує $\varphi \in \mathcal{D}(\mathbb{R}^n)$, що задовольняє $\sum_{k\in\mathbb{Z}^n} \varphi(x-k) = 1$ для довільного $x \in \mathbb{R}^n$ так, що

$$\|f|E\| \approx \Big(\sum_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k)\cdot f|E\|^q\Big)^{1/q}.$$

Вказані результати покращують та узагальнюють відповідні відомі результати для деяких функційних просторів.

Ключові слова і фрази: узагальнений простір Герца, простір Бєсова типу Герца, простір Трібеля-Лізоркіна типу Герца, властивість локалізації.